Adrian Fuchs and Gundelinde Wiegel

Mathematical foundations of information theory & Discrete Stochastics and Information Theory
SS 2018
Exercise Sheet 7

9^{th} of May 2018

Exercise 29 (2 Points)**.** Recall the *Weather in Oz* Markov chain from the lecture with state space $\mathcal{X} = \{b(\text{eautiful}), r(\text{ainy}), s(\text{nowy})\}$ and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} .$$

Today the weather is beautiful. Let X_n be the weather in n days.

(a) Compute the entropy rate of $(X_n)_{n\geq 0}$.

(b) Let $\varepsilon = 2^{-4}$. Consider the two sequences $w_1, w_2 \in \mathcal{X}^{20}$ given by

$$w_1 := brrssrrrsssbrsbsrbss$$
$$w_2 := brrrrrrrrrrrrrrbssss\,.$$

Are w_1 and w_2 in the typical set $A_\varepsilon^{(20)}$?

Exercise 30 (4 Points)**.** Let X be a random variable with values in $\mathcal{X} = \{1, 2, \ldots, m\}$ and distribution $p(x)$, and let $(X_i)_{i\geq 0}$ be i.i.d. random variables with the same distribution. Denote $\mu = \mathbb{E}(X)$ and $H = H(X) = -\sum p(x)\log_2 p(x)$. For $\varepsilon > 0$, consider the sets

$$A^{(n)} = \Big\{(x_1, \ldots, x_n) \in \mathcal{X}^n : \Big| -\frac{1}{n}\log_2 p_n(x_1, \ldots, x_n) - H \Big| \leq \varepsilon \Big\}$$

$$B^{(n)} = \Big\{(x_1, \ldots, x_n) \in \mathcal{X}^n : \Big| \frac{1}{n}\sum_{i=1}^{n} x_i - \mu \Big| \leq \varepsilon \Big\}.$$

(a) Does $\mathbb{P}[(X_1, \ldots, X_n) \in A^{(n)}] \to 1$ for $n \to \infty$?

(b) Does $\mathbb{P}[(X_1, \ldots, X_n) \in A^{(n)} \cap B^{(n)}] \to 1$ for $n \to \infty$?

(c) Show that $|A^{(n)} \cap B^{(n)}| \leq 2^{n(H+\varepsilon)}$ for all n.

(d) Show that $|A^{(n)} \cap B^{(n)}| \geq (\frac{1}{2})2^{n(H-\varepsilon)}$ for all n sufficiently large.

Exercise 31 (3 Points)**.** Let $(X_i)_{i\geq 1}$ be a stochastic process with finite state space $\mathcal{X}$. Show that if one of the limits

$$\lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n)$$
$$\lim_{n\to\infty} \frac{1}{n} H(X_{k+1}, \ldots, X_{k+n})$$
$$\lim_{n\to\infty} \frac{1}{n} H(X_{k+1}, \ldots, X_{k+n} \mid X_1, \ldots X_k)$$

exists, then also the other limits exist and all three coincide.