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Mathematical foundations of information theory
SS 2018
Exercise Sheet 4

18^{th} of April 2018

Exercise 17 (2 Points)**.** Let X be a discrete random variable with values $\{x_1, x_2, \ldots, x_n\}$ and the distribution $\mathbb{P}[X = x_i] = p_i$ where $\sum_{i=1}^{n} p_i = 1$. We denote $p = (p_1, \ldots, p_n)$. Show that

(a) $H(X) \geq 0$ and $H(X) = 0$ iff X is almost surely constant

(b) the maximum of $p \mapsto H(p)$ is reached iff X is uniformly distributed. (Hint: Use Langrange Optimization.)

Exercise 18 (1 Point)**.** Let X be a discrete random variable with values in $\{1, \ldots n\}$. P denotes the distribution on $\{1, \ldots n\}$ when $X \sim \text{bin}(n, p)$ and Q denotes the distribution on $\{1, \ldots n\}$ when $X \sim \text{bin}(n, q)$ for $p, q \in (0, 1)$. Compute the Kullback-Leibler-distance $D(P \,||\, Q)$.
We write $X \sim \text{bin}(n, p)$ if it is Binomial-distributed with parameters n, p, that is

$$\mathbb{P}[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \,.$$

Exercise 19 (2 Points)**.** Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space with $A, B \in \mathcal{A}$, $A \cap B = \emptyset$ and $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{4}$. Let $X, Y : \Omega \to \{0, 1, -1\}$ be random variables defined by

$$X(\omega) = \begin{cases} 1 & \omega \in A \\ -1 & \omega \in B \\ 0 & \text{else} \end{cases} \text{ and } Y(\omega) = \begin{cases} -1 & \omega \in A \\ 1 & \omega \in B \\ 0 & \text{else} \end{cases} .$$

(a) Are X and Y independent?

(b) Show that $H(X) = H(Y) = H(X, Y) = I(X; Y) = \frac{3}{2}$ and $H(X \mid Y) = H(Y \mid X) = 0$.

(c) Let $Z := XY$. Show that $H(Z) < H(X, Y) = H(X, Y, Z)$ and $H(Z \mid X) = 0$ but $H(X \mid Z) > 0$.

(d) Calculate $H(X \mid Z)$ and show that $H(X \mid Z) = I(X; Y \mid Z)$.

Exercise 20 (1 Point)**.** Let $X, Y : (\Omega, \mathcal{A}) \to \mathbb{Q}$ be random variables. Show that X is independent of Y if and only if $I(X; Y) = 0$.

Exercise 21 (2 Points)**.** Let X, Y be the values of two independently thrown dice. Calculate $H(X + Y)$ and $H(X) + H(Y)$ and compare the two values.