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Mathematical foundations of information theory & Discrete Stochastics and Information Theory
SS 2018
Exercise Sheet 9

23^{rd} of May 2018

Exercise 36 (2 Points)**.** Let $\mathcal{X} = \{a, b, c, d, e\}$.

(a) Give an example of a prefix code $C : \mathcal{X} \to \{0,1\}^*$ such that

$$\ell(C(a)) = 1, \qquad \ell(C(b)) = 2, \qquad \ell(C(c)) = 3, \qquad \ell(C(d)) = 4, \qquad \ell(C(e)) = 5\,.$$

(b) Show that the Kraft inequality is a strict inequality for this code, that is it holds

$$\sum_{x\in\mathcal{X}} D^{-\ell(C(x))} < 1.$$

(c) Give an example of an arbitrary binary sequence of length 5 which cannot be decoded.

(d) Give an example of a prefix code $C : \mathcal{X} \to \{0,1\}^*$ such that we have equality in the Kraft inequality.

Exercise 37 (1 Point)**.** Which of the following codes can not arise as Huffman-Codes?
(a) $\{01, 10\}$, (b) $\{0, 10, 11\}$, (c) $\{00, 01, 10, 110\}$

Exercise 38 (2 Points)**.** Let $X \in \{1, 2, 3, 4\}$ be a random variable for which following two codes are given:

$$C_1(1) = 0, C_1(2) = 10, C_1(3) = 110, C_1(4) = 111$$
$$C_2(1) = 00, C_2(2) = 01, C_2(3) = 10, C_2(4) = 11\,.$$

(a) Which of the two codes is the optimal for the distribution

$$\mathbb{P}(X = 1) = \frac{1}{4} \quad \mathbb{P}(X = 2) = \frac{1}{4} \quad \mathbb{P}(X = 3) = \frac{1}{4} \quad \mathbb{P}(X = 4) = \frac{1}{4}\,?$$

(b) Which of the two codes is the optimal for the distribution

$$\mathbb{P}(X = 1) = \frac{1}{2} \quad \mathbb{P}(X = 2) = \frac{1}{4} \quad \mathbb{P}(X = 3) = \frac{1}{8} \quad \mathbb{P}(X = 4) = \frac{1}{8}\,?$$

(c) Is one of the two codes a Huffman code (and therefore optimal) for the distribution

$$\mathbb{P}(X = 1) = \frac{3}{4} \quad \mathbb{P}(X = 2) = \frac{1}{12} \quad \mathbb{P}(X = 3) = \frac{1}{12} \quad \mathbb{P}(X = 4) = \frac{1}{12}\,?$$

Exercise 39 (3 Points)**.** Let$\mid \Sigma \mid= D$ and $C : \mathcal{X} \to \Sigma*$ be a prefix code. Show with an alternative proof (than the one from the lecture) that

$$L_C \geq H_D(X)\,,$$

where equality holds if and only if $p(x) = D^{-\ell(C(x))}$.
(Hint: Consider the length of the Code as a real variable, that is $\ell_x \in \mathbb{R}^+$ for all $x \in \mathcal{X}$. Use Langrange to minimize the expected length of the code under the condition that the code satisfies $\sum_{x\in\mathcal{X}} D^{-\ell(C(x))} = 1$ which means that equality holds in the Kraft-inequality.)