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Mathematical foundations of information theory
SS 2018

Exercise Sheet 3

11^{th} of April 2018

Exercise 13 (1 Point)**.** Let X be a discrete random variable with finite state space $\{0, 1, 2, 3\}$ and distribution

$$\mathbb{P}[X = k] = \mathrm{poi}(\lambda)(k) \text{ for } k = 1, \ldots 3 \text{ and } \mathbb{P}[X = 0] = 1 - \sum_{i=1}^{3} \mathrm{poi}(\lambda)(i)\,.$$

Find the entropy of X for $\lambda = \frac{1}{4}$. We write $X \sim \mathrm{poi}(\lambda)$ if it is Poisson-distributed with parameter λ, that is

$$\mathbb{P}[X = k] = \frac{\lambda^k}{k!} e^{-\lambda}\,.$$

Exercise 14 (2 Points)**.** Let $(X_n)_{n\in\mathbb{N}}$ a sequence of independent random variables with $\mathbf{E}(X_i) = 0$ and $\mathbf{Var}(X_i) = C < \infty$ for all $i \in \mathbb{N}$. Let $p > 1/2$ and $S_n = X_1 + \ldots + X_n$. Show that $\lim_{n\to\infty} S_n/n^p = 0$ in probability. (Hint: Use the Central Limit Theorem.)

Exercise 15 (2 Points)**.** (Jensen's inequality) Let $f : I \to \mathbb{R}$ be a convex function defined on an intervall $I \subseteq \mathbb{R}$. Let $x_1, \ldots, x_n \in I$ be points in the interval, and let $q_i \in [0, 1], i = 1, \ldots, n$ be coefficients such that $\sum_{i=1}^{n} q_i = 1$. Show that:

$$f\left(\sum_{i=1}^{n} q_i x_i\right) \leq \sum_{i=1}^{n} q_i \cdot f(x_i)\,.$$

We call a function $f : I \to \mathbb{R}$ convex, if it holds for all values $x, y \in I$ and $t \in [0, 1]$ that

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)\,.$$

(Hint: Use induction on n)

Exercise 16 (1 Point)**.** Consider the following two-stage experiment: We first throw the six-sided dice. If the result is 1, we draw a ball from urn A. Otherwise, we draw from urn B. Urn A contains 2 black balls and 3 white balls. Urn B contains 3 red balls and 4 blue balls. Let X be the color of the drawn ball. Calculate the entropy of X.