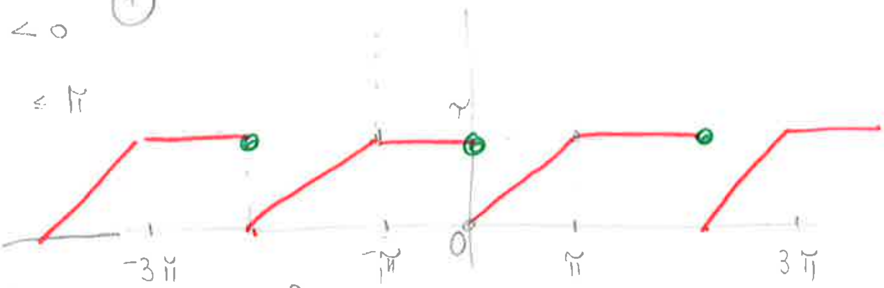
1

11 $f(x) = \begin{cases} \pi, & -\pi \le x < 0 \\ x, & 0 \le x \le \pi \end{cases}$



• f stetig auf $\mathbb{R} \setminus \{2k\pi,\ k \in \mathbb{Z}\}$

Fourierreihe : $F(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$

$$\begin{cases} a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx, & n = 0, 1, \dots \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx \end{cases}$$

$n=0$: $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \underbrace{\cos(0)}_{1}\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \pi\, dx + \frac{1}{\pi} \int_{0}^{\pi} x\, dx =$

$$= \frac{1}{\pi} \left(\pi x \Big|_{-\pi}^{0} + \frac{x^2}{2} \Big|_{0}^{\pi} \right) = \frac{1}{\pi} \left(\pi^2 + \frac{\pi^2}{2} \right) = \frac{3\pi}{2}$$

$\boxed{a_0 = \frac{3\pi}{2}}$

$n \ge 1$: $a_n = \frac{1}{\pi} \left(\int_{-\pi}^{0} \pi \cos(nx)\, dx + \int_{0}^{\pi} x \cos(nx)\, dx \right) = \frac{\sin(nx)}{n} \Big|_{-\pi}^{0} +$

$$+ \frac{1}{\pi} \underbrace{\int_{0}^{\pi} \left(\frac{\sin nx}{n} \right)' \cdot x\, dx}_{\text{PI}} = \frac{1}{n} \left[\underbrace{\sin 0}_{0} - \underbrace{\sin(-n\pi)}_{0} \right] + \frac{1}{\pi} \left[\frac{x}{n} \sin(nx) \Big|_{0}^{\pi} - \right.$$

$$\left. - \int_{0}^{\pi} \frac{\sin(nx)}{n}\, dx = \frac{1}{\pi} \left[\frac{\pi}{n} \underbrace{\sin(n\pi)}_{0} - 0 - \frac{1}{n} \int_{0}^{\pi} \left(-\frac{\cos(nx)}{n} \right)' dx \right] = \right.$$

$$= \frac{1}{\pi n^2} \cos(nx) \Big|_{0}^{\pi} = \frac{1}{\pi n^2} \left(\underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\cos 0}_{1} \right) = \boxed{\frac{(-1)^n - 1}{\pi n^2} =: a_n}$$

$n \ge 1$: $b_n = \frac{1}{\pi} \left(\int_{-\pi}^{0} \pi \sin(nx)\, dx + \int_{0}^{\pi} x \cdot \sin(nx)\, dx \right) = -\frac{\cos(nx)}{n} \Big|_{-\pi}^{0} +$

$$\frac{1}{\pi} \int_{0}^{\pi} x \left(-\frac{\cos(nx)}{n} \right)' dx = -\frac{1}{n} \left[1 - \underbrace{\cos(-n\pi)}_{\cos(n\pi) = (-1)^n} \right] + \frac{1}{\pi} \left[-x \frac{\cos(nx)}{n} \Big|_{0}^{\pi} + \int_{0}^{\pi} \frac{\cos(nx)}{n} \right.$$

$$= -\frac{1}{n} + \frac{(-1)^n}{n} + \frac{1}{\pi} \left[-\pi \frac{(-1)^n}{n} + \underbrace{\frac{1}{n} \frac{\sin(nx)}{n} \Big|_{0}^{\pi}}_{0} \right] = \frac{-1}{n} + \cancel{\frac{(-1)^n}{n}} - \cancel{\frac{(-1)^n}{n}} = \frac{-1}{n}$$

$\Rightarrow \boxed{b_n = -\frac{1}{n}}$

$\Rightarrow$ Fourierreihe

$$F(x) = \frac{3\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n - 1}{\pi n^2} \cos(nx) - \frac{1}{n} \sin(xn) \right)$$

$$\boxed{F(x) = f(x) \quad \forall x \in \mathbb{R} \setminus \{2k\pi, k \in \mathbb{Z}\}}$$

In $x = 2k\pi$

$$F(x) \to \frac{f(2\pi k_+) + f(2\pi k_-)}{2}$$

$\boxed{12}$ a) $\displaystyle\int_1^{\infty} \frac{1}{x + \sqrt{x} + \cos(x)}\, dx > \int_1^{+\infty} \underbrace{\frac{1}{2x+1}}_{g(x)}\, dx$

$$x + \underbrace{\sqrt{x}}_{<x} + \underbrace{\cos(x)}_{<1} < 2x + 1 \Rightarrow \frac{1}{x + \sqrt{x} + \cos x} > \frac{1}{2x+1}$$

$$\lim_{t\to\infty} \int_1^t g(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{2x+1}\, dx = \lim_{t\to\infty} \int_3^{2t+1} \frac{1}{2y}\, dy = \frac{1}{2} \lim_{t\to\infty} \int_3^{2t+1} \frac{1}{y}\, dy \Leftrightarrow$$

$2x+1 = y \Rightarrow 2\,dx = dy$

$x = 1 \Rightarrow y = 3$

$x = t \Rightarrow y = 2t+1$

$$\Leftrightarrow \frac{1}{2} \lim_{t\to\infty} \ln(y)\Big|_3^{2t+1} = +\infty \Rightarrow \int_1^{+\infty} \frac{1}{2x+1}\, dx \text{ divergente Minorante}$$

für $\displaystyle\int_1^{+\infty} \frac{1}{x + \sqrt{x} + \cos x}\, dx \Rightarrow$ divergent

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b) $\displaystyle\int_5^{+\infty} \frac{x^2 + x - 2}{x^3 + 3x + 2}\, dx$

$$\frac{x^2 + x - 2}{x^3 + 3x + 2} > \frac{x^2}{x^3 + 3x + 2} > \frac{x^2}{x^3 + x^2} = \frac{1}{x+1}$$

$$\cancel{x^3} + 3x + 2 < \cancel{x^3} + x^2$$

$$3x + 2 < x^2 \quad \checkmark$$

$$\int_5^{+\infty} \frac{1}{x+1}\, dx = \ln(x+1)\Big|_5^{+\infty} = +\infty \Rightarrow \int_5^{+\infty} \frac{1}{x+1}\, dx \text{ div minorante}$$

für $\displaystyle\int_5^{+\infty} \frac{x^2 + x - 2}{x^3 + 3x + 2} \to$ Div

$\boxed{13}$ (a) $\displaystyle\int_1^{+\infty} \frac{1}{x^2\sqrt{1+x^2}}\,dx \;=$

$x = \sinh t$, $dx = \cosh t\, dt$; $\cosh^2 t - \sinh^2 t = 1 \Rightarrow 1 + \underbrace{\sinh^2 t}_{x^2} = \cosh^2 t \Rightarrow \cosh t = \sqrt{1+x^2}$

$= \displaystyle\int \frac{1}{\sinh^2 t \cdot \cancel{\cosh t}} \cdot \cancel{\cosh t}\, dt = \int \frac{1}{\sinh^2 t}\, dt = -\frac{\cosh t}{\sinh t} =$

$-\cotanh x =$

$= -\dfrac{\sqrt{1+x^2}}{x}$

$\Rightarrow \displaystyle\int_1^{+\infty} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \lim_{t\to\infty}\left(-\frac{\sqrt{1+x^2}}{x}\,\Big|_1^t\right) = \lim_{t\to\infty}\left(-\frac{\sqrt{1+t^2}}{t} + \sqrt{2}\right) =$

$= \displaystyle\lim_{t\to\infty}\left(-\sqrt{1+\frac{1}{t^2}}\right) + \sqrt{2} = -1 + \sqrt{2} \Rightarrow$ Konv

(b) $\displaystyle\int_{\frac{1}{e}}^{1} \frac{1}{1+\ln(x)}\,dx = \left(\int_0^1 \frac{e^{y-1}}{y}\,dy\right) \to$ Konv oder div.

Subst: $1 + \ln x = y \Rightarrow \ln x = y - 1 \Rightarrow x = e^{y-1} \Rightarrow dx = e^{y-1}\,dy$

$x = \frac{1}{e} \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$\dfrac{e^{y-1}}{y} = \dfrac{1}{e}\cdot\dfrac{e^y}{y} > \dfrac{1}{ey}$

Für $y \in (0,1)$: $e^y > 1$

$\displaystyle\int_0^1 \frac{e^{y-1}}{y}\,dy > \frac{1}{e}\int_0^1 \frac{1}{y}\,dy = \frac{1}{e}\lim_{t\to 0_+}\left(\ln|y|\right)\Big|_t^1 = \underbrace{\frac{1}{e}\left(0-(-\infty)\right) = +\infty}_{\text{div}}$

$\Rightarrow \displaystyle\int_0^1 \frac{e^{y-1}}{y}\,dy = \int_{\frac{1}{e}}^{1} \frac{1}{1+\ln(x)}\,dx$ auch div.

$\boxed{14}$ $\sum_{n=1}^{\infty} \frac{k}{e^{k^2}}$

- $f : [1, +\infty) \to \mathbb{R}$
- $\begin{cases} f \text{ mon. fallend} \quad \checkmark \\ \lim_{x \to +\infty} f(x) = 0 \end{cases}$

Cauchy: $\sum_{n=1}^{\infty} \frac{k}{e^{k^2}}$ Konv $\Leftrightarrow$ $\int_1^{+\infty} \frac{x}{e^{x^2}} dx$ Konvergent.

$$\int_1^{+\infty} \frac{x}{e^{x^2}} dx = \lim_{t \to \infty} \int_1^t x e^{-x^2} dx$$

$$e^{-x^2} = y \Rightarrow -2x e^{-x^2} dx = dy$$

$$x = 1 \Rightarrow y = \frac{1}{e}$$

$$x = t \Rightarrow y = e^{-t^2}$$

$$\int_1^t x e^{-x^2} dx = -\frac{1}{2} \int_{\frac{1}{e}}^{\frac{1}{e^{t^2}}} \underbrace{-2x e^{-x^2} dx}_{dy} = \frac{1}{2} \int_{\frac{1}{e^{t^2}}}^{\frac{1}{e}} dy = \frac{1}{2} y \Big|_{\frac{1}{e^{t^2}}}^{\frac{1}{e}} =$$

$$= \frac{1}{2} \left(\frac{1}{e} - \underbrace{\frac{1}{e^{t^2}}}_{\to 0} \right) \xrightarrow[t \to \infty]{} \frac{1}{2e} \Rightarrow \int \frac{x}{e^{x^2}} \text{ Konvergiert} \Rightarrow$$

$$\sum \frac{k}{e^{k^2}} \to \text{Konvergent}$$