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Mathematical foundations of information theory & Discrete Stochastics and Information Theory SS 2018

Exercise Sheet 8

16^{th} of May 2018

Exercise 32 (2 Points)**.** Let

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{6}, \\ 8 & \text{with probability } \frac{1}{3}, \\ 9 & \text{with probability } \frac{1}{2}. \end{cases}$$

Let $X_1, X_2, \ldots$ be drawn independently and identically distributed (i.i.d.) according to the distribution of X. Find the limiting behavior of $(X_1 \cdot X_2 \cdots X_n)^{1/n}$ as $n \to \infty$.
Hint: How could the target limit be transformed/rewritten to apply the ergodic theorem or the law of large numbers?

Exercise 33 (2 Points)**.** Let $\mathcal{X} = \{a, b, c, d\}$ and

$$C(a) = 10, \;\; C(b) = 00, \;\; C(c) = 11, \;\; C(d) = 110\,.$$

(a) Show that this code is not prefix free but uniquely decodable.

(b) Show that there are arbitrarily long words $x_1 \cdots x_n$ such that $C(x_1 \cdots x_n)$ can only be decoded at the very end, that is for no $k < n$, the initial element x_1 can already be recovered from $C(x_1 \cdots x_k)$.

Exercise 34 (4 Points)**.** Let $\mathcal{X} = \{a, b, c, d\}$, and let $X : \Omega \to \mathcal{X}$ be a random variable with

$$\mathbb{P}[X = a] = \frac{3}{8}, \qquad \mathbb{P}[X = b] = \frac{2}{8}, \qquad \mathbb{P}[X = c] = \frac{2}{8}, \qquad \mathbb{P}[X = d] = \frac{1}{8}.$$

The elements of $\mathcal{X}$ are encoded as follows:

$$C(a) = 00, \qquad C(b) = 01, \qquad C(c) = 11, \qquad C(d) = 001\,.$$

(a) Is the code C (i) non-singular, (ii) prefix-free, (iii) uniquely decodable?

(b) Calculate the entropy $H(X)$ and the expected length $\mathbb{E}(\ell(C))$ where

$$\mathbb{E}(\ell(C)) = \sum_{x \in \mathcal{X}} \ell(C(x)) \cdot p(x)\,.$$

(c) Give a better code for this random variable (prefix-free, shorter expected length).

Exercise 35 (2 Points)**.** Give an example for a prefix free code on $\mathcal{X} = \{b(\text{eautiful}), r(\text{ain}),$ $s(\text{now}),\text{w(ind)}\}$ which is not postfix free. Use it to construct another example of a uniquely decodable code which is not instantaneous.