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Mathematical foundations of information theory
SS 2018
Exercise Sheet 5

25^{th} of April 2018

Exercise 20 (1 Point)**.** Let $X, Y : (\Omega, \mathcal{A}) \to \mathbb{Q}$ be random variables. Show that X is independent of Y if and only if $I(X;Y) = 0$.

Exercise 22 (2 Points)**.** (Chain rules) Let $X_1, \ldots, X_n, Y$ be random variables. Show that

(a) $H(X_1, \ldots, X_n) = \sum_{i=2}^n H(X_i \mid X_{i-1}, \ldots, X_1) + H(X_1)$ and

(b) $I((X_1, \ldots, X_n); Y) = \sum_{i=2}^n I(X_i; Y \mid X_{i-1}, \ldots X_1) + I(X_1; Y)$.

Exercise 23 (1 Point)**.** Show that (X, Y, Z) is Markovian if and only if X and Z are conditionally independent, that is

$$p_{(X,Z|Y=y)}(x, z) = p_{(X|Y=y)}(x) \cdot p_{(Z|Y=y)}(z)\,.$$

What can we say about $I(X; Z \mid Y)$?

Exercise 24 (1 Point)**.** For $i \in \mathbb{N}$ the random variable Y_i denotes the toss of a fair coin, that is

$$P[Y_i = 0] = P[Y_i = 1] = \frac{1}{2}$$

where 1 means "head". All tosses are repeated independently. For $n \geq 1$ let

$$X_n := Y_n + Y_{n-1}$$

be the numbers of heads in the $(n-1)$th and nth tosses.

(a) Calculate $\mathbb{P}(X_n = 1 \,|\, X_{n-1} = 1, X_{n-2} = 2)$ and $\mathbb{P}(X_n = 1 \,|\, X_{n-1} = 1\,)$.

(b) Calculate $\mathbb{P}(X_n = 2 \,|\, X_{n-1} = 1, X_{n-2} = 2)$ and $\mathbb{P}(X_n = 2 \,|\, X_{n-1} = 1\,)$.

(c) Is $(X_n)_{n \geq 1}$ a Markov Chain?

Exercise 25 (2 Points)**.** Let X, Y, W, Z be random variables on a common state space $\mathcal{X}$, which form the Markov chain $(X, Y, (W, Z))$ that is

$$p(x, y, z, w) = p(x)\, p(y \,|\, x)\, p(z, w \,|\, y)$$

for all $x, y, z, w \in \mathcal{X}$. Show that

$$I(X; Z) + I(X; W) \leq I(X; Y) + I(Z; W)\,.$$

Hint: Show that $I(X;Y) + I(Z;W) - I(X;Z) - I(X;W) = I(Z;W \mid X) - I(Z,W;X) + I(Y;X)$, and argue that $I(Y;X) - I(Z,W;X) \geq 0$.