Beispiel 4 $\vec{y}\,' = \underbrace{\begin{pmatrix}3 & -5\\ 2 & 1\end{pmatrix}}_{A}\vec{y}$

- $P(\lambda)=0 \iff \det|A-\lambda I| = \begin{vmatrix}3-\lambda & -5\\ 2 & 1-\lambda\end{vmatrix} = (3-\lambda)(1-\lambda)+10 =$

$$= \lambda^2-4\lambda+13=0 \Rightarrow \begin{cases}\lambda_{1,2}=2\pm 3i\\ \mu(\lambda_{1,2})=1\end{cases}$$

- Eigenvektor zu $\boxed{\lambda_1=2+3i}$

$(A-\lambda_1 I)\vec{v}^{(1)}=\vec{0} \iff$

$$\begin{pmatrix}1-3i & -5\\ 2 & -1-3i\end{pmatrix}\begin{pmatrix}v_1^{(1)}\\ v_2^{(1)}\end{pmatrix}=0 \Rightarrow$$

$\underline{I}\ (1-3i)\cdot v_1^{(1)} = 5v_2^{(1)} \Rightarrow$

$\Rightarrow v_2^{(1)} = \left(\frac{1}{5}-\frac{3}{5}i\right)v_1^{(1)}$

(Zeilen sind linear abhängig

$\underline{II}\ 2v_1^{(1)}-\frac{1}{5}(1+3i)(1-3i)v_1^{(1)}=0 \Rightarrow 2v_1^{(1)}-\frac{10}{5}v_1^{(1)}=0 \iff 0=0$

$$\Rightarrow \begin{cases}v_1^{(1)}=t\\ v_2^{(1)}=\left(\frac{1}{5}-\frac{3}{5}i\right)t\end{cases} \Rightarrow \vec{v}^{(1)}=t\begin{pmatrix}1\\ \frac{1}{5}-\frac{3}{5}i\end{pmatrix}$$

Wähle $\underline{t=5}$: $\boxed{\vec{v}^{(1)}=\begin{pmatrix}5\\ 1-3i\end{pmatrix}}$

Lösung: $\vec{y}^{[1]}$: $e^{(2+3i)x}\cdot\begin{pmatrix}5\\ 1-3i\end{pmatrix} = e^{2x}(\cos 3x+i\sin 3x)\left(\begin{pmatrix}5\\1\end{pmatrix}+i\begin{pmatrix}0\\-3\end{pmatrix}\right)$

$$= e^{2x}\left(\cos 3x\begin{pmatrix}5\\1\end{pmatrix}+i\cos 3x\begin{pmatrix}0\\-3\end{pmatrix}+i\sin 3x\begin{pmatrix}5\\1\end{pmatrix}\ \sin 3x\begin{pmatrix}0\\-3\end{pmatrix}\right)=$$

$$= \underbrace{e^{2x}\left(\cos 3x\begin{pmatrix}5\\1\end{pmatrix}-\sin 3x\begin{pmatrix}0\\3\end{pmatrix}\right)}_{\text{Lösung 1}} + i\underbrace{e^{2x}\left[\cos 3x\begin{pmatrix}0\\-3\end{pmatrix}+\sin 3x\begin{pmatrix}5\\1\end{pmatrix}\right]}_{\text{Lösung 2}} =$$

$$\vec{y}^{[1]}(x) = e^{2x}\left(\cos 3x\begin{pmatrix}5\\1\end{pmatrix}-\sin 3x\begin{pmatrix}0\\3\end{pmatrix}\right) + e^{2x}\left(\cos 3x\begin{pmatrix}0\\-3\end{pmatrix}+\sin 3x\begin{pmatrix}5\\1\end{pmatrix}\right)$$

Eigenvektor zu $\lambda_2 = 2-3i$: $\vec{v}^{(2)} = \overline{\vec{v}_1^{(1)}} = \begin{pmatrix}5\\1+3i\end{pmatrix}$

$$e^{(2-3i)x}\begin{pmatrix}5\\1+3i\end{pmatrix} = e^{2x}\Big(\underbrace{\cos(-3x)}_{\cos(3x)} + i\,\underbrace{\sin(-3x)}_{-\sin(3x)}\Big)\left[\begin{pmatrix}5\\1\end{pmatrix} + i\begin{pmatrix}0\\3\end{pmatrix}\right] =$$

$$= e^{2x}\left[\cos(3x)\begin{pmatrix}5\\1\end{pmatrix} + i\cos(3x)\begin{pmatrix}0\\3\end{pmatrix} - i\sin(3x)\begin{pmatrix}5\\1\end{pmatrix} + \sin 3x\begin{pmatrix}0\\3\end{pmatrix}\right]$$

$$= \underbrace{e^{2x}\left[\cos(3x)\begin{pmatrix}5\\1\end{pmatrix} + \sin(3x)\begin{pmatrix}0\\3\end{pmatrix}\right]}_{Re} + i\,\underbrace{e^{2x}\left[\cos(3x)\begin{pmatrix}0\\3\end{pmatrix} - \sin(3x)\begin{pmatrix}5\\1\end{pmatrix}\right]}_{Im}$$

$$\Rightarrow \vec{y}^{[2]} = e^{2x}\left[\cos(3x)\begin{pmatrix}5\\1\end{pmatrix} + \sin(3x)\begin{pmatrix}0\\3\end{pmatrix}\right] +$$
$$e^{2x}\left[\cos(3x)\begin{pmatrix}0\\3\end{pmatrix} - \sin(3x)\begin{pmatrix}5\\1\end{pmatrix}\right]$$

$$\vec{y}(x) = C_1\vec{y}^{[1]} + C_2\vec{y}^{[2]}$$

$\{\vec{y}^{[1]}, \vec{y}^{[2]}\} \rightarrow$ Fundamentalsystem.

Spezielle Lösung der inhomogenen Gleichung $\vec{y}\,' = A\vec{y} + \vec{b}(x)$

! Variation der Konstanten → machbar, aber aufwendig

! Ansatzmethode → Effizienter

Beispiel 1* → Beispiel 1) → Seite 4 mit $b(x) \neq \vec{0}$

$$\vec{y}\,' = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \vec{y} + \vec{b}(x)$$

$$\boxed{P(\lambda): \begin{cases} \lambda_1 = 1 & ; \mu(1) = 1 \\ \lambda_2 = 5 & ; \mu(5) = 1 \end{cases}}$$

$\boxed{A}$ $\vec{b}(x) = e^{2x} \begin{pmatrix} 1 \\ x \end{pmatrix}$

$\left.\begin{array}{l} p_1(x) = 1,\ \text{grad} = 0 \\ p_2(x) = x,\ \text{grad} = 1 \end{array}\right\}$ max grad $= 1 = m$ $\quad \Big\}\ m + \mu(2) = 1$

2 ist keine Nullstelle von $P(\lambda) \Rightarrow \mu(2) = 0$

<u>Ansatz</u>: $\boxed{\vec{y}_{sp}(x) = e^{2x} \begin{pmatrix} ax + b \\ cx + d \end{pmatrix}}$; $a, b, c, d = ?$

<u>Einsetzen und Koeffvergleich</u>:

$$\vec{y}_{sp}\,'(x) = 2e^{2x} \begin{pmatrix} ax + b \\ cx + d \end{pmatrix} + e^{2x} \begin{pmatrix} a \\ c \end{pmatrix} = e^{2x} \begin{pmatrix} 2ax + 2b + a \\ 2cx + 2d + c \end{pmatrix}$$

$$\Rightarrow e^{2x} \begin{pmatrix} 2ax + 2b + a \\ 2cx + 2d + c \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \cdot e^{2x} \begin{pmatrix} ax + b \\ cx + d \end{pmatrix} + e^{2x} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

$$\Rightarrow \begin{cases} 2ax + 2b + a = 3ax + 3b + 2cx + 2d + 1 \\ 2cx + 2d + c = 2ax + 2b + 3cx + 3d + x \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \begin{cases} 2a = 3a + 2c & \Rightarrow a = -2c \\ 2b + a = 3b + 2d + 1 & \Rightarrow -b + a = 2d + 1 \Rightarrow -b = 2d + 1 + \frac{2}{3} \end{cases} \\ \begin{cases} 2c = 2a + 3c + 1 & \Rightarrow -c = 2a + 1 \\ 2d + c = 2b + 3d \end{cases} \end{cases}$$

$\Rightarrow -c = 2(-2c) + 1 \Rightarrow 3c = 1 \Rightarrow \boxed{c = \frac{1}{3}}$

$a = -\frac{2}{3}$

$\circledast$ $-b = 2d + \frac{5}{3} \Rightarrow b = -2d - \frac{5}{3}$

IV gl. $2d + \frac{1}{3} = 2\left(-2d - \frac{5}{3}\right) + 3d \Rightarrow 2d + \frac{1}{3} = -4d - \frac{10}{3} + 3d \Rightarrow$

$3d = -\frac{11}{3} \Rightarrow \boxed{d = -\frac{11}{9}}$

$b = \frac{22}{9} - \frac{5}{3} = \frac{22 - 15}{9} = \frac{7}{9}$

$\boxed{a = -\frac{2}{3} \; ; \; b = \frac{7}{9} \; ; \; c = \frac{1}{3} \; ; \; d = -\frac{11}{9}}$

$$\Rightarrow \boxed{\vec{y}_{sp}(x) = e^{2x}\begin{pmatrix} -\frac{2}{3}x + \frac{7}{9} \\ \frac{7}{9}x - \frac{11}{9} \end{pmatrix}}$$

$\boxed{B}$ $\vec{b}(x) = e^{x}\begin{pmatrix} x \\ 2x+1 \end{pmatrix}$; $\left.\begin{array}{l} \mu(1) = 1 \\ m = 1 \end{array}\right\}$ $\mu + m = 2$

$$\vec{y}_{sp}(x) = e^{x}\begin{pmatrix} Ax^2 + Bx + C \\ Dx^2 + Ex + D \end{pmatrix}$$

$\boxed{C}$ $$\vec{b}(x) = \underbrace{e^{5x}\begin{pmatrix} 1 \\ x \end{pmatrix}}_{\vec{b}_1(x)} + \underbrace{e^{x}\begin{pmatrix} x \\ x^2 \end{pmatrix}}_{\vec{b}_2(x)} +$$

$\vec{b}_1(x) = e^{5x}\begin{pmatrix} 1 \\ x \end{pmatrix}$; $\mu(5) = 1 \Rightarrow \vec{y}_{sp,1}(x) = e^{5x}\begin{pmatrix} ax^2 + bx + c \\ dx^2 + ex + e \end{pmatrix}$

$\vec{b}_2(x) = e^{x}\begin{pmatrix} x \\ x^2 \end{pmatrix}$; $\mu(1) = 1 \Rightarrow \vec{y}_{sp,2}(x) = e^{x}\begin{pmatrix} a_1x^3 + b_1x^2 + c_1x + d_1 \\ a_2x^3 + b_2x^2 + c_2x + d_2 \end{pmatrix}$

$$\vec{y}_{sp}(x) = \vec{y}_{sp,1}(x) + \vec{y}_{sp,2}(x)$$

$\boxed{\text{Beispiel 6}}$ Man löse folgendes Anfangswertproblem:

$$(*)\begin{cases} y_1' = 2y_1 - y_2 + e^{2x}\cos(x) \\ y_2' = y_1 + 2y_2 \end{cases}$$

mit Anfangswerte : $y_1(0) = 1$; $y_2(0) = 2$

$$(*) \Longleftrightarrow \vec{y}'(x) = \underbrace{\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}}_{A} \vec{y}(x) + \underbrace{e^{2x}\begin{pmatrix} \cos(x) \\ 0 \end{pmatrix}}_{\vec{b}(x)}$$

I. Homogene Lösung

$$\boxed{\vec{y}'(x) = A\,\vec{y}(x)}$$

$$P(\lambda) : \det|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 + 1 = \lambda^2 - 4\lambda + 5$$

$$\begin{cases} \lambda_1 = \dfrac{4+\sqrt{-4}}{2} = 2+i \,; \\ \lambda_2 = 2 - i \end{cases}$$

$$\begin{cases} \mu(\lambda_1) = \nu(\lambda_1) = 1 \\ \mu(\lambda_2) = \nu(\lambda_2) = 1 \end{cases}$$

Eigenvektor zu $\lambda_1 = 2+i$

$$\big(A - (2+i)\big)\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}\begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_2^{(1)} = -i v_1^{(1)}$$

$$v^{(1)} = \begin{pmatrix} t \\ -it \end{pmatrix} \;;\quad \underline{\underline{t=1}} \Rightarrow v^{(1)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$\Rightarrow$ Eine Komplexe Lösung des DGLS $\vec{y}' = A\vec{y}$ als

$$\hat{\vec{y}}_1(x) = e^{(2+i)x}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{2x}(\cos x + i\sin x)\cdot\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right] =$$

$$= \underbrace{e^{2x}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix}}_{=\vec{y}_1(x)} + i\,\underbrace{e^{2x}\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix}}_{\vec{y}_2(x)}$$

- Aufspalten von $\vec{\hat{y}}_1(x)$ nach Real und Imaginärteil liefert 2 reelwertigen Lösungen $\vec{y}_1(x)$ und $\vec{y}_2(x)$ $\Rightarrow$ die vollständige Lösungen (reelwertig) :

$$\vec{y}_{hom}(x) = c_1\vec{y}_1(x) + c_2\vec{y}_2(x) = c_1 e^{2x}\begin{pmatrix}\cos x\\ \sin x\end{pmatrix} + c_2 e^{2x}\begin{pmatrix}\sin x\\ -\cos x\end{pmatrix}$$

II Spezielle Lösung :

$$\vec{b}(x) = e^{2x}\begin{pmatrix}\cos x\\ 0\end{pmatrix} = e^{2x}\left[\cos 1\cdot x\begin{pmatrix}1\\0\end{pmatrix} + \sin x\begin{pmatrix}0\\0\end{pmatrix}\right] =$$

$$= e^{(2+i)x}\left[\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix} - \frac{i}{2}\begin{pmatrix}0\\0\end{pmatrix}\right] + e^{(2-i)x}\left[\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix} + \frac{i}{2}\begin{pmatrix}0\\0\end{pmatrix}\right] =$$

$$= \underbrace{e^{(2+i)x}\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}}_{\vec{b}_1(x)} + \underbrace{e^{(2-i)x}\begin{pmatrix}\frac{1}{2}\\0\end{pmatrix}}_{\vec{b}_2(x)}$$

$[\ p_1(x) = 1\ ,\ p_2(x) = q_1(x) = 0\ ,\quad m = \max\{\text{grad}(p_i), \text{grad}(q_i)\} = 0$

Spezielle Lösung zu $\vec{b}_1(x)$ (Komplex-wertig)

zu System $\vec{y}\,' = A\vec{y} + \vec{b}_1(x)$

$\underline{\text{Ansatz}}$: $\vec{y}_{sp,1}(x) = e^{(2+i)x}\begin{pmatrix}\overbrace{Ax+B}^{\text{grad} = m+\mu(2+i) = 1}\\ Cx+D\end{pmatrix}$

$\mu(2+i) = 1$

- Komponenteweise Differentiation von $\vec{y}_{sp,1}(x)$ $\Rightarrow$

$$\vec{y}\,'_{sp,1}(x) = (2+i)\,e^{(2+i)x}\begin{pmatrix}Ax+B\\ Cx+D\end{pmatrix} + e^{(2+i)x}\cdot\begin{pmatrix}A\\C\end{pmatrix}$$

- Einsetzen in DGLS und Koeff. Vergleich $\Rightarrow$

$$\begin{cases} (2+i)e^{(2+i)x}(Ax+B) + e^{(2+i)x}A = 2e^{(2+i)x}(Ax+B) - e^{(2+i)x}(Cx+D) \\ \quad + \frac{1}{2}e^{(2+i)x} \\ (2+i)e^{(2+i)x}(Cx+D) + e^{(2+i)x}C = e^{(2+i)x}(Ax+B) + 2e^{(2+i)x}(Cx+D) \end{cases}$$

Kürzen durch $e^{(2+i)x}$ und sortieren nach Potenzen von $x \Rightarrow$

$$\begin{cases} (2+i)Ax + ((2+i)B + A) = (2A - C)x + (2B - D + \frac{1}{2}) \\ (2+i)Cx + ((2+i)D + C) = (A + 2C)x + (B + 2D) \end{cases}$$

Koeffizientenvergleich $\Rightarrow$

$$\begin{cases} 1. & (2+i)A = 2A - C \\ 2. & (2+i)B + A = 2B - D + \frac{1}{2} \\ 3. & (2+i)C = (A + 2C) \\ 4. & (2+i)D + C = (B + 2D) \quad \Rightarrow \quad B = iD + C \end{cases}$$

2. Gl $\Rightarrow A = -D + \frac{1}{2} - iB = -\cancel{D} + \frac{1}{2} + \cancel{D} - iC = \frac{1}{2} - iC$

1. Gl $\Rightarrow C = -iA = \frac{-i}{2} - C \Longrightarrow \boxed{\begin{matrix} C = -\frac{i}{4} \\ A = \frac{1}{4} \end{matrix}}$

$$\begin{matrix} \text{2. Gl} \Leftrightarrow \\ \text{4. Gl} \Leftrightarrow \end{matrix} \begin{cases} iB = -D + \frac{1}{4} \\ iD - \frac{i}{4} = B \end{cases} \Updownarrow \Rightarrow D \rightarrow \text{frei Wählbar}$$

z.B $D = 0$ und $B = -\frac{i}{4} \Rightarrow$

$$\boxed{\vec{y}_{sp,1}(x) = e^{(2+i)x}\begin{pmatrix} \frac{1}{4}x - \frac{i}{4} \\ -\frac{i}{4}x \end{pmatrix}}$$

Realteil von $\vec{y}_{Sp,1}(x)$ ist

$$\mathrm{Re}\left(\vec{y}_{sp,1}(x)\right) = \mathrm{Re}\left[e^{2x}\begin{pmatrix} -\frac{i}{4}\cos x + \frac{1}{4}\sin x + \frac{x}{4}\cos x + \frac{ix}{4}\sin x \\ -\frac{i}{4}x\cos x + \frac{x}{4}\sin x \end{pmatrix}\right]$$

$$= e^{2x}\begin{pmatrix} \frac{1}{4}\sin x + \frac{x}{4}\cos x \\ \frac{x}{4}\sin x \end{pmatrix}$$

$$\vec{y}_{Sp,2}(x) = \overline{\vec{y}_{1,Sp}(x)}$$

$$\vec{y}_{Sp}(x) = 2\,\mathrm{Re}\left(\vec{y}_{Sp,1}(x)\right) = 2e^{2x}\begin{pmatrix} \frac{1}{4}\sin x + \frac{x}{4}\cos x \\ \frac{x}{4}\sin x \end{pmatrix}$$

Allgemeine Lösung: des DGLS: $\vec{y}\,' = A\vec{y} + \vec{b}(x)$ ist

$$\vec{y}(x) = c_1 e^{2x}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + c_2 e^{2x}\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix} + 2e^{2x}\begin{pmatrix} \frac{1}{4}\sin x + \frac{x}{4}\cos x \\ \frac{x}{4}\sin x \end{pmatrix}$$

AWP: $y_1(0) = 1$; $y_2(0) = 2 \Rightarrow$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2\cdot 0 + 2\cdot 0 \\ c_1\cdot 0 + c_2(-1) + 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow$$

Lösung:

$$\vec{y}(x) = e^{2x}\begin{pmatrix} \cos x \\ \sin x \end{pmatrix} + 2e^{2x}\begin{pmatrix} \sin x \\ -\cos x \end{pmatrix} + 2e^{2x}\begin{pmatrix} \frac{1}{4}\sin x + \frac{x}{4}\cos x \\ \frac{x}{4}\sin x \end{pmatrix}$$

Beispiel 7 AWP :
$$\begin{cases} \vec{y}\,' = \begin{pmatrix} 1 & -4 \\ -1 & 1 \end{pmatrix} \vec{y} + e^{2x} \begin{pmatrix} x+1 \\ 3 \end{pmatrix} \\ \vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \quad ; \vec{y}(x) = ?$$

I: Homogene Lösung:

$$P(\lambda): \det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -4 \\ -1 & 1-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda_2 = 3 \quad ; \lambda_1 = -1$$

$$\mu(\lambda_2) = \mu(\lambda_1) = 1$$

Eigenvektor zu $\lambda_1 = -1$

$$(A + I)\,\vec{v}^{(1)} = \vec{0} \Rightarrow \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1^{(1)} = 2v_2^{(1)}$$

$$v^{(1)} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad ; t = 1 : \quad v^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Eigenvektor zu $\lambda_2 = 3$: $\vec{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix}$

$$(A - 3I)\,\vec{v}^{(2)} = \vec{0} \Rightarrow$$

$$\begin{pmatrix} -2 & -4 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1^{(2)} = -2v_2^{(2)} \Rightarrow v^{(2)} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\underline{\underline{t=1}} : \quad v^{(2)} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\text{Fund. system} \left\{ e^{-x} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; e^{3x} \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

$$\boxed{\vec{y}_{hom}(x) = c_1 e^{-x} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3x} \begin{pmatrix} -2 \\ 1 \end{pmatrix}}$$

II Spezielle Lösung

$$\vec{b}(x) = e^{2x} \begin{pmatrix} x+1 \\ 3 \end{pmatrix}$$

$$m = \max \{ \underbrace{\text{grad}(x+1)}_{1}, \underbrace{\text{grad}(3)}_{0} \} = 1 \; ; \; \mu(2) = 0$$

$$\vec{y}_{Sp}(x) = e^{2x}\begin{pmatrix} Ax+B \\ Cx+D \end{pmatrix}$$

Einsetzen und Koeffvergleich:

$$\vec{y}\,'_{sp}(x) = 2e^{2x}\begin{pmatrix} Ax+B \\ Cx+D \end{pmatrix} + e^{2x}\begin{pmatrix} A \\ C \end{pmatrix} = e^{2x}\begin{pmatrix} 2Ax+(2B+A) \\ 2Cx+(2D+C) \end{pmatrix}$$

$$\vec{y}\,'_{Sp}(x) = A\,\vec{y}_{sp}(x) + e^{2x}\begin{pmatrix} x+1 \\ 3 \end{pmatrix} \iff$$

$$\begin{cases} 2Ax + (2B+A) = (Ax+B) - 4(Cx+D) + (x+1) \\ 2Cx + (2D+C) = -(Ax+B) + (Cx+D) + 3 \end{cases}$$

$\Rightarrow$ Koeff. vergleich:

$$\begin{array}{l} \text{I} \\ \text{III} \end{array}\begin{cases} 2B+A = B-4D+1 \\ 2D+C = -B+D+3 \end{cases} \quad \text{und} \quad \begin{array}{l} \text{II} \\ \text{IV} \end{array}\begin{cases} 2A = A-4C+1 \\ 2C = C-A \end{cases}$$

IV : $C = -A$

II : $A = -4C + 1 \Rightarrow A = -4\cdot(-A)+1 \Rightarrow -3A = 1 \Rightarrow$ $\boxed{\begin{matrix} A = -\frac{1}{3} \\ C = \frac{1}{3} \end{matrix}}$

I $\quad B - \frac{1}{3} = -4D+1 \Rightarrow B = -4D + \frac{4}{3}$

III $\quad D + \frac{1}{3} = 4D - \frac{4}{3} + 3 \Rightarrow -3D = \frac{-5}{3} + 3 \Rightarrow$

$-3D = \frac{4}{3} \Rightarrow \boxed{D = \frac{-4}{9}} \Rightarrow B = \frac{16}{9} + \frac{12}{9} = \frac{28}{9}$

$\boxed{B = \frac{28}{9}}$

$$\Rightarrow \vec{y}_{Sp}(x) = e^{2x}\begin{pmatrix} -\frac{1}{3}x + \frac{28}{9} \\ \frac{1}{3}x - \frac{4}{9} \end{pmatrix}$$

$$\vec{y}(x) = c_1 e^{-x}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + e^{2x}\begin{pmatrix} -\frac{1}{3}x + \frac{28}{9} \\ \frac{1}{3}x - \frac{4}{9} \end{pmatrix}$$