Adrian Fuchs and Gundelinde Wiegel

Mathematical foundations of information theory
SS 2018
Exercise Sheet 6

2^{nd} of May 2018

Exercise 26 (1 Point)**.** Give an example to show that in general, a function of a Markov chain is not necessarily again a Markov chain (i.e., find a Markov chain $(X_n)_{n\geq 0}$ with finite spate space $\mathcal{X}$ and a function $f : \mathcal{X} \to \mathcal{Y}$ such that $(Y_n)_{n\geq 0}$ with $Y_n = f(X_n)$ is not a Markov chain).

Exercise 27 (4 Point)**.** Consider the Markov chain $(X_n)_{n\geq 0}$ on $\mathcal{X} = \{1, \ldots, 7\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix}$$

(a) Draw the transition graph. Is (X_n) irreducible?

(b) Calculate $\mathbb{P}(X_n = i \mid X_0 = 6)$ for $i \in \mathcal{X}$ and $n \in \{1, 2, 3\}$.

(c) Give a stationary distribution for X_n. Is it unique?

(d) Calculate $\lim_{n\to\infty} \mathbb{P}(X_n = i \mid X_0 = 6)$ for all $i \in \mathcal{X}$.

Exercise 28 (2 Point)**.** We consider the following two-stage problem: There are a fair and an unfair coin. With probability α we choose the fair coin and with probability $1 - \alpha$ we choose the unfair coin. Then we toss the chosen coin again and again independently. For $n \in \mathbb{N}$ the random variable X_n denotes the outcome of the n-th toss, that is $X_n = 1$ if the head shows up in the n-th toss and $X_n = 0$ if the number shows up.

(a) Is $(X_n)_{n\geq 0}$ stationary?.

(b) What is the entropy rate of $(X_n)_{n\geq 0}$?