zu 2) Spezielle Lösung:

1. Möglichkeit: Variation der Konstanten

2. Möglichkeit: Ansatzmethode

Falls $b(x) = e^{\lambda x} \begin{pmatrix} p_1(x) \\ \vdots \\ p_n(x) \end{pmatrix}$ mit $\max \deg p_i(x) = m$

$\rightarrow$ Ansatz: $y_{sp}(x) = e^{\lambda x} \begin{pmatrix} Q_1(x) \\ \vdots \\ Q_2(x) \end{pmatrix}$

wobei $Q_1(x), \ldots, Q_m(x)$ Polynome vom Grad $m + \mu(\lambda)$
mit unbekannten Koeffizienten

$\rightarrow$ y_{sp} einsetzen in DGL

$\rightarrow$ Durch Koeffizientenvergleich Polynome $Q_1, \ldots, Q_m$ bestimmen

$\rightarrow$ so erhält man $y_{sp}(x)$

Beispiel von oben:

$$y_1' = y_1 - 4y_2 + 2$$
$$y_2' = -y_1 + y_2 + 1$$

$\lambda_1 = -1, \quad \lambda_2 = 3$

$b(x) = e^{0 \cdot x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ also $m = 0$, $\mu(0) = 0$

Ansatz: $y_{sp}(x) = e^{0 \cdot x} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$

$y'_{sp}(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Einsetzen:

$0 = A - 4B + 2$ $\rightarrow$ $0 = B+1-4B+2 = -3B+3$

$\rightarrow B = 1$

$0 = -A + B + 1$ $\rightarrow$ $A = B+1$ $\rightarrow$ $A = 2$

$\rightarrow y_{sp}(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Falls $b(x) = e^{-t} \begin{pmatrix} x^2 \\ x \end{pmatrix}$ $m = 2$, $\mu(1) = 1$

Ansatz: $\vec{y}_{sp}(x) = e^{-t} \begin{pmatrix} A + Bx + Cx^2 + Dx^3 \\ E + Fx + Gx^2 + Hx^3 \end{pmatrix}$

Andere Störfunktion:

$$y_1' = y_1 - 4y_2 + e^{3x}$$
$$y_2' = -y_1 + y_2 + 2e^{3x} \qquad \text{also } b(x) = e^{3x}\begin{pmatrix}1\\2\end{pmatrix}$$

Ansatz: $\vec{y}_{s,p}(x) = e^{3x}\begin{pmatrix}A+Bx\\C+Dx\end{pmatrix}$ $\qquad m=0$, $\mu(3)=1$

$$\vec{y}_{s,p}'(x) = 3e^{3x}\begin{pmatrix}A+Bx\\C+Dx\end{pmatrix} + e^{3x}\begin{pmatrix}B\\D\end{pmatrix}$$

Einsetzen: $3A + 3Bx + B = A + Bx - 4C - 4Dx + 1$
$3C + 3Dx + D = -A - Bx + C + Dx + 2$

$$\Rightarrow \begin{cases} 3A + B = A - 4C + 1 \\ 3B = B - 4D \quad \rightarrow 2B = -4D \rightarrow \underline{B = -2D} \\ 3C + D = -A + C + 2 \\ 3D = -B + D \quad \rightarrow 2D = -B = 2D \rightarrow D = D \end{cases}$$

$2A + B = -4C + 1$
$2C + D = -A + 2 \quad \rightarrow A = 2 - 2C - D$
$\rightarrow 4 - \cancel{4C} - 2D - 2D = -\cancel{4C} + 1$
$\rightarrow 4 - 4D = 1 \quad \rightarrow 4D = 3$
$\rightarrow \underline{D = \frac{3}{4}}, \underline{B = -\frac{3}{2}}$

$\rightarrow$ ~~[illegible]~~

$$\Rightarrow \begin{cases} 2A - \frac{3}{2} = -4C + 1 \\ 3C + \frac{3}{4} = -A + C + 2 \quad \rightarrow A = 2 - 2C - \frac{3}{4} \end{cases}$$

$4A - 4C - \frac{3}{2} - \frac{3}{2} = -\cancel{4C} + 1$

$C = 0 \quad \Rightarrow A = 2 - \frac{3}{4} = \underline{\frac{5}{4}}$

$$y_{s,p}(x) = e^{3x}\begin{pmatrix}\frac{5}{4} - \frac{3}{2}x\\ \frac{3}{4}x\end{pmatrix}$$

$$y_{alle}(x) = C_1 e^{3x}\begin{pmatrix}-2\\1\end{pmatrix} + C_2 e^{-x}\begin{pmatrix}2\\1\end{pmatrix} + e^{3x}\begin{pmatrix}\frac{5}{4} - \frac{3}{2}x\\ \frac{3}{4}x\end{pmatrix}$$

AWP:
$y_1(0) = 0 \qquad 0 = -2C_1 + 2C_2 + \frac{5}{4}$
$y_2(0) = 1 \qquad 1 = C_1 + C_2 \quad \rightarrow C_1 = 1 - C_2$

in I) $0 = -2 + 2C_2 + 2C_2 + \frac{5}{4} \Rightarrow \frac{3}{4} = 4C_2 \Rightarrow \underline{C_2 = \frac{3}{16}}$
$\Rightarrow \underline{C_1 = \frac{13}{16}}$

~~Falls b(x)~~

Beispiel

$$y_1' = 2y_1 - y_2 + e^{2x}\cos x$$
$$y_2' = y_1 + 2y_2 + x$$

Also $\vec{y}' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \vec{y} + b(x)$

mit $b(x) = \cancel{b_1(x) + b_2(x)} =$

$$= \begin{pmatrix} e^{2x}\cos x \\ x \end{pmatrix} = \underbrace{e^{0\cdot x}\begin{pmatrix} 0 \\ x \end{pmatrix}}_{b_1(x)} + \underbrace{e^{2x}\begin{pmatrix} \cos x \\ 0 \end{pmatrix}}_{b_2(x)}$$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - (2+i))(\lambda - (2-i))$$

EV zu $\lambda_1 = 2+i$: $\quad v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\vec{y}(x) = e^{2x}(\cos x + i \sin x)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\longrightarrow \vec{y}_{hom} = C_1 e^{2x}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + C_2 e^{2x}\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}$$

Suchen spezielle Lösung zu $b_1(x)$

Ansatz $y_{sp,1}(x) = e^{2x}\begin{pmatrix} A\cos x + B\sin x \\ C\cos x + D\sin x \end{pmatrix}$

$$y_{sp}'(x) = 2e^{2x}\begin{pmatrix} A\cos x + B\sin x \\ C\cos x + D\sin x \end{pmatrix} + e^{2x}\begin{pmatrix} -A\sin x + B\cos x \\ -C\sin x + D\cos x \end{pmatrix}$$

$$\overset{!}{=} \begin{pmatrix} 2e^{2x}(A\cos x + B\sin x) - e^{2x}(C\cos x + D\sin x) + e^{2x}\cos x \\ e^{2x}(A\cos x + B\sin x) - 2e^{2x}(C\cos x + D\sin x) \end{pmatrix}$$

$$\Rightarrow \sin x\,(2B - A) + \cos x\,(2A + B) = \cos x\,(2A - C + 1) + \sin x\,(2B - D)$$
$$\sin x\,(2D - C) + \cos x\,(2C + D) = \cos x\,(A - 2C) + \sin x\,(B - 2D)$$

$$\Rightarrow \begin{cases} 2B - A = 2B - D \longrightarrow \underline{A = D} \\ \cancel{2A} + B = \cancel{2A} - C + 1 \longrightarrow \underline{B = 1 - C} \\ 2D - C = B - 2D = 1 - \cancel{C} - 2D \longrightarrow 4D = 1 \longrightarrow \underline{D = \tfrac{1}{4} = A} \\ 2C + D = A - 2C \longrightarrow 4C = A - D = 0 \longrightarrow \underline{C = 0} \end{cases}, \quad \underline{B = 1}$$

$$\longrightarrow y_{sp,1}(x) = e^{2x}\begin{pmatrix} \tfrac{1}{4}\cos x + \sin x \\ \tfrac{1}{4}\sin x \end{pmatrix}$$

Ansatz für $y_{sp,2}(x)$

$$y_{sp,2}(x) = e^{0\cdot x}\begin{pmatrix} A + Bx \\ C + Dx \end{pmatrix}$$

$$\longrightarrow \quad y'_{sp,2}(x) = \begin{pmatrix} B \\ D \end{pmatrix}$$

Einsetzen:

$$B = 2A + 2Bx - C - Dx$$
$$D = A + Bx + 2C + 2Dx + x$$

$$\Longrightarrow \quad 0 = 2B - D \longrightarrow \underline{D = 2B}$$

ii) $B = 2A - C$

$$0 = B + 2D + 1 \longrightarrow 0 = B + 4B + 1 \Longrightarrow \underline{B = -\tfrac{1}{5}}, \ \underline{D = -\tfrac{2}{5}}$$

iv) $D = A + 2C$

$$\longrightarrow \text{ii)} \ -\tfrac{2}{5} = 2A - C \longrightarrow C = \underline{C = 2A + \tfrac{2}{5}}$$

iv) $-\tfrac{2}{5} = A + 4A + \tfrac{4}{5} = 5A + \tfrac{4}{5} \Longrightarrow 5A = -\tfrac{6}{5} \longrightarrow \underline{A = -\tfrac{6}{25}}$

$$\longrightarrow C = -\tfrac{12}{25} + \tfrac{2}{5} = \underline{-\tfrac{8}{25}}$$

$$\longrightarrow \quad y_{sp,2}(x) = \begin{pmatrix} -\tfrac{6}{25} + -\tfrac{1}{5}x \\ -\tfrac{8}{25} - \tfrac{2}{5}x \end{pmatrix}$$

$$\longrightarrow \quad y_{sp} = y_{sp,1} + y_{sp,2}$$

Beispiel:

$$y_1' = 2y_1 - y_2 + e^{2x}\cos x$$
$$y_2' = y_1 + 2y_2$$

Also $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$$p(\lambda) = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda - (2+i))(\lambda - (2-i))$$

EV zu $\lambda_1 = 2+i$: $\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\vec{y}_1(x) = e^{2x}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{2x}(\cos x + i \sin x)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\Rightarrow \vec{y}_{hom}(x) = C_1 e^{2x}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + C_2 e^{2x}\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}$$

Suchen spezielle Lösung

Ansatz: $\vec{y}_{1,sp}(x) = e^{(2+i)x}\begin{pmatrix} A + Bx \\ C + Dx \end{pmatrix}$

$$b_1(x) = e^{(2+i)x}\begin{pmatrix} \frac{1}{2} - 0\frac{i}{2} \\ 0 - 0\frac{i}{2} \end{pmatrix} = e^{(2+i)x}\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

$$\vec{y}_{1,sp}'(x) = (2+i)e^{(2+i)x}\begin{pmatrix} A + Bx \\ C + Dx \end{pmatrix} + e^{(2+i)x}\begin{pmatrix} B \\ D \end{pmatrix}$$

Einsetzen:

$$\begin{cases} (2+i)A + (2+i)Bx + B = 2A + 2Bx - C - Dx + \frac{1}{2} \\ (2+i)C + (2+i)Dx + D = A + Bx + 2C + 2Dx \end{cases}$$

$$\Rightarrow \begin{cases} (2+i)A + B = 2A - C + \frac{1}{2} & \cancel{\rightarrow iA = 2A - C + \frac{1}{2} \rightarrow C = \frac{1}{2} - iA} \\ (2+i)B = 2B - D & \rightarrow iB = -D \\ (2+i)C + D = A + 2C & \\ (2+i)D = B + 2D & \rightarrow iD = B \end{cases}$$

$\cancel{\rightarrow D = iB = -i^2 D = D \rightarrow 2D = 0}$

D beliebig $\rightarrow B = iD$

z.B $D = 0 \rightarrow B = 0$

$\cancel{\rightarrow iC = A \rightarrow C = \frac{1}{2} - i^2 C}$

$\cancel{iA + iD = -C + \frac{1}{2} \Rightarrow C = \frac{1}{2} - iA + iD} = \frac{1}{2} - iD + C + iC$

$iC + D = A \rightarrow A = D + iC$

$\Rightarrow iC = -\frac{1}{2} + iD \quad |\cdot(-i)$

$\Rightarrow C = \frac{i}{2} + D = \frac{i}{2}$

$\rightarrow A = 2D - \frac{1}{2} = -\frac{1}{2}$

$$\Rightarrow \begin{cases} iA + B = -C + \frac{1}{2} \\ iB = -D \\ iC + D = A \\ iD = B \longrightarrow -i^2 B = B \end{cases}$$

$$A = iC + D$$

$$B = -C + \tfrac{1}{2} - iA = \cancel{-C} + \tfrac{1}{2} + \cancel{C} - iD$$

$$B = \tfrac{1}{2} - iD$$

$$iD = \tfrac{1}{2} - iD \longrightarrow 2iD = \tfrac{1}{2}$$

$$\longrightarrow D = \frac{1}{4i} = -\frac{i}{4}$$

$$\longrightarrow B = \frac{1}{2} - i\,\frac{-i}{4} = \frac{1}{4}$$

$$A = iC - \frac{i}{4} = -\frac{i}{4}$$

$$\longrightarrow \vec{y}_{p,1}(x) = e^{(2+i)x} \begin{pmatrix} -\frac{i}{4} + \frac{1}{4}x \\ -\frac{i}{4}x \end{pmatrix}$$

$$\mathrm{Re}\ \vec{y}_{p}(x) = \mathrm{Re}\ e^{2x} \begin{pmatrix} -\frac{i}{4}\cos x + \frac{1}{4}\sin x + \frac{1}{4}x\cos x + \frac{i}{4}x\sin x \\ -\frac{i}{4}x\cos x + \frac{1}{4}x\sin x \end{pmatrix}$$

$$= e^{2x} \begin{pmatrix} \frac{1}{4}\sin x + \frac{1}{4}x\cos x \\ \frac{1}{4}x\sin x \end{pmatrix}$$

$$\vec{y}_{p}(x) = 2\, e^{2x} \begin{pmatrix} \frac{1}{4}\sin x + \frac{1}{4}x\cos x \\ \frac{1}{4}x\sin x \end{pmatrix}$$