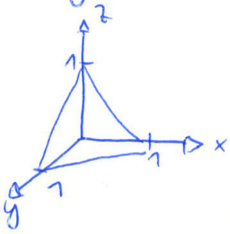
Konversatorium - Mathe B : 8. Juni 2011

(1) $x+y+z=1$



$$Vol = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 1 \, dz\, dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1-x} z \Big|_0^{1-x-y} dy\, dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} 1-x-y \, dy\, dx = \int_{x=0}^{1} \left[y - xy - \tfrac{1}{2}y^2\right]\Big|_0^{1-x} dx$$

$$= \int_0^1 1-x-x+x^2-\tfrac{1}{2}+x-\tfrac{1}{2}x^2 \, dx = \int_0^1 \tfrac{1}{2}x^2 - x + \tfrac{1}{2}\, dx = \tfrac{1}{6}x^3 - \tfrac{1}{2}x^2 + \tfrac{1}{2}x \Big|_0^1$$

$$= \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{2} = \tfrac{1}{6}$$

Schwerpunkt: $x_s = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} x \, dz\, dy\, dx = \int_{x=0}^{1}\int_{y=0}^{1-x} zx \Big|_0^{1-x-y} dy\, dx$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} x - x^2 - yx \, dy\, dx = \int_{x=0}^{1} xy - x^2y - \tfrac{x}{2}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 x - x^2 - \cancel{x^2} + x^3 - \tfrac{x}{2} + \cancel{x^2} - \tfrac{x^3}{2}\, dx = \int_0^1 \tfrac{x^3}{2} - x^2 + \tfrac{x}{2}\, dx = \tfrac{x^4}{8} - \tfrac{1}{3}x^3 + \tfrac{x^2}{4}\Big|_0^1$$

$$= \tfrac{1}{8} - \tfrac{1}{3} + \tfrac{1}{4} = \tfrac{3-8+6}{24} = \underline{\underline{\tfrac{1}{24}}}$$

Wegen Symmetrie (d.h. Ebene schneidet z-Achse & y-Achse & x-Achse jeweils bei 1)

$\Rightarrow \underline{y_s = z_s = \tfrac{1}{24}}$

(2) $I = \iiint_B (x^2+y^2)\, z \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz$

Kugelkoordinaten: $x = r\cos\varphi \sin\vartheta$ allgemein: $r \geq 0$, $\varphi \in [0, 2\pi]$,

$y = r \sin\varphi \sin\vartheta$ $\vartheta \in [0, \pi)$

$z = r\cos\vartheta$

$B = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq x,y,z,\ x^2+y^2+z^2 \leq 2\}$

Neue Grenzen: *) $x^2+y^2+z^2 = \underbrace{r^2\cos^2\varphi\sin^2\vartheta + r^2\sin^2\varphi\sin^2\vartheta}_{} + r^2\cos^2\vartheta$

$= r^2\sin^2\vartheta + r^2\cos^2\vartheta = r^2 \leq 2 \Rightarrow r \in [0, 2]$

*) $z \geq 0 \Rightarrow r\cos\vartheta \geq 0 \overset{r\geq 0}{\Rightarrow} \cos\vartheta \geq 0 \Rightarrow \vartheta \in [0, \frac{\pi}{2}]$

*) $y \geq 0 \Rightarrow r\sin\varphi\sin\vartheta \geq 0 \underset{r\geq 0}{\overset{\sin\vartheta\geq 0}{\Longrightarrow}} \sin\varphi \geq 0 \Rightarrow \varphi \in [0, \pi]$

*) $x \geq 0 \Rightarrow r\cos\varphi\sin\vartheta \geq 0 \underset{r\geq 0}{\overset{\sin\vartheta\geq 0}{\Longrightarrow}} \cos\varphi \geq 0$ & $\Longrightarrow \varphi \in [0, \frac{\pi}{2}]$

Jacobi-Determinante: $r^2\sin\vartheta$

$$\Rightarrow I = \int_{\varphi=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2}\int_{r=0}^{\sqrt{2}} \underbrace{r^2\sin^2\vartheta}_{=x^2+y^2} \cdot \underbrace{r\cos\vartheta}_{z} \cdot \underbrace{r}_{=\sqrt{x^2+y^2+z^2}} \cdot \overbrace{r^2\sin\vartheta}\, dr\, d\vartheta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2}\int_{r=0}^{\sqrt{2}} r^6\sin^3\vartheta\cos\vartheta\, dr\, d\vartheta\, d\varphi = \int_{\varphi=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2} \frac{r^7}{7}\sin^3\vartheta\cos\vartheta\Big|_0^{\sqrt{2}} d\vartheta\, d\varphi$$

$$= \int_{\varphi=0}^{\pi/2}\int_{\vartheta=0}^{\pi/2} \frac{8}{7}\sqrt{2}\sin^3\vartheta\cos\vartheta\, d\vartheta\, d\varphi = \int_{\varphi=0}^{\pi/2} \frac{2}{7}\sqrt{2}\sin^4\vartheta\Big|_0^{\pi/2} d\varphi$$

$$= \int_{\varphi=0}^{\pi/2} \frac{2}{7}\sqrt{2}\, d\varphi = \frac{2}{7}\sqrt{2}\varphi\Big|_0^{\pi/2} = \frac{\sqrt{2}}{7}\pi$$

③ $I = \iiint_B 2yz\sqrt{x^2+y^2}\,dx\,dy\,dz$

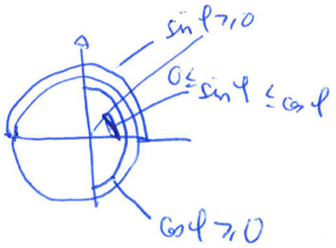
Zylinderkoordinaten: $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$ — allgemein: $r \geq 0$, $\varphi \in [0, 2\pi)$

Jacobi-Det: r

Neuer Integrationsbereich: $B = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \leq y \leq x,\ \sqrt{x^2+y^2} \leq z \leq 2\}$

*) $0 \leq y \leq x \Rightarrow 0 \leq r\sin\varphi \leq r\cos\varphi$

$\overset{r>0}{\Longrightarrow} 0 \leq \sin\varphi \leq \cos\varphi$



i) $0 \leq \sin\varphi \Rightarrow \varphi \in [0, \pi]$

ii) $0 \leq \cos\varphi \Rightarrow \varphi \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi)$ $\Rightarrow \varphi \in [0, \frac{\pi}{2}]$

iii) zusätzlich muß gelten: $\sin\varphi \leq \cos\varphi \Rightarrow \underline{\underline{\varphi \in [0, \frac{\pi}{4}]}}$ (Skizze!)

*) $\sqrt{x^2+y^2} \leq z \leq 2 \Rightarrow r \leq z \leq 2$

$\Rightarrow \underline{\underline{r \in [0,2]}}$ & $\underline{\underline{z \in [r,2]}}$

$$\Rightarrow I = \int_{\varphi=0}^{\pi/4}\int_{r=0}^{2}\int_{z=r}^{2} \overbrace{2r\sin\varphi}^{y} \cdot z \cdot \overbrace{r}^{\sqrt{x^2+y^2}} \cdot \overbrace{r}^{\text{Jacobi}}\,dz\,dr\,d\varphi$$

$$= \int_{\varphi=0}^{\pi/4}\int_{r=0}^{2}\int_{z=r}^{2} 2r^3\sin\varphi\, z\,dz\,dr\,d\varphi = \int_{\varphi=0}^{\pi/4}\int_{r=0}^{2} r^3\sin\varphi\, z^2\Big|_r^2\,dr\,d\varphi$$

$$= \int_{\varphi=0}^{\pi/4}\int_{r=0}^{2} 4r^3\sin\varphi - r^5\sin\varphi\,dr\,d\varphi = \int_{\varphi=0}^{\pi/4} r^4\sin\varphi - \frac{r^6}{6}\sin\varphi\Big|_0^2\,d\varphi$$

$$= \int_{\varphi=0}^{\pi/4} 16\sin\varphi - \frac{64}{6}\sin\varphi\,d\varphi = -16\cos\varphi + \frac{32}{3}\cos\varphi\Big|_0^{\pi/4}$$

$$= -16\frac{\sqrt{2}}{2} + \frac{32}{3}\frac{\sqrt{2}}{2} - \left(-16 + \frac{32}{3}\right) = \sqrt{2}\cdot\frac{32-48}{6} + \frac{48-32}{3} = \underline{\underline{\frac{16}{3} - \frac{8\sqrt{2}}{3}}}$$

(4) $y' = (x^2+2)y^2 + \frac{y}{x+1}$ mit $y(0)=1$

Typ Bernoulli! mit $\alpha = 2$

Substitution: $z = y^{1-\alpha} = y^{-1} = \frac{1}{y} \Rightarrow y = \frac{1}{z}$

Überführung in lin. DGL: (Formel aus Vorlesung!)

$$z' = -\frac{1}{x+1} y - (x^2+2)$$

$$z_{hom}(x) = C \cdot e^{\int -\frac{1}{x+1} dx} = C \cdot e^{-\ln(x+1)} = C (x+1)^{-1} = \underline{\frac{C}{x+1}}$$

Spezielle Lösung: $z_p(x) = C(x) \cdot \frac{1}{x+1}$

Formel aus Vorlesung: $C(x) = \int -(+x^2+2) \cdot e^{\int \frac{1}{x+1} dx} dx$

$$= \int (-x^2-2) \cdot (x+1)\, dx = \int -x^3 - 2x - x^2 - 2\, dx$$

$$= -\frac{1}{4}x^4 - x^2 - \frac{1}{3}x^3 - 2x$$

$$\Rightarrow z_{sp}(x) = \frac{1}{x+1}\left(-\frac{1}{4}x^4 - x^2 - \frac{1}{3}x^3 - 2x\right)$$

$$\Rightarrow z_{alle}(x) = \frac{C}{x+1} + \frac{1}{x+1}\left(-\frac{1}{4}x^4 - x^2 - \frac{1}{3}x^3 - 2x\right)$$

$$\Rightarrow y_{alle}(x) = \frac{1}{z_{alle}(x)} = \frac{x+1}{C - \frac{1}{4}x^4 - x^2 - \frac{1}{3}x^3 - 2x}$$

AWP: $y(0) \overset{!}{=} 1$

$$\Rightarrow 1 = y_{alle}(0) = \frac{0+1}{C - \frac{1}{4}\cdot 0^4 - 0^2 - \frac{1}{3}\cdot 0^3 - 2\cdot 0} = \frac{1}{C} \Rightarrow \underline{\underline{C=1}}$$

$$\Rightarrow \boxed{y(x) = \frac{x+1}{1 - \frac{1}{4}x^4 - x^2 - \frac{1}{3}x^3 - 2x}}$$