Beispiel: $y_1' = y_1 - 4y_2 + e^{2x}(x+1)$
$y_2' = -y_1 + y_2 + 3e^{2x}$

AWP: $y_1(0) = 1$, $y_2(0) = 0$

$$P(\lambda) = \det\begin{pmatrix} 1-\lambda & -4 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$\lambda_{1,2} = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} \quad \begin{matrix} \nearrow 3 \\ \searrow -1 \end{matrix}$$

Eigenwerte: $\lambda_1 = -1$, $\lambda_2 = 3$

EV zu $\lambda_1 = -1$: $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \vec{0} \longrightarrow u = 2v$, setze $v = 1$: EV: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

EV zu $\lambda_2 = 3$: $\begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \vec{0} \longrightarrow u = -2v$, setze $v = 1$: EV: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\vec{y}_{hom}(x) = C_1 e^{-x}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3x}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Spezielle Lösung: $\vec{b}(x) = e^{2x}\begin{pmatrix} x+1 \\ 3 \end{pmatrix}$ also $m = 1$, $\mu(2) = 0$, da 2 kein EW

Ansatz: $\vec{y}_{sp}(x) = e^{2x}\begin{pmatrix} A + Bx \\ C + Dx \end{pmatrix}$

$$\longrightarrow \vec{y}_{sp}\,'(x) = 2e^{2x}\begin{pmatrix} A + Bx \\ C + Dx \end{pmatrix} + e^{2x}\begin{pmatrix} B \\ D \end{pmatrix} = e^{2x}\begin{pmatrix} 2A + B + Bx \\ 2C + Dx + D \end{pmatrix}$$

Einsetzen:
$$\begin{cases} e^{2x}(2A + B + Bx) = e^{2x}(A + Bx) - 4e^{2x}(C + Dx) + e^{2x}(x+1) \\ e^{2x}(2C + D + Dx) = -e^{2x}(A + Bx) + e^{2x}(C + Dx) + 3e^{2x} \end{cases} \Bigg| : e^{2x}$$

$$\Longleftrightarrow \begin{cases} (2A + B) + Bx = (A - 4C + 1) + x(B - 4D + 1) \\ (2C + D) + Dx = (-A + C + 3) + x(-B + D) \end{cases}$$

Koeffizientenvergleich: $2A + B = A - 4C + 1$ & $B = B - 4D + 1 \longrightarrow D = \frac{1}{4}$
$2C + D = -A + C + 3$ & $D = -B + D \longrightarrow B = 0$

$\longrightarrow A = -4C + 1$ $\quad$ $C + \frac{1}{4} = A + 3 = -4C + 4 \longrightarrow C = \frac{4}{5}$, $A = -\frac{16}{5} + 1 = -\frac{11}{5}$

$$\vec{y}_{all}(x) = C_1 e^{-x}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{3x}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^{2x}\begin{pmatrix} -\frac{11}{5}x \\ \frac{4}{5} + \frac{1}{4}x \end{pmatrix}$$

AWP: $1 = 2C_1 - 2C_2$ $\quad\longrightarrow 1 = -2C_2 - \frac{8}{5} - 2C_2 = -4C_2 - \frac{8}{5}$

$0 = C_1 + C_2 + \frac{4}{5} \longrightarrow C_1 = -C_2 - \frac{4}{5}$ $\quad\longrightarrow C_2 = -\frac{13}{20}$ & $C_1 = \frac{13}{20} - \frac{4}{5} = -\frac{7}{20}$