(23) X_1 = Augenzahl 1. Wurf ; $X_1 \in \{1, \dots, 6\}$

X_2 = Augenzahl 2. Wurf $X_2 \in \{1, \dots 6\}$

Sei $X = \max\{X_1, X_2\}$

$X \in \{1, 2, \dots, 6\}$

x	1	2	3	4	5	6
$\mathbb{P}[X=x] = p_X(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$
$\mathbb{P}[X \leq x] = F_X(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{9}{36}$	$\frac{16}{36}$	$\frac{25}{36}$	1

$$p_X(1) = \mathbb{P}[\max\{X_1, X_2\} = 1] = \mathbb{P}[(1,1)] = \frac{1}{36}$$

$$p_X(2) = \mathbb{P}[(2,2), (1,2), (2,1)] = \frac{3}{36}$$

$$p_X(3) = \mathbb{P}\begin{bmatrix}(1,3) & (2,3) & (3,3) \\ (3,1) & (3,2) & \end{bmatrix} = \frac{5}{36}$$

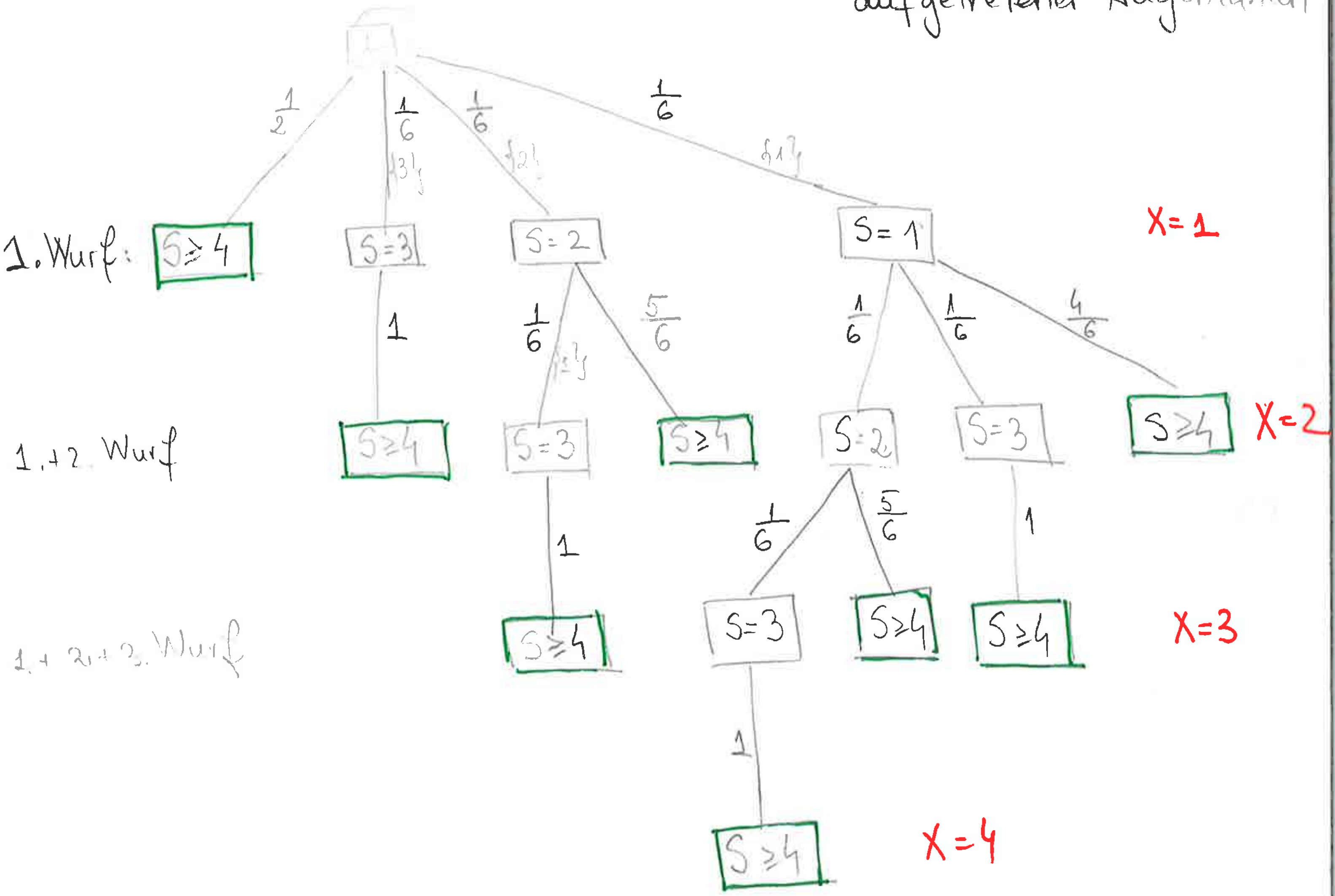
$$p_X(4) = \mathbb{P}\begin{bmatrix}(1,4), & (2,4) & (3,4) & (4,4) \\ (4,1) & (4,2) & (4,3) & \end{bmatrix} = \frac{7}{36}$$

$$p_X(5) = \mathbb{P}\begin{bmatrix}(1,5), & (2,5), & (3,5), & (4,5), & (5,5) \\ (5,1) & (5,2) & (5,3) & (5,4) & \end{bmatrix} = \frac{9}{36}$$

$$p_X(6) = \frac{5+6}{36} = \frac{11}{36}$$

$$F_X(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{36}, & 1 \leq x < 2 \\ \frac{4}{36}, & 2 \leq x < 3 \\ \frac{9}{36}, & 3 \leq x < 4 \\ \frac{16}{36}, & 4 \leq x < 5 \\ \frac{25}{36}, & 5 \leq x < 6 \\ 1, & x \geq 6 \end{cases}$$

24 X = # notwendigen Würfen ; S = Summe der aufgetretener Augenzahlen



$$p_X(1) = \mathbb{P}[X=1] = \frac{1}{2}$$

$$p_X(2) = \mathbb{P}[X=2] = \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{4}{6} = \frac{6+5+4}{36} = \frac{15}{6^2}$$

$$p_X(3) = \mathbb{P}[X=3] = \frac{1}{6}\cdot\frac{1}{6}\cdot 1 + \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{5}{6} + \frac{1}{6}\cdot\frac{1}{6}\cdot 1 = \frac{6+5+6}{6^3}$$

$$= \frac{17}{6^3}$$

$$p_X(4) = \mathbb{P}[X=4] = \frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6} = \frac{1}{6^3}$$

$$\sum_{i=1}^{4} p_X(i) = \frac{1}{2} + \underbrace{\frac{15}{6^2} + \frac{17}{6^3} + \frac{1}{6^3}}_{\frac{18}{6^2} = \frac{3\cdot \cancel{6}}{6^{\cancel{2}}} =} = \frac{1}{2} + \frac{1}{2}$$

x	1	2	3	4	Σ
$p_X(x)$	$\frac{1}{2}$	$\frac{15}{6^2}$	$\frac{17}{6^3}$	$\frac{1}{6^3}$	1

$$\mathbb{E}[X] = 1\cdot\frac{1}{2} + 2\cdot\frac{15}{6^2} + 3\cdot\frac{17}{6^3} + 4\cdot\frac{1}{6^3} = \frac{1}{2} + \frac{30}{6^2} + \frac{51+4}{6^3} =$$

$$= \frac{1}{2} + \frac{30\cdot 6 + 55}{6^3} = \frac{1}{2} + \frac{235}{6^3} = 1{,}5307$$

$$Var(X) = \mathbb{E}[X^2] - \underbrace{\left(\mathbb{E}[X]\right)^2}_{2{,}343} = 2{,}949 - 2{,}343 = 0{,}606$$

$$\mathbb{E}[X^2] = 1\cdot\frac{1}{2} + 4\cdot\frac{15}{6^2} + 9\cdot\frac{17}{6^3} + 16\cdot\frac{1}{6^3} = \frac{1}{2} + \frac{360+153+16}{6^3} =$$

$$= \frac{1}{2} + \frac{529}{\underbrace{6^3}_{216}} = 2{,}9490$$

$$\boxed{\begin{array}{l} \mathbb{E}[X] = 1{,}5307 \\ Var(X) = 0{,}606 \end{array}}$$

25 X mit Dichte f_X ; $c > 1$

$$f_X(x) = \begin{cases} \frac{1}{3}x^3 & , \quad 0 \le x \le 1 \\ \frac{1}{3} & , \quad 1 \le x \le c \\ 0 & , \quad \text{sonst} \end{cases}$$

a) $f_X(x) \geq 0$ ✓

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_0^1 \frac{1}{3}x^3\,dx + \int_1^c \frac{1}{3}\,dx = \frac{1}{3}\frac{x^4}{4}\Big|_0^1 + \frac{1}{3}x\Big|_1^c =$$

$$= \frac{1}{12} + \frac{1}{3}(c-1) = \frac{1}{12} - \frac{1}{3} + \frac{1}{3}c \overset{!}{=} 1 \Rightarrow \frac{1}{3}c - \frac{1}{4} = 1 \Rightarrow$$

$$\frac{1}{3}c = \frac{1}{4} + 1 \Rightarrow \frac{1}{3}c = \frac{5}{4} \Rightarrow c = \frac{5}{4}\cdot\frac{3}{1} = \frac{15}{4}$$

$$\boxed{c = \frac{15}{4}}$$

$$f_X(x) = \begin{cases} \frac{1}{3}x^3 , & 0 \le x \le 1 \\ \frac{1}{3} , & 1 \le x \le \frac{15}{4} \\ 0 , & \text{sonst} \end{cases}$$

b) $F_X(x) = ?$

$x < 0$: $F_X(x) = 0$

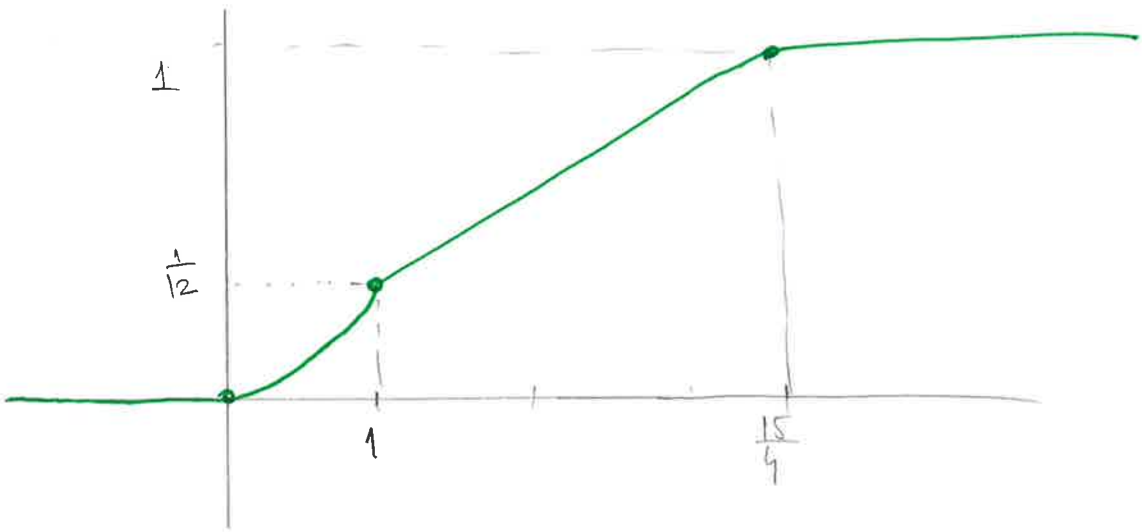
$x \in [0,1]$: $F_X(x) = \int_0^x \frac{1}{3}t^3\,dt = \frac{1}{3}\frac{t^4}{4}\Big|_0^x = \frac{x^4}{12}$

$x \in [1, \frac{15}{4}]$: $F_X(x) = \int_0^1 \frac{1}{3}t^3\,dt + \int_1^x \frac{1}{3}\,dt = \frac{1}{12}t^4\Big|_0^1 + \frac{1}{3}t\Big|_1^x =$

$= \frac{1}{12} + \frac{1}{3}(x-1) = \frac{1}{12} + \frac{1}{3}x - \frac{1}{3} = \frac{1}{3}x - \frac{1}{4}$

$x \ge 4$: $F_X(x) = 1$

$$F_X(x) = \begin{cases} 0 , & x \le 0 \\ \frac{x^4}{12} , & 0 < x \le 1 \\ \frac{1}{3}x - \frac{1}{4} , & 1 < x \le \frac{15}{4} \\ 1 , & x > 4 \end{cases}$$



(5)

c) $\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \frac{1}{3}\int_0^1 x \cdot x^3\,dx + \frac{1}{3}\int_1^{\frac{15}{4}} x\,dx = \frac{1}{3}\frac{x^5}{5}\Big|_0^1 + \frac{1}{3}\frac{x^2}{2}\Big|_1^{\frac{15}{4}}$

$= \frac{1}{15} + \frac{1}{6}\left(\frac{225}{16} - \frac{16}{16}\right) = \underbrace{\frac{1}{15}}_{0{,}066} + \underbrace{\frac{209}{96}}_{2{,}177} = 2{,}243$

$Var(X) = \mathbb{E}[X^2] - \underbrace{\left(\mathbb{E}[X]\right)^2}_{5{,}032} = 0{,}771$

$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \frac{1}{3}\int_0^1 x^5\,dx + \frac{1}{3}\int_1^{\frac{15}{4}} x^2\,dx = \frac{1}{18} +$

$\frac{1}{9} x^3 \Big|_1^{\frac{15}{4}} = \frac{1}{18} + \frac{1}{9}\left(\underbrace{\left(\frac{15}{4}\right)^3}_{52{,}734} - 1\right) = \frac{1}{18} + 5{,}748 = 5{,}803$

$\boxed{Var(X) = 0{,}771}$

d) $\mathbb{P}\left[\frac{1}{2} < X \le \frac{5}{4}\right] = F_X\left(\frac{5}{4}\right) - F_X\left(\frac{1}{2}\right) = \left(\frac{1}{3}\cdot\frac{5}{4} - \frac{1}{4}\right) - \left(\frac{1}{2}\right)^4 \cdot \frac{1}{12} =$

$= \frac{5}{12} - \frac{3}{12} - \frac{1}{16}\cdot\frac{1}{12} = \frac{2}{12} - \frac{1}{12}\cdot\frac{1}{16} = \frac{1}{12}\left(2 - \frac{1}{16}\right) =$

$= \frac{1}{12}\cdot\frac{31}{16} = \frac{31}{\underbrace{12\cdot 16}_{192}} = \underline{0{,}1614}$

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(26) $F_X(x) = \begin{cases} 0, & x \le 0 \\ cx, & 0 < x \le 1 \\ 1 - \frac{c}{x}, & 1 < x \end{cases}$

a)

- $F_X(x)$ muss rechtsseitig stetig sein.

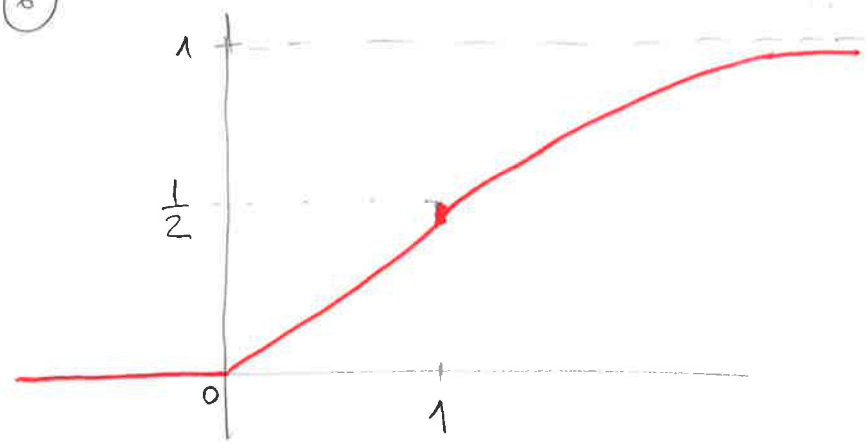
$0 = F_X(0) = \lim_{x \downarrow 0} F_X(x) = \lim_{x \downarrow 0} cx = 0$

$c \cdot 1 = F_X(1) = \lim_{x \downarrow 1} F_X(x) = \lim_{x \downarrow 1}\left(1 - \frac{c}{x}\right) = 1 - c \Rightarrow c = 1 - c \Rightarrow$

$2c = 1 \Rightarrow \boxed{c = \frac{1}{2}}$

(b)

$$F_X(x) = \begin{cases} 0 & , x \le 0 \\ \frac{1}{2}x & , 0 < x \le 1 \\ 1 - \frac{1}{2x} & , x > 1 \end{cases}$$



(c) $$f_X(x) = F_X'(x) = \begin{cases} 0 & , x \le 0 \\ \frac{1}{2} & , 0 < x \le 1 \\ \frac{1}{2x^2} & , x > 1 \end{cases}$$

$$\left(-\frac{1}{2x}\right)' = \frac{1' \cdot 2x - 1(2x)'}{(2x)^2} = \frac{-\cancel{2}}{\cancel{4}_2 x^2} = -\frac{1}{2x^2}$$

(d) $$P\left[\frac{1}{2} \le X < \frac{3}{2}\right] = F_X\left(\frac{3}{2}\right) - F_X\left(\frac{1}{2}\right) = \left(1 - \frac{1}{\cancel{2} \cdot \frac{3}{\cancel{2}}}\right) - \frac{1}{2} \cdot \frac{1}{2} =$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

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27 $$f_X(x) = \begin{cases} 0 & , x \le 0 \\ \frac{(x+1)}{\lambda(\lambda+1)} e^{-\frac{x}{\lambda}} & , x > 0 \end{cases}$$

a) $f_X(x) \ge 0$ ✓

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \frac{1}{\lambda(\lambda+1)} \int_0^{+\infty} (x+1) e^{-\frac{x}{\lambda}}\,dx = \frac{1}{\lambda(\lambda+1)} \int_0^{+\infty} x e^{-\frac{x}{\lambda}}\,dx - \frac{1}{\lambda+1} \int_0^{+\infty} \underbrace{-\frac{1}{\lambda} e^{-\frac{x}{\lambda}}}_{\left(e^{-\frac{x}{\lambda}}\right)'}\,dx$$

$$= \frac{1}{\lambda(\lambda+1)} \int_0^{+\infty} x e^{-\frac{x}{\lambda}}\,dx \underbrace{- \frac{1}{\lambda+1} e^{-\frac{x}{\lambda}} \Big|_0^{+\infty}}_{\frac{1}{\lambda+1}}$$

$$\cdot \int_0^{+\infty} x e^{-\frac{x}{\lambda}}\,dx = -\lambda \int_0^{+\infty} x \left(e^{-\frac{x}{\lambda}}\right)' dx = -\lambda \left[x \cdot e^{-\frac{x}{\lambda}} \Big|_0^{+\infty} - \int_0 e^{-\frac{x}{\lambda}}\,dx \right] =$$

Partielle Integration

7

$$= -\lambda\left[0 + \lambda\int_0^{+\infty}\left(e^{-\frac{x}{\lambda}}\right)' dx\right] = -\lambda^2 e^{-\frac{x}{\lambda}}\Big|_0^{+\infty} = -\lambda^2(0-1) = \lambda^2$$

$$\Rightarrow \int_{-\infty}^{+\infty} f_X(x)\,dx = \frac{1}{\lambda(\lambda+1)}\cdot\lambda^2 + \frac{1}{\lambda+1} = \frac{\lambda^2+\lambda}{\lambda(\lambda+1)} = 1 \checkmark$$

b) $\underline{x < 0}$: $F_X(x) = 0$

$\boxed{x \geq 0}$: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{t+1}{\lambda(\lambda+1)} e^{-\frac{t}{\lambda}}\,dt =$

$$\frac{1}{\lambda(\lambda+1)}\underbrace{\int_0^x t e^{-\frac{t}{\lambda}}\,dt}_{A} + \frac{1}{\lambda(\lambda+1)}\int_0^x e^{-\frac{t}{\lambda}}\,dt$$

$$\frac{1}{\lambda(\lambda+1)}\int_0^x e^{-\frac{t}{\lambda}}\,dt = -\frac{1}{\lambda+1} e^{-\frac{t}{\lambda}}\Big|_0^x = -\frac{1}{\lambda+1}\left[e^{-\frac{x}{\lambda}} - 1\right]$$

$$A = \int_0^x t e^{-\frac{t}{\lambda}}\,dt = -\lambda\left[t e^{-\frac{t}{\lambda}}\Big|_0^x + \lambda e^{-\frac{t}{\lambda}}\Big|_0^x\right] = -\lambda\left[x e^{-\frac{x}{\lambda}} + \lambda e^{-\frac{x}{\lambda}} - \lambda\right]$$

$$\Rightarrow F_X(x) = \frac{1}{\lambda(\lambda+1)}\cdot(-\lambda)\left(x e^{-\frac{x}{\lambda}} + \lambda e^{-\frac{x}{\lambda}} - \lambda\right) - \frac{1}{\lambda+1}\left(e^{-\frac{x}{\lambda}} - 1\right) =$$

$$= \frac{-1}{\lambda+1} x e^{-\frac{x}{\lambda}} - \frac{\lambda}{\lambda+1} e^{-\frac{x}{\lambda}} + \underbrace{\frac{\lambda}{\lambda+1} - \frac{1}{\lambda+1} e^{-\frac{x}{\lambda}} + \frac{1}{\lambda+1}}_{1}$$

$$= \frac{-1}{\lambda+1} x e^{-\frac{x}{\lambda}} - e^{-\frac{x}{\lambda}} + 1$$

$$\Rightarrow F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\frac{x}{\lambda}} - \frac{1}{\lambda+1} x e^{-\frac{x}{\lambda}}, & x > 0 \end{cases}$$