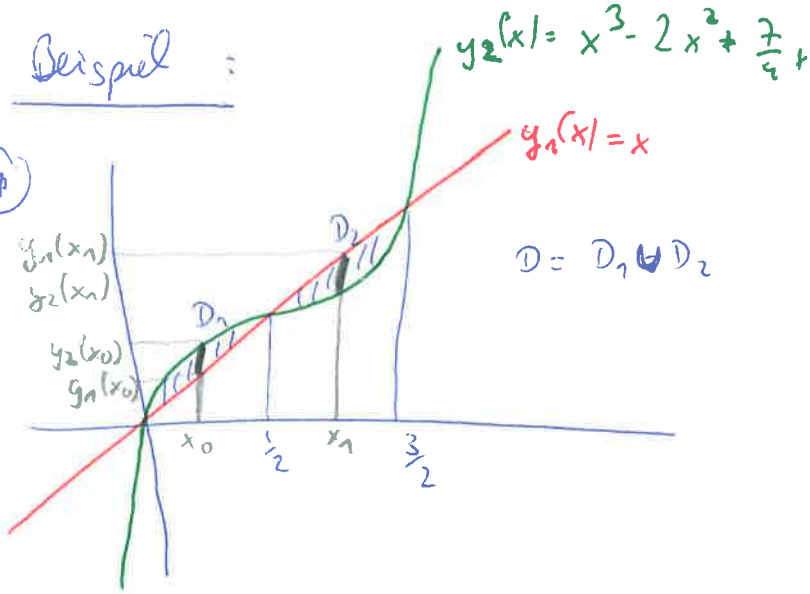
Beispiel:



$D = D_1 \cup D_2$

Gesucht: Fläche von D

$$\int_D 1 \, dx\, dy = \underbrace{\int_{D_1} 1 \, dx\, dy}_{F_1} + \underbrace{\int_{D_2} 1 \, dx\, dy}_{F_2}$$

F_1: a) x läuft in D_1 zwischen 0 und $\frac{1}{2}$

b) Sei $x_0 \in [0, \frac{1}{2}]$. Dann läuft y in D_1 zwischen $y_1(x_0)$ und $y_2(x_0)$

$$\Rightarrow F_1 = \int_{D_1} 1 \, dx\, dy = \int_{x=0}^{\frac{1}{2}} \int_{y=y_1(x)}^{y_2(x)} 1 \, dy\, dx = \int_{x=0}^{\frac{1}{2}} y_2(x) - y_1(x) \, dx$$

$$= \int_{x=0}^{\frac{1}{2}} x^3 - 2x^2 + \frac{3}{4}x \, dx = \frac{x^4}{4} - \frac{2}{3}x^3 + \frac{3}{8}x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{64} - \frac{2}{24} + \frac{3}{32} = \underline{\frac{5}{192}}$$

F_2: a) x läuft in D_2 zwischen $\frac{1}{2}$ und $\frac{3}{2}$

b) Sei $x_1 \in [\frac{1}{2}, \frac{3}{2}]$. Dann läuft y in D_1 zwischen $y_2(x_1)$ und $y_1(x_1)$

$$\Rightarrow F_2 = \int_{D_2} 1 \, dx\, dy = \int_{x=\frac{1}{2}}^{\frac{3}{2}} \int_{y=y_2(x)}^{y_1(x)} 1 \, dy\, dx = \int_{x=\frac{1}{2}}^{\frac{3}{2}} y_1(x) - y_2(x) \, dx$$

$$= \int_{x=\frac{1}{2}}^{\frac{3}{2}} -x^3 + 2x^2 - \frac{3}{4}x = -\frac{x^4}{4} + \frac{2}{3}x^3 - \frac{3}{8}x^2 \Big|_{\frac{1}{2}}^{\frac{3}{2}} = -\frac{81}{64} + \frac{54}{24} - \frac{27}{32} - \frac{5}{192} = \frac{11}{96}$$

$$\Rightarrow \int_D 1 \, dx\, dy = \frac{5}{192} + \frac{11}{96} = \underline{\underline{\frac{6}{94}}}$$