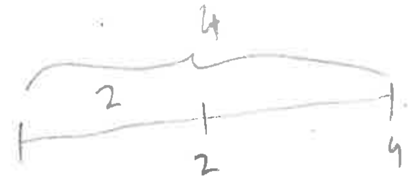
62) $(N_t)$ Rate $\lambda$

a) $\mathbb{P}[N_3=6]=\frac{(3\lambda)^6}{6!}e^{-3\lambda}$

$\mathbb{P}[N_{2,6}\le 3]=\sum_{k=0}^{3}\frac{(2.6\lambda)^k}{k!}e^{-2.6\lambda}$




b) $\mathbb{P}[N_4=4\mid N_2=2]=$

$=\frac{\mathbb{P}[N_4=4,\ N_2=2]}{\mathbb{P}[N_2=2]}$

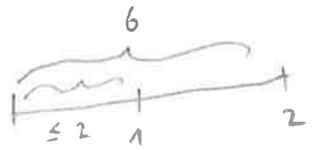
$=\frac{\mathbb{P}[N_4-N_2=2,\ N_2=2]}{\mathbb{P}[N_2=2]}=\frac{\mathbb{P}[N_2=2]^2}{\mathbb{P}[N_2=2]}=\mathbb{P}[N_2=2]=\frac{(2\lambda)^2}{2!}e^{-2\lambda}$

c) $\mathbb{P}[N_7-N_3=2\mid N_3=2]=\frac{\mathbb{P}[N_7-N_3=2,\ N_3=2]}{\mathbb{P}[N_3=2]}=\mathbb{P}[N_7-N_3=2]$

$\mathbb{P}[N_7-N_3=2]=\mathbb{P}[N_4=2]=\frac{(4\lambda)^2}{2!}e^{-4\lambda}$

63) $(N_t)$ Rate 2

a) $\mathbb{P}[N_1\ge 3]=1-\mathbb{P}[N_1\le 2]=1-\sum_{k=0}^{2}\frac{2^k}{k!}e^{-2}=1-e^{-2}(1+2+2)=1-5e^{-2}=0.323$




b) $\mathbb{P}[N_1\le 2,\ N_2=6]$

$=\sum_{k=0}^{2}\mathbb{P}[N_1=k,\ N_2-N_1=6-k]$

$=\sum_{k=0}^{2}\mathbb{P}[N_1=k]\cdot\mathbb{P}[N_1=6-k]=\sum_{k=0}^{2}\frac{2^k}{k!}e^{-2}\cdot\frac{2^{6-k}}{(6-k)!}e^{-2}$

$=2^6e^{-4}\sum_{k=0}^{2}\frac{1}{k!(6-k)!}=2^6e^{-4}\cdot\frac{11}{360}=0.036$

c) $\mathbb{P}[N_1 \le 2 \mid N_3 = 6] = \frac{\mathbb{P}[N_1 \le 2, N_3 = 6]}{\mathbb{P}[N_3 = 6]} = \frac{0{,}109}{0{,}161} \approx 0{,}68$

$$\mathbb{P}[N_1 \le 2, N_3 = 6] = \sum_{k=0}^{2} \mathbb{P}[N_1 = k, N_3 - N_1 = 6-k]$$

$$= \sum_{k=0}^{2} \mathbb{P}[N_1 = k] \cdot \mathbb{P}[N_3 - N_1 = 6-k]$$

$$= \sum_{k=0}^{2} \frac{2^k}{k!} e^{-2} \cdot \frac{(2\cdot 2)^{6-k}}{(6-k)!} e^{-2\cdot 2}$$

$$= e^{-6}\left(\frac{2^0}{0!} \frac{4^6}{6!} + \frac{2^1}{1!} \cdot \frac{4^5}{5!} + \frac{2^2}{2!} \cdot \frac{4^4}{4!}\right)$$

$$= e^{-6} \cdot \frac{1984}{45} \approx 0.109$$

$$\mathbb{P}[N_3 = 6] = \frac{(2\cdot 3)^6}{6!} e^{-2\cdot 3} \approx 0{,}161$$

d) $\mathbb{P}[N_3 = 6 \mid N_1 \le 2] = \frac{\mathbb{P}[N_1 \le 2, N_3 = 6]}{\mathbb{P}[N_1 \le 2]} = \frac{0.109}{5e^{-2}} \approx 0{,}162$

G4

$s<t$

$$Cov(N_t, N_s) = \mathbb{E}[N_s N_t] - \underbrace{\mathbb{E}[N_s]}_{\lambda s}\underbrace{\mathbb{E}[N_t]}_{\lambda t}$$

$$\mathbb{E}[N_s N_t] = \mathbb{E}[N_s(N_t - N_s + N_s)] =$$
$$= \mathbb{E}[N_s(N_t - N_s) + N_s^2] =$$
$$= \mathbb{E}[N_s]\mathbb{E}[N_t - N_s] + \mathbb{E}[N_s^2]$$
$$= \lambda s \, \lambda(t-s) + (\lambda s)^2 + \lambda s = \lambda^2 ts + \lambda s = \lambda s(1 + \lambda t)$$

$X \sim Poi(\lambda)$

$\Rightarrow \mathbb{E}[X] = \lambda$

$\mathbb{E}[X^2] = \lambda^2 + \lambda$

$$\Rightarrow Cov(N_t, N_s) = \lambda^2 ts + \lambda s - \lambda^2 ts = \underline{\lambda s}$$

$$Cov(N_t, N_t) = Var(N_t) = \lambda t$$

Im Allgemeinen

$$\Rightarrow Cov(N_t, N_s) = \lambda \min\{s, t\}$$

$X \sim Poi(\lambda)$

$$\mathbb{E}[X^2] = \sum_{k \geq 0} k \mathbb{P}(X = k) = \sum_{k \geq 0} k^2 \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{(k-1)!} \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$(x \cdot e^x)' = \left(\sum_{k=0}^{\infty} \frac{x^{k+1}}{k!}\right)' \Leftrightarrow e^x + xe^x = \sum_{k=0}^{\infty} \frac{(k+1)x^k}{k!} = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{(k-1)!} \quad \Big| \cdot x$$

$$\Rightarrow e^x(x + x^2) = \sum_{k=1}^{\infty} \frac{k x^k}{(k-1)!} = \sum_{k=0}^{\infty} \frac{k x^k}{(k-1)!}$$

$$\Rightarrow \mathbb{E}[X^2] = e^{-\lambda} e^{\lambda}(\lambda + \lambda^2) = \lambda + \lambda^2$$

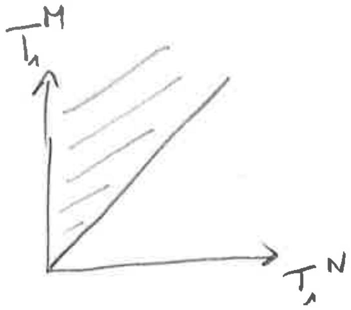
65| $\left.\begin{array}{l}(N_t)_{t\geq 0} \text{ mit Rate } \lambda \\ (M_t)_{t\geq 0} \text{ mit Rate } \mu\end{array}\right\}$ unabhängig

Sei $T_n^N$ ... Zeit des n-ten Ereignisses von $N_t$
$T_n^M$ ... —— " —— von $M_t$

$\left.\begin{array}{l}T_1^N \sim Exp(\lambda) \\ T_1^M \sim Exp(\mu)\end{array}\right\}$ unabh.

$\Rightarrow$ die gemeinsame Dichte von $(T_1^N, T_1^M)$ ist

$$f(x,y) = \begin{cases} \lambda \cdot \mu\, e^{-\lambda x} e^{-\mu y} & \text{wenn } x,y \geq 0 \\ 0 & \text{sonst} \end{cases}$$




$$\mathbb{P}[T_1^N < T_1^M] = \int_0^\infty \int_x^\infty f(x,y)\, dy\, dx$$

$$= \int_0^\infty \int_x^\infty \lambda\mu\, e^{-\lambda x} e^{-\mu y}\, dy\, dx$$

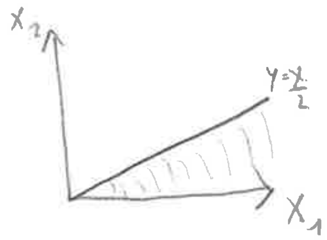
$$= \lambda\mu \int_0^\infty e^{-\lambda x} \int_x^\infty e^{-\mu y}\, dy\, dx = -\lambda \int_0^\infty e^{-\lambda x} \left(e^{-\mu y}\right)\Big|_{y=x}^{\infty} dx$$

$$= \lambda \int_0^\infty e^{-(\lambda+\mu)x}\, dx$$

$$= -\frac{\lambda}{\lambda+\mu}\, e^{-(\lambda+\mu)x}\Big|_0^\infty = \frac{\lambda}{\lambda+\mu}$$

66) $(N_t)_{t \geq 0}$ $\quad \lambda = 4 \quad$ $X_1, X_2 \sim Exp(4)$ unabh.

$$\mathbb{P}[X_1 \geq 2X_2] = \int_0^\infty 4e^{-4x} \int_0^{x/2} 4e^{-4y}\,dy\,dx$$




$$= 16 \int_0^\infty e^{-4x} \left(-\frac{1}{4}\right) e^{-4y}\Big|_0^{x/2} =$$

$$-4 \int_0^\infty e^{-4x} \left(e^{-2x} - 1\right) dx =$$

$$= -4 \int_0^\infty e^{-6x} - e^{-4}\,dx = -4\left(-\frac{1}{6} e^{-6x}\Big|_0^\infty + \frac{1}{4} e^{-4x}\Big|_0^\infty\right)$$

$$= -\frac{4}{6} + 1 = \frac{1}{3}$$

**67** $(N_t)_{t\geq 0}$ Poisson-Prozess mit Rate 2

**a)**


$$P[N_1 \geq 2, N_2 - N_1 \leq 1] = P[N_1 \geq 2]\, P[N_2 - N_1 \leq 1]$$
$$= (1 - P[N_1 \leq 1])\, P[N_1 \leq 1] = (1 - 3e^{-2})\, 3e^{-2} = 3e^{-2} - 9e^{-4} \approx 0{,}241$$

$$P[N_1 \leq 1] = P[N_1 = 0] + P[N_1 = 1] = \frac{2^0}{0!} e^{-2} + \frac{2^1}{1!} e^{-2} = 3e^{-2}$$

**b)** $$P\left[T_2 - T_1 \geq \frac{10}{12}\right] = P\left[X_2 \geq \frac{10}{12}\right] = e^{-2\frac{10}{12}} = e^{-\frac{5}{3}} = 0{,}19$$

**c)**


$$P[N_2 \leq 3] = \sum_{k=0}^{3} P[N_2 = k] = \sum_{k=0}^{3} \frac{(2\cdot 2)^k}{k!} e^{-2\cdot 2} = e^{-4}\left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!}\right)$$
$$= e^{-4}\left(1 + 4 + 8 + \frac{32}{3}\right) = e^{-4}\left(\frac{3 + 12 + 24 + 32}{3}\right) = e^{-4}\,\frac{71}{3}$$
$$\approx 0{,}433$$

**d)**


$$P[N_1 \geq 4 \text{ oder } N_2 - N_1 \geq 4] = 1 - P[N_1 \leq 3, N_2 - N_1 \leq 3]$$
$$= 1 - P[N_1 \leq 3] \cdot P[N_2 - N_1 \leq 3] = 1 - P[N_1 \leq 3]^2 = 1 - \left(\frac{19}{3}\right)^2 e^{-4} = 0{,}265$$

$$P[N_1 \leq 3] = \sum_{k=0}^{3} \frac{2^k}{k!} e^{-2} = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right) = e^{-2}\left(1 + 2 + 2 + \frac{4}{3}\right)$$
$$= e^{-2}\left(\frac{3 + 6 + 6 + 4}{3}\right) = \frac{19}{3} e^{-2}$$