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On the ellipticity of coupled finite element and one–equation boundary element methods for boundary value problems

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Abstract

In this paper we extend some recent results on the stability of the Johnson–Nédelec coupling of finite and boundary element methods in the case of boundary value problems. In [11, 13, 15], the case of a free–space transmission problem was considered, and sufficient and necessary conditions are stated which ensure the ellipticity of the bilinear form for the coupled problem. The proof was based on considering the energies which are related to both the interior and exterior problem. In the case of boundary value problems for either interior or exterior problems, additional estimates are required to bound the energy for the solutions of related subproblems. Moreover, several techniques for the stabilization of the coupled formulations are analysed. Applications involve boundary value problems with either hard or soft inclusions, exterior boundary value problems, and macro–element techniques.

1 Introduction

The coupling of finite and boundary element methods plays an important role in particular when considering physical models which couple linear and nonlinear partial differential equations, including problems in unbounded exterior domains. From a mathematical point of view, the use of the so–called symmetric coupling of finite and boundary element methods [3, 5, 10, 18] ensures stability and optimal a priori error estimates, and allows the construction of almost optimal preconditioned iterative solution strategies. Although there are efficient implementations available, the use of the symmetric formulation is still not very popular in engineering and for more advanced applications. Instead, the one–equation or Johnson–Nédélec coupling [9] is used, which relies on the use of single and double layer boundary integral operators only, and which allows for both Galerkin and collocation schemes for discretization. While for the general case a rigorous mathematical analysis was not available for some time, numerical examples indicated the stability of this coupling scheme for more general situations [4]. In the case of free space transmission problems, the stability of the Johnson–Nédélec coupling of finite and boundary elements was first established in [13], and in [15] this approach was extended to prove ellipticity of the related bilinear form. This result was further refined in [11] where a sufficient and necessary condition on the finite element coefficient matrix and on the contraction property of the shifted double layer boundary integral operator was given. As in the proof of the ellipticity of the single layer boundary integral operator [8], the proofs in [11, 13, 15] are based on considering the energies which are related to interior and exterior boundary value problems. So the application to the analysis of free space transmission problems was more or less straightforward.

In this paper we extend the previous results as given in [11] to the Johnson–Nédélec coupling of finite and boundary elements in the case of boundary value problems. As a model problem we consider an interior Dirichlet boundary value problem of a diffusion equation with variable coefficients, but with a Laplace equation within a given inclusion. While the formulation of the finite and boundary element coupling approach follows as for a free space transmission problem, the ellipticity and stability analysis requires the consideration of some exterior eigenvalue problem to relate the energy of the bounded finite element problem to the energy of some related problem in an unbounded exterior domain. In addition we have to do a careful analysis of certain orthogonal splittings which are necessary in the two–dimensional case. The main result of this paper is given in Theorem 2.2 where sufficient conditions are given to ensure ellipticity of the coupled bilinear form. For the particular example of a circular inclusion in a circular domain we are able to compute all involved constants explicitely, and to confirm the theoretical results by numerical examples. The approach presented for this model problem can be extended to the analysis of coupled schemes for the solution of other boundary value problems in a rather similar way. Examples include interior boundary value problems with soft inclusions, exterior boundary value problems, and macro–element formulations [7]. While the methodologies to prove ellipticity in all of these cases are more or less the same, minimal eigenvalues of different eigenvalue problems enter the sufficient and necessary conditions which are required to ensure ellipticity. While we restrict our considerations to the case of scalar linear diffusion equations, extensions to systems of partial differential equations, as, e.g., in linear elasticity [6, 16], and to nonlinear partial differential equations [1] follow the same lines.

2 Boundary value problems with hard inclusions

2.1 Model problem and variational formulation

As a model problem we consider the Dirichlet boundary value problem

$$-\mathrm{div}[A(x)\nabla u(x)] = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma := \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded Lipschitz domain. Let $\Omega_0 \subset \Omega$ be some inclusion, again Lipschitz, with boundary $\Gamma_0 = \partial\Omega_0$, $\Gamma \cap \Gamma_0 = \emptyset$, where we assume a material behavior which is different from that in the remainder $\Omega_1 = \Omega \backslash \overline{\Omega}_0$. In particular we assume $A(x) = I$ and $f(x) = 0$ for $x \in \Omega_0$, see Fig. 1.

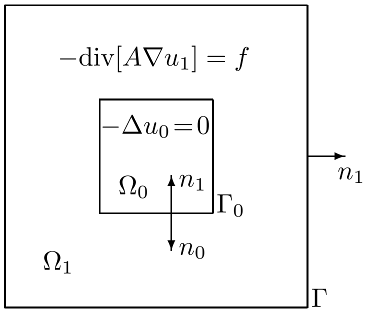

Figure 1: Boundary value problem with hard inclusion.

We further assume that the symmetric coefficient matrix $A(x)$ is uniform positive definite in Ω_1, i.e., there is a positive constant

$$\lambda_{\min} := \inf_{x \in \Omega_1} \min_{k=1,\ldots,n} \lambda_k(A(x)) > 0. \tag{2.2}$$

Moreover, let $f \in L_2(\Omega_1)$, and in the two–dimensional case we finally assume $\operatorname{diam} \Omega_0 < 1$.

Instead of the global problem (2.1) we consider the local partial differential equations

$$-\Delta u_0(x) = 0 \quad \text{for } x \in \Omega_0, \quad -\mathrm{div}[A(x)\nabla u_1(x)] = f(x) \quad \text{for } x \in \Omega_1,$$

with the boundary and transmission conditions

$$u_1(x) = 0 \quad \text{for } x \in \Gamma, \quad u_1(x) = u_0(x), \quad n_1 \cdot A(x)\nabla u_1(x) + \frac{\partial}{\partial n_0} u_0(x) = 0 \quad \text{for } x \in \Gamma_0.$$

Note that n_i denote the exterior normal vectors with respect to the subdomains Ω_i, $i = 0, 1$, and $n_1 = -n_0$ on Γ_0. By introducing

$$t(x) := \frac{\partial}{\partial n_0} u_0(x) = -n_1 \cdot A(x)\nabla u_1(x) \quad \text{for } x \in \Gamma_0,$$

the variational formulation of the subproblem in Ω_1 is to find $u_1 \in H^1_0(\Omega_1, \Gamma)$ such that

$$\int_{\Omega_1} [A(x)\nabla u_1(x)] \cdot \nabla v(x)\, dx + \int_{\Gamma_0} t(x) v(x)\, ds_x = \int_{\Omega_1} f(x) v(x)\, dx \tag{2.3}$$

is satisfied for all $v \in H_0^1(\Omega_1, \Gamma) := \{v \in H^1(\Omega_1) : v_{|\Gamma} = 0\}$, while the boundary integral equation related to the Laplace equation in Ω_0 reads, by using $u_0 = u_1$ on Γ_0,

$$(Vt)(x) = (\frac{1}{2}I + K)u_1(x) \quad \text{for almost all } x \in \Gamma_0 . \tag{2.4}$$

Recall that

$$(Vt)(x) = \int_{\Gamma_0} U^*(x,y)t(y)\, ds_y, \quad (Ku)(x) = \int_{\Gamma_0} \frac{\partial}{\partial n_0} U^*(x,y)u(y)\, ds_y \quad \text{for } x \in \Gamma_0 \tag{2.5}$$

denote the single and double layer boundary integral operators, and $U^*(x,y)$ is the fundamental solution of the Laplace operator, i.e.,

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & \text{for } n = 2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|} & \text{for } n = 3. \end{cases}$$

By combining (2.3) and the weak formulation of the boundary integral equation (2.4), we obtain a variational problem to find $(u_1, t) \in H_0^1(\Omega_1, \Gamma) \times H^{-1/2}(\Gamma_0)$ such that

$$a(u_1, t; v, \tau) = \langle f, v \rangle_{\Omega_1} \tag{2.6}$$

is satisfied for all $(v, \tau) \in H_0^1(\Omega_1, \Gamma) \times H^{-1/2}(\Gamma_0)$, where the associated bilinear form is given by

$$a(u, t; v, \tau) := \langle A\nabla u, \nabla v \rangle_{L_2(\Omega_1)} + \langle t, v \rangle_{\Gamma_0} + \langle Vt, \tau \rangle_{\Gamma_0} - \langle (\frac{1}{2}I + K)u_{|\Gamma_0}, \tau \rangle_{\Gamma_0}. \tag{2.7}$$

Since the boundary integral equation (2.4) holds for any solution of the Laplace equation in Ω_0, it follows for a constant solution $\overline{u}_0 \equiv 1$ that $(\frac{1}{2}I + K)\overline{u}_0 = 0$ almost everywhere on Γ_0. By using the symmetry relation $KV = VK'$, see, e.g., [14], the solution $t \in H^{-1/2}(\Gamma)$ of the boundary integral equation (2.4) satisfies

$$\langle t, 1 \rangle_{\Gamma_0} = \langle Vt, V^{-1}\overline{u}_0 \rangle_{\Gamma_0} = \langle (\frac{1}{2}I + K)u_1, V^{-1}\overline{u}_0 \rangle_{\Gamma_0} = \langle u_1, V^{-1}(\frac{1}{2}I + K)\overline{u}_0 \rangle_{\Gamma_0} = 0, \tag{2.8}$$

hence we may introduce

$$H_*^{-1/2}(\Gamma_0) = \left\{ \tau \in H^{-1/2}(\Gamma_0) : \langle \tau, 1 \rangle_{\Gamma_0} = 0 \right\}.$$

Therefore, instead of (2.6) we may also consider the alternative variational problem to find $(u_1, t) \in H_0^1(\Omega_1, \Gamma) \times H_*^{-1/2}(\Gamma_0)$ such that

$$a(u_1, t; v, \tau) = \langle f, v \rangle_{\Omega_1} \tag{2.9}$$

is satisfied for all $(v, \tau) \in H_0^1(\Omega_1, \Gamma) \times H_*^{-1/2}(\Gamma_0)$.

One possibility to ensure unique solvability of the variational formulations (2.6) and (2.9) is to prove ellipticity of the bilinear form (2.7) either in $H^1_0(\Omega_1, \Gamma) \times H^{-1/2}(\Gamma_0)$, or in $H^1_0(\Omega_1, \Gamma) \times H^{-1/2}_*(\Gamma_0)$. Then, by using standard arguments, we can also prove existence, uniqueness, and quasi–optimality of related Galerkin approximations when using conforming finite and boundary element methods, see, e.g., [12, 14].

Although both variational formulations (2.6) and (2.9) are equivalent to each other, in the two–dimensional case $n = 2$ the ellipticity results for the bilinear form (2.7) are rather different when considering the bilinear form for $(u_1, t) \in H^1_0(\Omega, \Gamma) \times H^{-1/2}(\Gamma_0)$, or for $(u_1, t) \in H^1_0(\Omega, \Gamma) \times H^{-1/2}_*(\Gamma_0)$, see Theorem 2.2. Note that the variational problem (2.9) can be reformulated either by using a Lagrange multiplier, or by using a modified bilinear form.

The proof of ellipticity for the bilinear form (2.7) follows similar to the case of a free space transmission problem [11, 15]. However, to estimate the energy which is related to the boundary value problem in the bounded domain Ω_1, we need to introduce some artificial eigenvalue problem which is related to an unbounded exterior domain.

2.2 An exterior eigenvalue problem

For any given $v \in H^{1/2}(\Gamma_0)$ let $w_1 \in H^1_0(\Omega_1, \Gamma)$ be the unique solution of the Dirichlet boundary value problem

$$-\Delta w_1(x) = 0 \quad \text{for } x \in \Omega_1, \quad w_1(x) = 0 \quad \text{for } x \in \Gamma, \quad w_1(x) = v(x) \quad \text{for } x \in \Gamma_0. \tag{2.10}$$

Then we can introduce the Dirichlet to Neumann map

$$(S_1 v)(x) := \frac{\partial}{\partial n_1} w_1(x) \quad \text{for } x \in \Gamma_0,$$

where $S_1 : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ is the related Steklov–Poincaré operator. From its definition we obtain for the weak solution w_1 of the Dirichlet boundary value problem (2.10), by using $w_1 = 0$ on Γ,

$$\langle S_1 v, v\rangle_{\Gamma_0} = \int_{\partial\Omega_1} \frac{\partial}{\partial n_1} w_1(x) w_1(x)\, ds_x = \int_{\Omega_1} |\nabla w_1(x)|^2\, dx.$$

For the chosen $v \in H^{1/2}(\Gamma_0)$ we may also consider the exterior Dirichlet boundary value problem

$$-\Delta w_\infty(x) = 0 \quad \text{for } x \in \Omega_0^c := \mathbb{R}^n \backslash \overline{\Omega}_0, \quad w_\infty(x) = v(x) \quad \text{for } x \in \Gamma_0 \tag{2.11}$$

satisfying the radiation condition

$$w_\infty(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to \infty. \tag{2.12}$$

The solution of the exterior boundary value problem (2.11) and (2.12) is given by the representation formula

$$w_\infty(x) = -\int_{\Gamma_0} U^*(x,y)\frac{\partial}{\partial n_0} w_\infty(y)\, ds_y + \int_{\Gamma_0} \frac{\partial}{\partial n_0} U^*(x,y) v(y)\, ds_y \quad \text{for } x \in \Omega_0^c,$$

where in the two–dimensional case we require the scaling condition

$$\int_{\Gamma_0} \frac{\partial}{\partial n_0} w_\infty(y)\, ds_y = 0. \tag{2.13}$$

to ensure the radiation condition (2.12). In fact, $t_\infty = \dfrac{\partial}{\partial n_0} w_\infty$ is the unique solution of the boundary integral equation

$$(Vt_\infty)(x) = (-\frac{1}{2}I + K)v(x) \quad \text{for } x \in \Gamma_0. \tag{2.14}$$

The scaling condition (2.13) and using (2.8) imply

$$0 = \langle t_\infty, 1\rangle_{\Gamma_0} = \langle Vt_\infty, V^{-1}\overline{u}_0\rangle_{\Gamma_0} = \langle(-\frac{1}{2}I + K)v, V^{-1}\overline{u}_0\rangle_{\Gamma_0} = -\langle v, V^{-1}\overline{u}_0\rangle_{\Gamma_0} = -\langle v, t_{\text{eq}}\rangle_{\Gamma_0},$$

where $t_{\text{eq}} \in H^{-1/2}(\Gamma_0)$ is the unique solution of the boundary integral equation $Vt_{\text{eq}} = \overline{u}_0$ on Γ_0. In particular in the two–dimensional case we therefore need to consider $v \in H^{1/2}_*(\Gamma_0)$ where

$$H^{1/2}_*(\Gamma_0) := \left\{ v \in H^{1/2}(\Gamma_0) : \langle v, t_{\text{eq}}\rangle_{\Gamma_0} = 0 \right\}.$$

The solution of the exterior Dirichlet boundary value problem (2.11) and (2.12) induces the Dirichlet to Neumann map

$$(S^{\text{ext}}v)(x) := \frac{\partial}{\partial n_1} w_\infty(x) = -\frac{\partial}{\partial n_0} w_\infty(x) \quad \text{for } x \in \Gamma_0,$$

where the exterior Steklov–Poincaré operator is given as

$$S^{\text{ext}} = V^{-1}(\frac{1}{2}I - K) = D + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K), \tag{2.15}$$

and

$$(Du)(x) = -\frac{\partial}{\partial n_{0,x}} \int_{\Gamma_0} \frac{\partial}{\partial n_{0,y}} U^*(x,y) u(y)\, ds_y \quad \text{for } x \in \Gamma_0$$

is the so–called hypersingular boundary integral operator.

Since the Steklov–Poincaré operators $S_1, S^{\text{ext}} : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ are both elliptic, we can consider the spectral equivalence inequalities

$$\mu_{\min}\, \langle S^{\text{ext}}v, v\rangle_{\Gamma_0} \le \langle S_1 v, v\rangle_{\Gamma_0} \quad \text{for all } v \in \begin{cases} H^{1/2}_*(\Gamma_0), & n = 2, \\ H^{1/2}(\Gamma_0), & n = 3. \end{cases} \tag{2.16}$$

Note that $\mu_{\min}$ is characterized as minimal eigenvalue of a related eigenvalue problem of the underlying partial differential equations. In Sect. 3 we present a case study for a particular situation where we can compute $\mu_{\min}$ explicitly.

Since the Dirichlet boundary value problem (2.10) is well defined for all $v \in H^{1/2}(\Gamma_0)$ also in the two–dimensional case, we introduce the decomposition $H^{1/2}(\Gamma_0) = H_*^{1/2}(\Gamma_0) \oplus \{\overline{u}_0\}$ as follows: For an arbitrary $v \in H^{1/2}(\Gamma_0)$ let

$$v = v_0 + \alpha, \quad \alpha = \frac{\langle v, t_{\text{eq}}\rangle_{\Gamma_0}}{\langle 1, t_{\text{eq}}\rangle_{\Gamma_0}}, \quad v_0 \in H_*^{1/2}(\Gamma_0). \tag{2.17}$$

The constraint $v_0 \in H_*^{1/2}(\Gamma_0)$ implies $\langle v_0, t_{\text{eq}}\rangle_{\Gamma_0} = \langle v_0, V^{-1}\overline{u}_0\rangle_{\Gamma_0} = 0$, i.e. orthogonality in $H^{1/2}(\Gamma_0)$. Since

$$\|v\|_{V^{-1}}^2 = \langle V^{-1}v, v\rangle_{\Gamma_0}, \quad \|v\|_{S_1}^2 = \langle S_1 v, v\rangle_{\Gamma_0}$$

define equivalent norms in $H^{1/2}(\Gamma_0)$, i.e. there exist positive constants γ_1 and γ_2 such that

$$\gamma_1 \, \|v\|_{S_1}^2 \le \|v\|_{V^{-1}}^2 \le \gamma_2 \, \|v\|_{S_1}^2 \quad \text{for all } v \in H^{1/2}(\Gamma_0),$$

we conclude that there exists a positive constant

$$c_S \le 1 - \frac{\gamma_1}{\gamma_2} < 1$$

such that the strengthened Cauchy–Schwarz inequality

$$\langle S_1 \overline{u}_0, v_0\rangle_{\Gamma_0} \le c_S \, \|\overline{u}_0\|_{S_1} \|v_0\|_{S_1} \quad \text{for all } v_0 \in H_*^{1/2}(\Gamma_0) \tag{2.18}$$

is satisfied. Then there holds

$$(1 - c_S) \left[\langle S_1 v_0, v_0\rangle_{\Gamma_0} + \alpha^2 \langle S_1 \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}\right] \le \langle S_1 v, v\rangle_{\Gamma_0} \quad \text{for all } v \in H^{1/2}(\Gamma_0), \tag{2.19}$$

where $(v_0, \alpha) \in H_*^{1/2}(\Gamma_0) \times \mathbb{R}$ is given by (2.17).

2.3 Norm equivalence inequalities

The ellipticity estimate of the bilinear form (2.7) is based on different representations of the involved Steklov–Poincaré operators, or equivalently, on different representations of the related energies for solutions of both interior and exterior problems. These results are partially based on the contraction property of the double layer boundary integral operator, see [17], i.e.,

$$\|(\frac{1}{2}I + K)v\|_{V^{-1}} \le c_K \, \|v\|_{V^{-1}} \quad \text{for all } v \in H^{1/2}(\Gamma_0), \tag{2.20}$$

where

$$c_K = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^V c_1^D} < 1.$$

Lemma 2.1 *Let $S^{\text{ext}} : H_*^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ be the Steklov–Poincaré operator as defined in* (2.15)*. Then there holds*

$$\frac{1}{c_K} \, \|(\frac{1}{2}I - K)v\|^2_{V^{-1}} \leq \langle S^{ext} v, v\rangle_{\Gamma_0} \quad \textit{for all } v \in H_*^{1/2}(\Gamma_0). \tag{2.21}$$

Proof. As in [17, Corollary 5.1] we start with the estimates

$$(1 - c_K)\|v\|_{V^{-1}} \leq \|(\frac{1}{2}I - K)v\|_{V^{-1}} \leq c_K \, \|v\|_{V^{-1}} \quad \text{for all } v \in H_*^{1/2}(\Gamma_0).$$

Moreover, we have

$$\langle Dv, v\rangle_{\Gamma_0} \geq c_K(1 - c_K)\|v\|^2_{V^{-1}} \quad \text{for all } v \in H_*^{1/2}(\Gamma_0).$$

Hence we conclude, by using the symmetric representation of the exterior Steklov–Poincaré operator (2.15), the estimate

$$\langle S^{\text{ext}} v, v\rangle_{\Gamma_0} \;=\; \langle Dv, v\rangle_{\Gamma_0} + \|(\frac{1}{2}I - K)v\|^2_{V^{-1}} \geq \frac{1}{c_K} \, \|(\frac{1}{2}I - K)v\|^2_{V^{-1}}.$$

∎

2.4 Ellipticity of the coupled bilinear form

Now we are in a position to state the main result of this paper.

Theorem 2.2 *Let $\lambda_{\min} > 0$ be the minimal eigenvalue of the coefficient matrix A as defined in* (2.2)*, let $\mu_{\min} > 0$ be as defined in* (2.16)*, and let $c_S \in [0, 1)$ be the constant of the strengthened Cauchy–Schwarz inequality* (2.18)*.*

i. In the two–dimensional case $n = 2$ there holds the ellipticity estimate

$$a(v, \tau; v, \tau) \geq c_1^A \Big[\|\nabla v\|^2_{L_2(\Omega_1)} + \|\tau\|^2_V \Big] \quad \textit{for all } (v, \tau) \in H_0^1(\Omega_1, \Gamma) \times H^{-1/2}(\Gamma_0)$$

with some positive constant $c_1^A > 0$ if the condition

$$\lambda_{\min} > \frac{1}{4} \frac{1}{1 - c_S} \max \left\{ \frac{c_K}{\mu_{\min}}, \frac{\langle t_{eq}, 1\rangle_{\Gamma_0}}{\langle S_1 \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}} \right\} \tag{2.22}$$

is satisfied.

ii. In the two–dimensional case $n = 2$ there also holds the ellipticity estimate

$$a(v, \tau; v, \tau) \geq \widetilde{c}_1^A \Big[\|\nabla v\|^2_{L_2(\Omega_1)} + \|\tau\|^2_V \Big] \quad \textit{for all } (v, \tau) \in H_0^1(\Omega_1, \Gamma) \times H_*^{-1/2}(\Gamma_0)$$

with some positive constant $\widetilde{c}_1^A > 0$ if the condition

$$\lambda_{\min} > \frac{1}{4} \frac{1}{1 - c_S} \frac{c_K}{\mu_{\min}} \tag{2.23}$$

is satisfied.

iii. In the three–dimensional case $n=3$ there holds the ellipticity estimate

$$a(v,\tau;v,\tau) \geq \widehat{c}_1^A \Big[\|\nabla v\|^2_{L_2(\Omega_1)} + \|\tau\|^2_V \Big] \quad \textit{for all } (v,\tau) \in H^1_0(\Omega_1,\Gamma) \times H^{-1/2}(\Gamma_0)$$

with some positive constant $\widehat{c}_1^A > 0$ if the condition

$$\lambda_{\min} > \frac{1}{4} \frac{c_K}{\mu_{\min}} \tag{2.24}$$

is satisfied.

Proof. For arbitrary $(v,\tau) \in H^1_0(\Omega_1,\Gamma) \times H^{-1/2}(\Gamma_0)$ we first consider

$$\begin{aligned} a(v,\tau;v,\tau) &= \int_{\Omega_1} A(x)\nabla v(x) \cdot \nabla v(x)\, dx + \langle V\tau,\tau\rangle_{\Gamma_0} + \langle (\frac{1}{2}I - K)v,\tau\rangle_{\Gamma_0} \\ &\geq \lambda_{\min} \int_{\Omega_1} |\nabla v(x)|^2\, dx + \|\tau\|^2_V + \langle (\frac{1}{2}I - K)v,\tau\rangle_{\Gamma_0}. \end{aligned}$$

For an arbitrary but fixed $v \in H^1_0(\Omega_1,\Gamma)$, we introduce the splitting $v = v_{\Gamma_0} + \widetilde{v}$ where v_{Γ_0} is the harmonic extension of $v_{|\Gamma_0}$, i.e. $v_{\Gamma_0} \in H^1_0(\Omega_1,\Gamma)$ is the weak solution of the Dirichlet boundary value problem

$$-\Delta v_{\Gamma_0}(x) = 0 \quad \text{for } x \in \Omega_1, \quad v_{\Gamma_0}(x) = v(x) \quad \text{for } x \in \Gamma_0, \quad v_{\Gamma_0}(x) = 0 \quad \text{for } x \in \Gamma,$$

i.e.

$$\int_{\Omega_1} \nabla v_{\Gamma_0}(x) \cdot \nabla z(x) dx = 0 \quad \text{for all } z \in H^1_0(\Omega_1).$$

By construction we have $\widetilde{v} \in H^1_0(\Omega_1)$. Hence we obtain, by applying Green's first formula,

$$\begin{aligned} \int_{\Omega_1} |\nabla v(x)|^2\, dx &= \int_{\Omega_1} |\nabla v_{\Gamma_0}(x)|^2\, dx + \int_{\Omega_1} |\nabla \widetilde{v}(x)|^2\, dx \\ &= \int_{\Gamma_0} \frac{\partial}{\partial n_1} v_{\Gamma_0}(x) v_{\Gamma_0}(x)\, ds_x + \int_{\Omega_1} |\nabla \widetilde{v}(x)|^2\, dx \\ &= \langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} + \int_{\Omega_1} |\nabla \widetilde{v}(x)|^2\, dx. \end{aligned} \tag{2.25}$$

i. In the two–dimensional case we introduce the splitting (2.17), i.e. for $v_{|\Gamma_0} \in H^{1/2}(\Gamma_0)$ we have

$$v_{|\Gamma_0} = v_0 + \alpha, \quad \alpha = \frac{\langle v_{|\Gamma_0}, t_{\text{eq}}\rangle_{\Gamma_0}}{\langle 1, t_{\text{eq}}\rangle_{\Gamma_0}}, \quad \langle v_0, t_{\text{eq}}\rangle_{\Gamma_0} = 0.$$

Hence, by using (2.19) we obtain from (2.25)

$$\int_{\Omega_1} |\nabla v(x)|^2\, dx \geq (1-c_S)\Big[\alpha^2 \langle S_1 \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\Big] + \|\nabla \widetilde{v}\|^2_{L_2(\Omega_1)}.$$

For $\tau \in H^{-1/2}(\Gamma_0)$ we consider the splitting

$$\tau = \tau_0 + \frac{\langle \tau, 1\rangle_{\Gamma_0}}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}}\, t_{\rm eq}, \quad \langle \tau_0, 1\rangle_{\Gamma_0} = 0, \quad V t_{\rm eq} = 1 \text{ on } \Gamma_0,$$

which implies

$$\|\tau\|_V^2 = \|\tau_0\|_V^2 + \frac{[\langle \tau, 1\rangle_{\Gamma_0}]^2}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}}.$$

Hence we conclude

$$\langle (\frac{1}{2} I - K) v_{|\Gamma_0}, \tau\rangle_{\Gamma_0} = \langle (\frac{1}{2} I - K) v_0, \tau_0\rangle_{\Gamma_0} + \alpha \langle \tau, 1\rangle_{\Gamma_0}.$$

With this we obtain in the two–dimensional case, by using (2.21) and (2.16),

$$\begin{aligned}
a(v,\tau;v,\tau) \;\; \geq \;\; & \lambda_{\min}\Big[(1-c_S)\left[\alpha^2\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\right] + \|\nabla \widetilde{v}\|^2_{L_2(\Omega_1)}\Big] \\
& + \|\tau_0\|_V^2 + \frac{[\langle \tau, 1\rangle_{\Gamma_0}]^2}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}} + \langle (\frac{1}{2} I - K) v_0, \tau_0\rangle_{\Gamma_0} + \alpha\langle 1, \tau\rangle_{\Gamma_0} \\
\geq \;\; & \lambda_{\min}\Big[(1-c_S)\left[\alpha^2\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\right] + \|\nabla \widetilde{v}\|^2_{L_2(\Omega_1)}\Big] \\
& + \|\tau_0\|_V^2 + \frac{[\langle \tau, 1\rangle_{\Gamma_0}]^2}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}} - \|(\frac{1}{2} I - K) v_0\|_{V^{-1}}\|\tau_0\|_V + \alpha\langle 1, \tau\rangle_{\Gamma_0} \\
\geq \;\; & \lambda_{\min}\Big[(1-c_S)\left[\alpha^2\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\right] + \|\nabla \widetilde{v}\|^2_{L_2(\Omega_1)}\Big] \\
& + \|\tau_0\|_V^2 + \frac{[\langle \tau, 1\rangle_{\Gamma_0}]^2}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}} - \sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_0, v_0\rangle_{\Gamma_0}}\,\|\tau_0\|_V + \alpha\,\langle 1, \tau\rangle_{\Gamma_0} \\
= \;\; & \lambda_{\min}\|\nabla \widetilde{v}\|^2_{L_2(\Omega_1)} + \frac{1}{2}\left(\frac{1}{\gamma_1}\sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_0, v_0\rangle_{\Gamma_0}} - \gamma_1\|\tau_0\|_V\right)^2 \\
& + \frac{1}{2}\left(\frac{1}{\gamma_2}\alpha + \gamma_2\langle 1, \tau\rangle_{\Gamma_0}\right)^2 + \left(1 - \frac{1}{2}\gamma_2^2\langle t_{\rm eq}, 1\rangle_{\Gamma_0}\right)\frac{[\langle 1, \tau\rangle_{\Gamma_0}]^2}{\langle t_{\rm eq}, 1\rangle_{\Gamma_0}} \\
& + \left(1 - \frac{1}{2}\gamma_1^2\right)\|\tau_0\|_V^2 + \left(\lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma_1^2}\frac{c_K}{\mu_{\min}}\right)\langle S_1 v_0, v_0\rangle_{\Gamma_0} \\
& + \left(\lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma_2^2}\frac{1}{\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0}}\right)\alpha^2\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0} \\
\geq \;\; & \frac{1}{2}\min\left\{\lambda_{\min}, 1 - \frac{1}{2}\gamma_1^2, 1 - \frac{1}{2}\gamma_2^2\langle t_{\rm eq}, 1\rangle_{\Gamma_0}\right\}\left(\|\nabla v\|^2_{L_2(\Omega_1)} + \|\tau\|_V^2\right)
\end{aligned}$$

if the conditions

$$1 - \frac{1}{2}\gamma_1^2 = \lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma_1^2}\frac{c_K}{\mu_{\min}} > 0$$

and

$$1 - \frac{1}{2}\gamma_2^2\langle t_{\rm eq}, 1\rangle_{\Gamma_0} = \lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma_2^2}\frac{1}{\langle S_1\overline{u}_0,\overline{u}_0\rangle_{\Gamma_0}} > 0$$

are satisfied. Hence we find

$$\gamma_1^2 = -[\lambda_{\min}(1-c_S)] - 1] + \sqrt{[\lambda_{\min}(1-c_S)] - 1]^2 + \frac{c_K}{\mu_{\min}}}$$

and

$$1 - \frac{1}{2}\gamma_1^2 = \frac{1}{2}\left[1 + \lambda_{\min}(1-c_S) - \sqrt{[\lambda_{\min}(1-c_S) - 1]^2 + \frac{c_K}{\mu_{\min}}}\right] > 0$$

is satisfied for

$$\lambda_{\min}(1-c_S)\mu_{\min} > \frac{1}{4}c_K\,.$$

On the other hand we obtain

$$\gamma_2^2 = \frac{1}{\langle t_{\text{eq}}, 1\rangle_{\Gamma_0}}\left[-[\lambda_{\min}(1-c_S) - 1] + \sqrt{[\lambda_{\min}(1-c_S) - 1]^2 + \frac{\langle t_{\text{eq}}, 1\rangle_{\Gamma_0}}{\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}}}\right]$$

and therefore

$$1 - \frac{1}{2}\gamma_2^2\langle t_{\text{eq}}, 1\rangle_{\Gamma_0} = \frac{1}{2}\left[1 + \lambda_{\min}(1-c_S) - \sqrt{[\lambda_{\min}(1-c_S) - 1]^2 + \frac{\langle t_{\text{eq}}, 1\rangle_{\Gamma_0}}{\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}}}\right] > 0$$

is satisfied for

$$\lambda_{\min}(1-c_S) > \frac{1}{4}\frac{\langle t_{\text{eq}}, 1\rangle_{\Gamma_0}}{\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}}.$$

With this we conclude the first ellipticity estimate for $(v,\tau) \in H_0^1(\Omega_1,\Gamma)\times H^{-1/2}(\Gamma_0)$ in the two–dimensional case when assuming condition (2.22).

ii. Next we consider, for $n = 2$, the case $\tau \in H_*^{-1/2}(\Gamma)$, i.e. $\langle 1,\tau\rangle_{\Gamma_0} = 0$. In this case, the splitting of the norm in $H^{-1/2}(\Gamma_0)$ is not required anymore. Then we have, by using (2.21) and (2.16),

$$\begin{aligned}
a(v,\tau;v,\tau) \;\geq\; & \lambda_{\min}\Big[(1-c_S)\left[\alpha^2\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\right] + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] \\
& + \|\tau\|_V^2 - \|(\tfrac{1}{2}I - K)v_0\|_{V^{-1}}\|\tau\|_V \\
\geq\; & \lambda_{\min}\Big[(1-c_S)\left[\alpha^2\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} + \langle S_1 v_0, v_0\rangle_{\Gamma_0}\right] + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] \\
& + \|\tau\|_V^2 - \sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_0, v_0\rangle_{\Gamma_0}}\,\|\tau\|_V \\
\geq\; & \lambda_{\min}\Big[(1-c_S)\alpha^2\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] + \frac{1}{2}\left(\frac{1}{\gamma}\sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_0, v_0\rangle_{\Gamma_0}} - \gamma\|\tau\|_V\right)^2 \\
& + \left(\lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma^2}\frac{c_K}{\mu_{\min}}\right)\langle S_1 v_0, v_0\rangle_{\Gamma_0} + \left(1 - \frac{1}{2}\gamma^2\right)\|\tau\|_V^2 \\
\geq\; & \lambda_{\min}\Big[(1-c_S)\alpha^2\langle S_1\overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] + \left(1 - \frac{1}{2}\gamma_*^2\right)\left(\langle S_1 v_0, v_0\rangle_{\Gamma_0} + \|\tau\|_V^2\right)
\end{aligned}$$

if

$$\lambda_{\min}(1-c_S) - \frac{1}{2}\frac{1}{\gamma_*^2}\frac{c_K}{\mu_{\min}} = 1 - \frac{1}{2}\gamma_*^2 > 0$$

is satisfied. From this we find

$$\gamma_*^2 = -[\lambda_{\min}(1-c_S)-1] + \sqrt{[\lambda_{\min}(1-c_S)-1]^2 + \frac{c_K}{\mu_{\min}}}$$

and therefore

$$1 - \frac{1}{2}\gamma_*^2 = \frac{1}{2}\left[1 + \lambda_{\min}(1-c_S) - \sqrt{[\lambda_{\min}(1-c_S)-1]^2 + \frac{c_K}{\mu_{\min}}}\right] > 0$$

for

$$\lambda_{\min}\mu_{\min}(1-c_S) > \frac{1}{4}c_K,$$

i.e. condition (2.23).

iii. In the three–dimensional case the exterior Steklov–Poincaré operator (2.15) is well defined for all $v_{|\Gamma_0} \in H^{1/2}(\Gamma)$. Hence there is no need to use the splitting (2.17). Using (2.25) we therefore have

$$\begin{aligned}
a(v,\tau;v,\tau) \;\; &\geq \;\; \lambda_{\min}\Big[\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] + \|\tau\|_V^2 - \|(\frac{1}{2}I - K)v_{|\Gamma_0}\|_{V^{-1}}\|\tau\|_V \\
&\geq \lambda_{\min}\Big[\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} + \|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)}\Big] + \|\tau\|_V^2 - \sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0}}\|\tau\|_V \\
&= \lambda_{\min}\|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)} + \frac{1}{2}\left(\frac{1}{\gamma}\sqrt{\frac{c_K}{\mu_{\min}}\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0}} - \gamma\|\tau\|_V\right)^2 \\
&\qquad + \left(\lambda_{\min} - \frac{1}{2}\frac{1}{\gamma^2}\frac{c_K}{\mu_{\min}}\right)\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} + \left(1 - \frac{1}{2}\gamma^2\right)\|\tau\|_V^2 \\
&\geq \lambda_{\min}\|\nabla\widetilde{v}\|^2_{L_2(\Omega_1)} + \left(1 - \frac{1}{2}\gamma_*^2\right)\left(\langle S_1 v_0, v_0\rangle_{\Gamma_0} + \|\tau\|_V^2\right)
\end{aligned}$$

if

$$\lambda_{\min} - \frac{1}{2}\frac{1}{\gamma_*^2}\frac{c_K}{\mu_{\min}} = 1 - \frac{1}{2}\gamma_*^2 > 0$$

is satisfied. From this we find

$$\gamma_*^2 = -[\lambda_{\min}-1] + \sqrt{[\lambda_{\min}-1]^2 + \frac{c_K}{\mu_{\min}}}$$

and therefore

$$1 - \frac{1}{2}\gamma_*^2 = \frac{1}{2}\left[1 + \lambda_{\min} - \sqrt{[\lambda_{\min}-1]^2 + \frac{c_K}{\mu_{\min}}}\right] > 0$$

for

$$\lambda_{\min}\mu_{\min} > \frac{1}{4}c_K,$$

i.e. condition (2.24). ■

Remark 2.1 *The conditions* (2.22)–(2.24) *are not only sufficient to ensure ellipticity of the bilinear form* $a(\cdot,\cdot)$ *but also necessary, see the discussion in* [11] *in the case of a free space transmission problem.*

3 Eigenvalue problem in a ring domain

In general, the constants as used in Theorem 2.2 are not known explicitly. However, in what follows we consider an example where we are able to determine all involved constants explicitly. For this we consider the boundary value problem (2.1) in the particular two–dimensional case when $\Omega = B_R(0)$ is a circular domain with the inclusion $\Omega_0 = B_r(0)$, $r < R$, and $r < 1$. Although the assumption $\operatorname{diam}\Omega_0 < 1$ is not satisfied for $r \in [\frac{1}{2}, 1)$, the condition $r < 1$ ensures the ellipticity of the single layer integral operator V in this particular case.

Recall that in polar coordinates (ϱ, φ) the general solution of the Laplace equation is given as

$$u(\varrho,\varphi) = A_0 + B_0 \log \varrho + \sum_{k=1}^{\infty} \varrho^{-k}[A_k \cos k\varphi + B_k \sin k\varphi] + \sum_{k=1}^{\infty} \varrho^{k}[\widetilde{A}_k \cos k\varphi + \widetilde{B}_k \sin k\varphi].$$

For the solution w_∞ of the exterior Dirichlet boundary value problem (2.11)–(2.12) we first find $A_0 = B_0 = 0$ and $\widetilde{A}_k = \widetilde{B}_k = 0$ due to the radiation condition (2.12). In addition we need to assume

$$\frac{1}{2\pi}\int_0^{2\pi} g(\varphi)d\varphi = 0$$

which corresponds to the condition $g \in H^{1/2}_*(\Gamma)$. From the Fourier expansion of g and using the Dirichlet boundary condition in (2.11) we further obtain

$$r^{-k}A_k = \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\cos k\varphi d\varphi =: g_{k,\cos}, \quad r^{-k}B_k = \frac{1}{\pi}\int_0^{2\pi} g(\varphi)\sin k\varphi d\varphi =: g_{k,\sin}.$$

Hence we have

$$w_\infty(\varrho,\varphi) = \sum_{k=1}^{\infty}\left(\frac{r}{\varrho}\right)^k \Big[g_{k,\cos}\cos k\varphi + g_{k,\sin}\sin k\varphi\Big] \quad \text{for } \varrho > r,$$

from which

$$S^{\text{ext}}g = \frac{\partial}{\partial n_1}w_\infty = -\frac{\partial}{\partial\varrho}w_\infty(\varrho,\varphi)_{|\varrho=r} = \sum_{k=1}^{\infty}\frac{k}{r}\Big[g_{k,\cos}\cos k\varphi + g_{k,\sin}\sin k\varphi\Big]$$

follows. With this we conclude

$$\langle S^{\text{ext}} g, g\rangle_{\Gamma_0} = \int_0^{2\pi} \Big[-\frac{\partial}{\partial \varrho} w_\infty(\varrho,\varphi)_{|\varrho=r} w_\infty(r,\varphi) \Big] \, r \, d\varphi = \pi \sum_{k=1}^\infty k \Big[g_{k,\cos}^2 + g_{k,\sin}^2 \Big]. \tag{3.1}$$

For the solution w_1 of the Dirichlet boundary value problem (2.10) we find analoguesly, still using $g_0 = 0$ due to $g \in H_*^{1/2}(\Gamma_0)$,

$$w_1(\varrho,\varphi) = \sum_{k=1}^\infty \frac{R^{2k} - \varrho^{2k}}{R^{2k} - r^{2k}} \frac{r^k}{\varrho^k} \Big[g_{k,\cos} \cos k\varphi + g_{k,\sin} \sin k\varphi \Big],$$

and

$$S_1 g = \frac{\partial}{\partial n_1} w_1 = -\frac{\partial}{\partial \varrho} w_1(\varrho,\varphi)_{|\varrho=r} = \sum_{k=1}^\infty \frac{k}{r} \frac{R^{2k} + r^{2k}}{R^{2k} - r^{2k}} \Big[g_{k,\cos} \cos k\varphi + g_{k,\sin} \sin k\varphi \Big].$$

Hence we obtain

$$\langle S_1 g, g\rangle_{\Gamma_0} = \pi \sum_{k=1}^\infty k \frac{R^{2k} + r^{2k}}{R^{2k} - r^{2k}} \Big[g_{k,\cos}^2 + g_{k,\sin}^2 \Big]. \tag{3.2}$$

From (3.1) and (3.2) we then conclude

$$\langle S^{\text{ext}} g, g\rangle_{\Gamma_0} \le \langle S_1 g, g\rangle_{\Gamma_0} \le \frac{R^2 + r^2}{R^2 - r^2} \langle S^{\text{ext}} g, g\rangle_{\Gamma_0} \quad \text{for all } g \in H_*^{1/2}(\Gamma),$$

i.e.

$$\mu_{\min} = 1.$$

It remains to determine the constant of the strengthened Cauchy–Schwarz inequality (2.18). The solution of the Dirichlet boundary value problem

$$-\Delta u(\varrho,\varphi) = 0 \quad \text{for } r < \varrho < R, \ \varphi \in [0, 2\pi), \quad u(r,\varphi) = 1, \quad u(R,\varphi) = 0,$$

is given by

$$u(\varrho,\varphi) = \frac{\log \varrho - \log R}{\log r - \log R},$$

and therefore

$$S_1 u = -\frac{\partial}{\partial \varrho} u(\varrho,\varphi)_{|\varrho=r} = \frac{1}{\log R - \log r} \frac{1}{r}.$$

Hence we find in this case $c_S = 0$, i.e. orthogonality. Recall that in the case of a circular domain the single layer integral operator has a constant eigenfunction with a related eigenvalue $-r \log r$. With this we obtain

$$\langle S_1 \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} = \frac{2\pi}{\log R - \log r}, \quad \langle t_{\text{eq}}, 1\rangle_{\Gamma_0} = \langle V^{-1} \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0} = -\frac{2\pi}{\log r}.$$

In the case of a circular domain we finally have $c_K = \frac{1}{2}$.

Hence we can state the ellipticity condition (2.22) as

$$\lambda_{\min} > \max\left\{\frac{1}{8}, \frac{1}{4}\frac{\log r - \log R}{\log r}\right\} \quad \text{for } (v,\tau) \in H^1_0(\Omega_1,\Gamma) \times H^{-1/2}(\Gamma_0), \tag{3.3}$$

while for the ellipticity condition (2.23) we obtain

$$\lambda_{\min} > \frac{1}{8} \quad \text{for } (v,\tau) \in H^1_0(\Omega_1,\Gamma) \times H^{-1/2}_*(\Gamma_0).$$

We observe that in this case condition (2.23) corresponds to the result of the free space transmission problem [11], while condition (2.22) gives an additional restriction for a sufficient large R.

4 Numerical results

In this section we provide two numerical examples to support the theoretical results of Theorem 2.2. In particular, we are interested in the two–dimensional case due to the more involved lower bound on $\lambda_{\min}$ to ensure the ellipticity of the bilinear form (2.7). We first consider the ring domain, where we were able to derive explicit conditions in Sect. 3 analytically, and a square domain with a square inclusion.

4.1 Ring domain with circular inclusion

As in Sect. 3 we consider the two–dimensional circular domain $\Omega = B_R(0)$ with the circular inclusion $\Omega_0 = B_r(0)$ of radius $r = 0.2 < R$, and $A(x) = \lambda I$ for $x \in \Omega_1 := \Omega \setminus \overline{\Omega}_0$ with $\lambda \in (0,1]$. To check the estimates of Theorem 2.2, we compute discrete approximations of the ellipticity constants c_1^A and $\widetilde{c}_1^A$ by the Rayleigh quotients

$$\begin{aligned} c_1^A &= \inf_{(0,0)\neq(v,\tau)\in H^1_0(\Omega_1,\Gamma)\times H^{-1/2}(\Gamma_0)} \frac{a(v,\tau;v,\tau)}{\|\nabla v\|^2_{L_2(\Omega_1)} + \langle V\tau,\tau\rangle_{\Gamma_0}} \\ &= \inf_{(0,0)\neq(v,\tau)\in H^1_0(\Omega_1,\Gamma)\times H^{-1/2}(\Gamma_0)} \frac{a_S(v,\tau;v,\tau)}{\|\nabla v\|^2_{L_2(\Omega_1)} + \langle V\tau,\tau\rangle_{\Gamma_0}}, \\ \widetilde{c}_1^A &= \inf_{(0,0)\neq(v,\tau)\in H^1_0(\Omega_1,\Gamma)\times H^{-1/2}_*(\Gamma_0)} \frac{a_S(v,\tau;v,\tau)}{\|\nabla v\|^2_{L_2(\Omega_1)} + \langle V\tau,\tau\rangle_{\Gamma_0}} \end{aligned}$$

with the symmetrized bilinear form

$$\begin{aligned} a_S(u,t;v,\tau) \;:=\; & \lambda\int_{\Omega_1} \nabla u(x)\cdot\nabla v(x)\,dx + \frac{1}{2}\langle t,v\rangle_{\Gamma_0} + \frac{1}{2}\langle u,\tau\rangle_{\Gamma_0} \\ & + \langle Vt,\tau\rangle_{\Gamma_0} - \frac{1}{2}\langle(\frac{1}{2}I+K)u,\tau\rangle_{\Gamma_0} - \frac{1}{2}\langle t,(\frac{1}{2}I+K)v\rangle_{\Gamma_0}. \end{aligned}$$

The approximations of the ellipticity constants c_1^A and $\widetilde{c}_1^A$ are then given by the minimal eigenvalues of the algebraic eigenvalue problem

$$\begin{pmatrix} \mu A_h & \frac{1}{4}M_h^\top - \frac{1}{2}K_h^\top \\ \frac{1}{4}M_h - \frac{1}{2}K_h & V_h + \delta\, \underline{a}\,\underline{a}^\top \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \sigma \begin{pmatrix} A_h & \\ & V_h \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix}, \tag{4.1}$$

where $\delta = 0$ in the case of c_1^A and $\delta = 1$ in the case of $\widetilde{c}_1^A$. For the discretization we used a globally quasi–uniform triangular finite element mesh in Ω_1 with piecewise linear continuous basis functions φ_i, and the related boundary element mesh on Γ_0 with piecewise constant basis functions ψ_k. The blocks are then given by

$$M_h[\ell, i] = \langle \varphi_{i|\Gamma}, \psi_\ell \rangle_{\Gamma_0}, \qquad A_h[j, i] = \int_\Omega \nabla\varphi_i(x) \cdot \nabla\varphi_j(x) dx,$$
$$K_h[\ell, i] = \langle K\varphi_{i|\Gamma}, \psi_\ell \rangle_{\Gamma_0}, \qquad V_h[\ell, k] = \langle V\psi_k, \psi_\ell \rangle_{\Gamma_0}$$

for $i, j = 1, \ldots, \widetilde{M}$, $k, \ell = 1, \ldots, N$, where $\widetilde{M}$ denotes the number of non Dirichlet nodes of the finite element mesh and N is the number of boundary elements on the interface Γ_0. The vector $\underline{a}$ with

$$a[\ell] = \langle \psi_\ell, 1 \rangle_{\Gamma_0} \qquad \text{for } \ell = 1, \ldots, N$$

is related to the stabilization to enforce $\tau \in H_*^{-1/2}(\Gamma_0)$ in the case of $\widetilde{c}_1^A$.

In Fig. 2, we plot the minimal eigenvalues of the eigenvalue problem (4.1) for a sequence of coefficients $\lambda_i = \frac{i}{100}$, $i = 1, \ldots, 100$. The minimal eigenvalues were computed by some appropriate eigenvalue solver, i.e. the LAPACK routine for generalized symmetric eigenvalue problems with a positive definite matrix on the right–hand side.

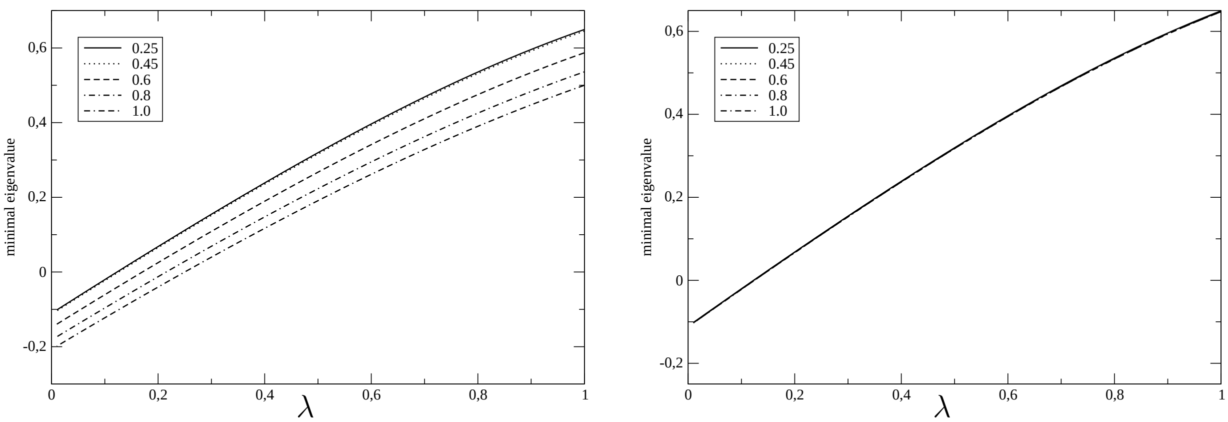

Figure 2: The minimal eigenvalues of (4.1) as a function of the coefficient λ for the ring domain and several choices of the radius $R \in \{0.25, 0.45, 0.6, 0.8, 1.0\}$ of the exterior boundary in the case without (left) and with stabilization (right).

While in the stabilized case (Fig. 2, right) the curves of the minimal eigenvalues as approximations of the ellipticity constants $\widetilde{c}_1^A$ are on top of each other for all considered exterior radii R, in the case without stabilization (Fig. 2, left) the minimal eigenvalues as approximations of the ellipticity constants c_1^A depend on the radius R. For $R \in [0.25, 0.44]$ the curves are on top of each other in both cases, but for larger values of λ we are in the case of

$$\frac{\langle t_{\text{eq}}, 1\rangle_{\Gamma_0}}{\langle S_1 \overline{u}_0, \overline{u}_0\rangle_{\Gamma_0}} = \frac{1}{4}\frac{\log r - \log R}{\log r}, \quad \lambda_{\min} > \max\left\{\frac{1}{8}, \frac{1}{4}\frac{\log r - \log R}{\log r}\right\}$$

as described in (2.22) and (3.3). This effect can be seen even better in Fig. 3, where we determined the minimal coefficient λ, i.e. $\lambda_{\min}$ such that the bilinear form $a(\cdot,\cdot;\cdot,\cdot)$ is elliptic for $R = 0.2 + 0.05j$ and $j = 1, \dots, 16$. Having in mind that we are limited to about 10000 degree of freedoms due to the use of the LAPACK routines, we observe a good agreement of the approximations of $\lambda_{\min}$ and the theoretical values 0.125 and $(\ln r - \ln R)(4 \ln r)^{-1}$ of (3.3). In particular, we observe that the estimates of Sect. 3 and Theorem 2.2 are sharp.

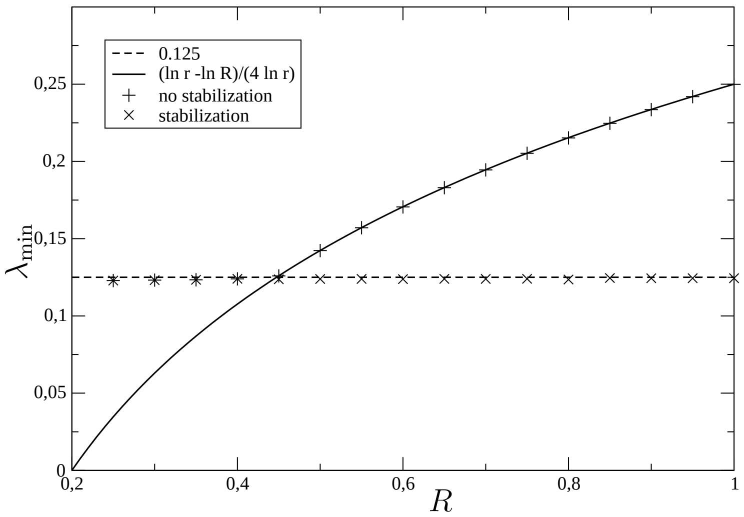

Figure 3: Approximations of $\lambda_{\min}$ for the ring domain where $r = 0.2$ and $R = 0.2 + 0.05j$ with $j = 1, \dots, 16$.

4.2 Square domain with square inclusion

As a second example we consider a square domain $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ with a square inclusion $\Omega_0 = (-a, a) \times (-a, a)$ for several values of $a \in [0.025, 0.4]$. In this case we are not able to find explicit values for the involved constants c_K, $\mu_{\min}$, and c_S. As before, we compute approximations of the ellipticity constants c_1^A and $\widetilde{c}_1^A$ as the minimal eigenvalues of the problem (4.1).

In Fig. 4, we plot the minimal eigenvalues of the eigenvalue problem (4.1) for a sequence of coefficients $\lambda_i = \frac{i}{100}$, $i = 1, \dots, 100$ for several values of the parameter a, where $2a$ is the side length of the inclusion Ω_0.

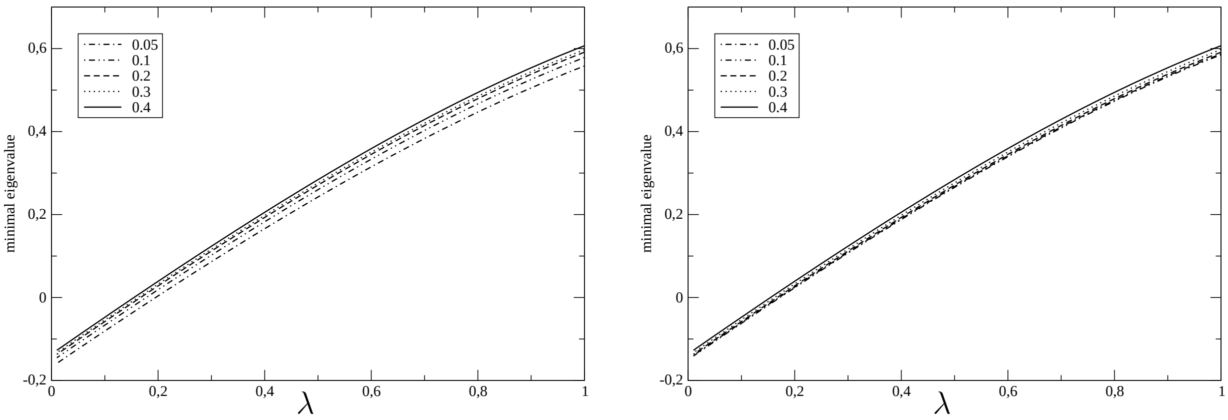

Figure 4: The minimal eigenvalues of (4.1) as a function of the coefficient λ for the square domain and several choices of the parameter $a \in \{0.05, 0.1, 0.2, 0.3, 0.4\}$ of the inclusion in the case without (left) and with stabilization (right).

As we do not extend the exterior domain as for the ring domain but shrink the inclusion, we observe the marching curves for decreasing values of a in the case of the non–stabilized setting. For the stabilized version the curves are not on top of each other as for the ring domain but there seems to be a limit for decreasing values of a. This can been seen even better from Fig. 5 where we determined the minimal coefficient λ, i.e. $\lambda_{\min}$ such that the bilinear form $a(\cdot, \cdot)$ is elliptic for $a \in \{0.025, 0.05, 0.075, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4\}$. Since we are looking at a small range of values $\lambda_{\min}$ there are still some small effects visible due to the varying meshes for different values of a. Nevertheless we can observe the diverse behaviour of the stabilized and the non–stabilized setting for small values of the parameter a.

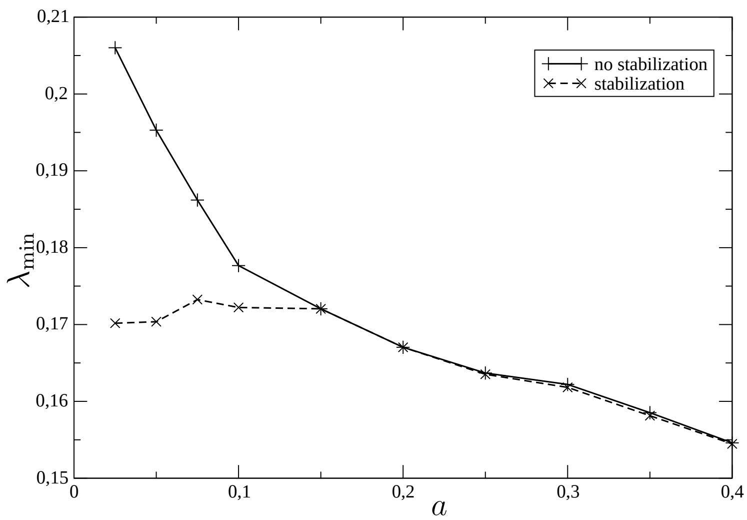

Figure 5: Approximations of $\lambda_{\min}$ for the square with square inclusion of side length $2a$.

5 Extensions

The approach as outlined in the previous sections can be applied also to other coupled finite and boundary element formulations to solve related boundary value problems. In what follows we consider three particular applications, namely the solution of a boundary value problem with soft inclusions, the solution of exterior boundary value problems, and the use of boundary element macro–elements within a finite element formulation.

5.1 Boundary value problems with soft inclusions

As in (2.1) we consider the Dirichlet boundary value problem

$$-\mathrm{div}[A(x)\nabla u(x)] = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma, \tag{5.1}$$

but now we assume $A(x) = I$ and $f(x) = 0$ in the surrounding domain $\Omega_1 = \Omega\backslash\overline{\Omega}_0$, while we consider the potential equation in the inclusion $\Omega_0 \subset \Omega$, see Fig. 6.

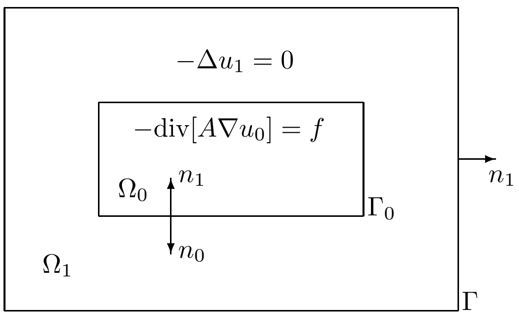

Figure 6: Boundary value problem with soft inclusion.

Instead of the global problem (5.1) we consider the local partial differential equations

$$-\mathrm{div}[A(x)\nabla u_0(x)] = f(x) \quad \text{for } x \in \Omega_0, \quad -\Delta u_1(x) = 0 \quad \text{for } x \in \Omega_1,$$

with the boundary and transmission conditions

$$u_1(x) = 0 \quad \text{for } x \in \Gamma, \quad u_1(x) = u_0(x), \quad \frac{\partial}{\partial n_1} u_1(x) + n_0 \cdot A(x)\nabla u_0(x) = 0 \quad \text{for } x \in \Gamma_0.$$

As in (2.3)–(2.4) we first consider the variational formulation to find $u_0 \in H^1(\Omega_0)$ satisfying

$$\int_{\Omega_0} [A(x)\nabla u_0(x)] \cdot \nabla v(x)\, dx + \int_{\Gamma_0} t_{1|\Gamma_0}(x) v(x)\, ds_x = \int_{\Omega_0} f(x) v(x)\, dx \tag{5.2}$$

for all $v \in H^1(\Omega_0)$, and the boundary integral equation which is related to a Dirichlet boundary value problem in Ω_1,

$$(V_1 t_1)(x) = (\frac{1}{2}I + K_1)u_1(x) \quad \text{for } x \in \Gamma_1 = \partial\Omega_1, \quad t_1(x) = \frac{\partial}{\partial n_1}u_1(x) \quad \text{for } x \in \Gamma_1, \quad (5.3)$$

where the boundary integral operators are defined as

$$(V_1 t_1)(x) = \int_{\Gamma_1} U^*(x,y)t_1(y)ds_y, \quad (K_1 u_1)(x) = \int_{\Gamma_1} \frac{\partial}{\partial n_1}U^*(x,y)u_1(y)ds_y \quad \text{for } x \in \Gamma_1.$$

Let $t_{\text{eq}} \in H^{-1/2}(\Gamma_1)$ be the unique solution of the boundary integral equation

$$(V_1 t_{\text{eq}})(x) = \begin{cases} 1 & \text{for } x \in \Gamma_0, \\ 0 & \text{for } x \in \Gamma. \end{cases}$$

By chosing $(v, \tau) = (1, t_{\text{eq}})$ we then obtain from (5.2) and (5.3)

$$\begin{aligned} \langle f, 1\rangle_{\Omega_0} &= \langle t_{1|\Gamma_0}, 1\rangle_{\Gamma_0} = \langle t_1, V_1 t_{\text{eq}}\rangle_{\Gamma_1} = \langle V_1 t_1, t_{\text{eq}}\rangle_{\Gamma_1} \\ &= \langle (\frac{1}{2}I + K_1)u_1, t_{\text{eq}}\rangle_{\Gamma_1} = \langle u_1, V_1^{-1}(\frac{1}{2}I + K_1)V_1 t_{\text{eq}}\rangle_{\Gamma_1}. \end{aligned}$$

Recall that, for $x \in \Gamma_1$,

$$\begin{aligned} [(\frac{1}{2}I + K_1)V_1 t_{\text{eq}}](x) &= \frac{1}{2}(V_1 t_{\text{eq}})(x) + \int_{\Gamma_1} \frac{\partial}{\partial n_1}U^*(x,y)(V_1 t_{\text{eq}})(y)ds_y \\ &= \frac{1}{2}(V_1 t_{\text{eq}})(x) + \int_{\Gamma_0} \frac{\partial}{\partial n_1}U^*(x,y)ds_y. \end{aligned}$$

In particular for $x \in \Gamma_0$ we therefore have, by using $n_1 = -n_0$ and the jump relation of the double layer potential,

$$(\frac{1}{2}I + K_1)(V_1 t_{\text{eq}})(x) = \frac{1}{2} - \int_{\Gamma_0} \frac{\partial}{\partial n_0}U^*(x,y)ds_y = 1,$$

while for $x \in \Gamma$

$$(\frac{1}{2}I + K_1)(V_1 t_{\text{eq}})(x) = -\int_{\Gamma_0} \frac{\partial}{\partial n_0}U^*(x,y)ds_y = 0$$

follows. From this we further conclude

$$V_1^{-1}(\frac{1}{2}I + K_1)V_1 t_{\text{eq}} = t_{\text{eq}},$$

and due to $u_1 = u_0$ on Γ_0 and $u_1 = 0$ on Γ we finally find the scaling condition

$$\langle f, 1\rangle_{\Omega_0} = \langle u_1, t_{\text{eq}}\rangle_{\Gamma_1} = \langle u_0, t_{\text{eq}|\Gamma_0}\rangle_{\Gamma_0}.$$

Since the domain bilinear form in (5.2) defines only a semi–norm in $H^1(\Omega_0)$, similar as in [15] we now introduce the splitting

$$u_0(x) = \overline{u}_0 + \widetilde{u}_0(x) \quad \text{for } x \in \Omega_0, \quad \langle \widetilde{u}_0, t_{\mathrm{eq}|\Gamma_0}\rangle_{\Gamma_0} = 0, \quad \overline{u}_0 = \frac{\langle f, 1\rangle_{\Omega_0}}{\langle 1, t_{\mathrm{eq}|\Gamma_0}\rangle_{\Gamma_0}},$$

and we define the factor space $H^1_*(\Omega_0) := \{v \in H^1(\Omega_0) : \langle v_{|\Gamma_0}, t_{\mathrm{eq}|\Gamma_0}\rangle_{\Gamma_0} = 0\}$ where $\|\nabla v\|_{L_2(\Omega_0)}$ implies an equivalent norm. By convention we set $v_{|\Gamma} \equiv 0$ for all $v \in H^1_*(\Omega_1)$. Then we end up with a modified variational problem to find $(\widetilde{u}_0, t_1) \in H^1_*(\Omega_0) \times H^{-1/2}(\Gamma_0)$ such that

$$\int_{\Omega_0} [A(x)\nabla\widetilde{u}_0(x)] \cdot \nabla v(x)\, dx + \int_{\Gamma_0} t_{1|\Gamma_0}(x) v(x)\, ds_x = \int_{\Omega_0} f(x)v(x)\, dx, \tag{5.4}$$

$$\langle V_1 t_1, \tau\rangle_{\Gamma_1} - \langle (\frac{1}{2}I + K_1)\widetilde{u}_{0|\Gamma_0}, \tau\rangle_{\Gamma_1} = \overline{u}_0 \, \langle 1, \tau\rangle_{\Gamma_0} \tag{5.5}$$

is satisfied for all $(v, \tau) \in H^1_*(\Omega_0) \times H^{-1/2}(\Gamma_1)$. To establish ellipticity of the related bilinear form we consider

$$\begin{aligned}
a(v,\tau; v,\tau) &= \int_{\Omega_0} [A(x)\nabla v(x)] \cdot \nabla v(x)\, dx + \langle V_1\tau, \tau\rangle_{\Gamma_1} + \langle (\frac{1}{2}I - K_1)v_{|\Gamma_0}, \tau\rangle_{\Gamma_1} \\
&\geq \lambda_{\min} \|\nabla v\|^2_{L_2(\Omega_0)} + \|\tau\|^2_{V_1} - \|(\frac{1}{2}I - K_1)v_{|\Gamma_0}\|_{V_1^{-1}} \|\tau\|_{V_1} \\
&= \lambda_{\min} \|\nabla \widetilde{v}\|^2_{L_2(\Omega_0)} + \lambda_{\min}\langle S_0 v_{\Gamma_0}, v_{\Gamma_0}\rangle_{\Gamma_0} + \|\tau\|^2_{V_1} - \|(\frac{1}{2}I - K_1)v_{|\Gamma_0}\|_{V_1^{-1}} \|\tau\|_{V_1} \\
&\geq \lambda_{\min} \|\nabla \widetilde{v}\|^2_{L_2(\Omega_0)} + \lambda_{\min}\langle S_0 v_{\Gamma_0}, v_{\Gamma_0}\rangle_{\Gamma_0} - \frac{1}{2}\gamma\|(\frac{1}{2}I - K_1)v_{|\Gamma_0}\|^2_{V_1^{-1}} + \left(1 - \frac{1}{2}\frac{1}{\gamma}\right) \|\tau\|^2_{V_1}
\end{aligned}$$

for some positive constant $\gamma \in \mathbb{R}$, where we have used a similar splitting as in (2.25), and where $S_0 = V^{-1}(\frac{1}{2}I + K)$ is the Steklov–Poincaré operator which is related to the Dirichlet boundary value problem in Ω_0, see the boundary integral equation (2.4). Similar as in Lemma 2.1 we have

$$\|(\frac{1}{2}I - K_1)v_{|\Gamma_0}\|^2_{V_1^{-1}} \leq c_K \, \langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_1} \quad \text{for all } v \in H^1_*(\Omega_0)$$

with the Steklov–Poincaré operator $S_1 := D_1 + (\frac{1}{2}I - K_1')V_1^{-1}(\frac{1}{2}I - K_1)$. In analogy to (2.16) we assume the equivalence inequality

$$\mu_{\min}\langle S_1 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} \leq \langle S_0 v_{|\Gamma_0}, v_{|\Gamma_0}\rangle_{\Gamma_0} \quad \text{for all } v \in H^1_*(\Omega_0)$$

for some $\mu_{\min} > 0$. With this we conclude the ellipticity estimate

$$\begin{aligned}
a(v,\tau; v,\tau) &\geq \lambda_{\min} \|\nabla\widetilde{v}\|^2_{L_2(\Omega_0)} + \left(\lambda_{\min} - \frac{1}{2}\frac{c_K}{\mu_{\min}}\gamma\right) \langle S_0 v_{\Gamma_0}, v_{\Gamma_0}\rangle_{\Gamma_0} + \left(1 - \frac{1}{2}\frac{1}{\gamma}\right) \|\tau\|^2_{V_1} \\
&\geq \left(1 - \frac{1}{2}\frac{1}{\gamma^*}\right) \left[\|\nabla\widetilde{v}\|^2_{L_2(\Omega_0)} + \langle S_0 v_{\Gamma_0}, v_{\Gamma_0}\rangle_{\Gamma_0} + \|\tau\|^2_{V_1}\right]
\end{aligned}$$

if

$$\lambda_{\min} - \frac{1}{2}\frac{c_K}{\mu_{\min}}\gamma^* = 1 - \frac{1}{2}\frac{1}{\gamma^*} > 0$$

is satisfied. From this we find

$$\gamma^* = \frac{\mu_{\min}}{c_K}(\lambda_{\min} - 1) + \sqrt{\frac{\mu_{\min}^2}{c_K^2}(\lambda_{\min} - 1)^2 + \frac{\mu_{\min}}{c_K}},$$

and

$$\lambda_{\min} - \frac{1}{2}\frac{c_K}{\mu_{\min}}\gamma^* = \frac{1}{2}\left[1 + \lambda_{\min} - \sqrt{(\lambda_{\min} - 1)^2 + \frac{c_K}{\mu_{\min}}}\right] > 0$$

is satisfied for

$$\lambda_{\min} > \frac{1}{4}\frac{c_K}{\mu_{\min}}.$$

This condition is rather similar as condition (2.23) in the two–dimensional case of the boundary value problem (2.1) with a hard inclusion, but now the orthogonal splitting is included in the definition of $H^1_*(\Omega_1)$. While the determination of the constant $\mu_{\min}$ only differs in the use of the exterior and interior Steklov–Poincaré operators, and in fact, it is the same in the case of the ring domain, the contraction constant c_K now corresponds to Ω_1. In particular for the ring domain as considered in Sect. 3 we find $\mu_{\min} = 1$ and $c_K = \frac{r+R}{2R}$.

5.2 Exterior boundary value problems

As a second extension we consider the model problem of an exterior boundary boundary value problem, see Fig. 7,

$$-\text{div}[A(x)\nabla u_i(x)] = f(x) \quad \text{for } x \in \Omega_i, \quad -\Delta u_e(x) = 0 \quad \text{for } x \in \Omega_e, \tag{5.6}$$

with the boundary and transmission conditions

$$u_i(x) = 0 \text{ for } x \in \Gamma,\ u_i(x) = u_e(x),\ n_i \cdot A(x)\nabla u_i(x) + \frac{\partial}{\partial n_e} u_e(x) = 0 \text{ for } x \in \Gamma_I, \tag{5.7}$$

and the radiation condition

$$u_e(x) = \mathcal{O}\Big(\frac{1}{|x|}\Big) \quad \text{as } |x| \to \infty.$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$, where its exterior is decomposed into two non–overlapping subdomains Ω_i and Ω_e with interface Γ_I, i.e.

$$\Omega^c := \mathbb{R}^n \backslash \overline{\Omega} = \Omega_e \cup \Gamma_I \cup \Omega_i.$$

As before we assume $f \in L_2(\Omega_i)$, and that $A(x)$ is uniform positive definite in Ω_i.

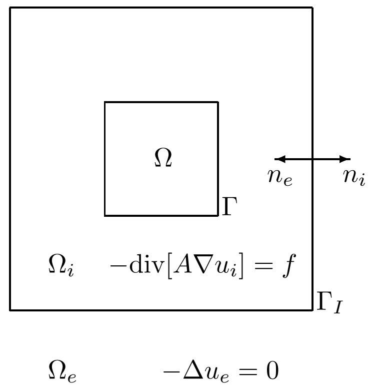

Figure 7: Exterior boundary value problem.

The variational problem of the exterior Dirichlet boundary value problem (5.6)–(5.7) is to find $(u_i, t_e) \in H_0^1(\Omega_i, \Gamma) \times H_*^{-1/2}(\Gamma_I)$ such that

$$\int_{\Omega_i} [A(x)\nabla u_i(x)] \cdot \nabla v(x)]\, dx - \int_{\Gamma_i} t_e(x)v(x)\, ds_x = \int_{\Omega_i} f(x)v(x)\, dx$$

is satisfied for all $v \in H_0^1(\Omega_i, \Gamma) := \left\{ v \in H^1(\Omega_i) : v_{|\Gamma} = 0 \right\}$ and

$$\langle V t_e, \tau \rangle_{\Gamma_I} = \langle (-\frac{1}{2} I + K) u_{i|\Gamma_I}, \tau \rangle_{\Gamma_I} \quad \text{for all } \tau \in H_*^{-1/2}(\Gamma_I).$$

Note that the requirement $t_e = \frac{\partial}{\partial n_i} u_e \in H_*^{-1/2}(\Gamma_I)$ ensures the correct radiation condition for the solution u_e of the exterior problem.

The related bilinear form is now given by

$$a(u, t; v, \tau) := \langle A\nabla u, \nabla v \rangle_{L_2(\Omega_i)} - \langle t, v \rangle_{\Gamma_I} + \langle Vt, \tau \rangle_{\Gamma_I} + \langle (\frac{1}{2} I - K) u, \tau \rangle_{\Gamma_I},$$

and we can proceed as in the proof of Theorem 2.2 to obtain, for $(v, \tau) \in H_0^1(\Omega_i, \Gamma) \times H_*^{-1/2}(\Gamma)$,

$$\begin{aligned} a(v, \tau; v, \tau) &= \int_{\Omega_i} A(x)\nabla v(x) \cdot \nabla v(x)\, dx + \langle V\tau, \tau \rangle_{\Gamma_I} - \langle (\frac{1}{2} I + K) v_{|\Gamma}, \tau \rangle_{\Gamma} \\ &\geq \lambda_{\min} \Big[\|\nabla \widetilde{v}\|^2_{L_2(\Omega_i)} + \langle S_i v_\Gamma, v_\Gamma \rangle_{\Gamma_I} \Big] + \|\tau\|_V^2 - \|(\frac{1}{2} I + K) v_{|\Gamma}\|_{V^{-1}} \|\tau\|_V, \end{aligned}$$

where S_i is the Steklov–Poincaré operator which is related to the Dirichlet boundary value problem

$$-\Delta w_i(x) = 0 \quad \text{for } x \in \Omega_i, \quad w_i(x) = 0 \quad \text{for } x \in \Gamma, \quad w_i(x) = v_\Gamma(x) \quad \text{for } x \in \Gamma_I,$$

i.e.

$$S_i v_\Gamma = \frac{\partial}{\partial n_i} w_i(x) \quad \text{for } x \in \Gamma_I.$$

Instead of (2.16) we now consider the spectral equivalence inequality

$$\mu_{\min} \langle S^{\text{int}} v, v \rangle_{\Gamma_I} \leq \langle S_i v, v \rangle_{\Gamma_I} \quad \text{for all } v \in H^{1/2}(\Gamma_I),$$

where S^{int} is the Steklov–Poincaré operator which is related to the Dirichlet boundary value problem

$$-\Delta w(x) = 0 \quad \text{for } x \in \Omega_i \cup \Gamma \cup \Omega, \quad w(x) = v_\Gamma(x) \quad \text{for } x \in \Gamma_I.$$

Now, by using

$$\frac{1}{c_K} \| (\frac{1}{2} I + K) v_{|\Gamma} \|^2_{V^{-1}} \leq \langle S^{\text{int}} v_{|\Gamma}, v_{|\Gamma} \rangle_{\Gamma_I} \quad \text{for all } v \in H^{1/2}(\Gamma_I),$$

we further obtain

$$a(v, \tau; v, \tau) \;\; \geq \;\; \lambda_{\min} \Big[\| \nabla \widetilde{v} \|^2_{L_2(\Omega_i)} + \langle S_i v_\Gamma, v_\Gamma \rangle_{\Gamma_I} \Big] + \| \tau \|^2_V - \sqrt{\frac{c_K}{\mu_{\min}} \langle S^{\text{int}} v_\Gamma, v_\Gamma \rangle_{\Gamma_I}} \| \tau \|_V,$$

and as in the case $n = 3$ of Theorem 2.2 we conclude ellipticity for

$$\lambda_{\min} \mu_{\min} > \frac{1}{4} c_K.$$

5.3 Macro–elements

As in [7] we finally consider the case that the boundary element subdomain is considered as a macro–element within a finite element discretization, e.g., to model singularities more accurately. In this situation, the boundaries of both subdomains have a piece of the boundary $\Gamma = \partial\Omega$ where Dirichlet boundary conditions are predescribed, see Fig. 8. In particular, we consider the Dirichlet boundary value problem

$$-\text{div}[A(x) \nabla u(x)] = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma := \partial\Omega, \tag{5.8}$$

where we assume $A(x) = I$, $f(x) = 0$ in Ω_0, $\Gamma_1 \cap \Gamma \neq \emptyset$, and $\Gamma_0 \cap \Gamma \neq \emptyset$.

Instead of the global problem (5.8) we consider the local subproblems

$$-\text{div}[A(x) \nabla u_1(x)] = f(x) \quad \text{for } x \in \Omega_1, \quad -\Delta u_0(x) = 0 \quad \text{for } x \in \Omega_0,$$

with the boundary and transmission conditions

$$u_i(x) = 0 \text{ for } x \in \Gamma, \quad u_1(x) = u_0(x), \quad \frac{\partial}{\partial n_0} u_0(x) + n_1 \cdot A(x) \nabla u_1(x) = 0 \text{ for } x \in \Gamma_C,$$

where $\Gamma_C := (\Gamma_1 \cap \Gamma_2) \setminus \Gamma$ denotes the interface of the two subdomains.

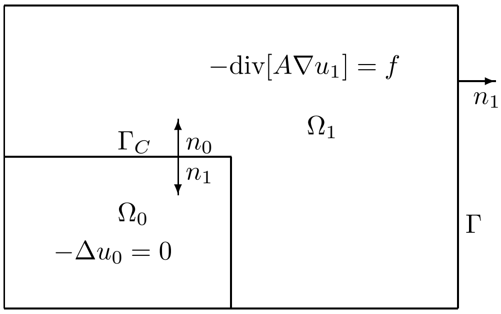

Figure 8: Boundary value problem with a macro–element.

As before, we consider the variational formulation for the diffusion equation in Ω_1, and the weakly singular boundary integral (2.4) for the Laplace equation in Ω_0. Thus we end up with the variational problem to find $(u_1, t_0) \in H^1_0(\Gamma_1, \Gamma_1 \cap \Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\int_{\Omega_1} [A(x)\nabla u_1(x)] \cdot \nabla v(x)\, dx + \int_{\Gamma_C} t_{0|\Gamma_C}(x) v(x)\, ds_x = \int_{\Omega_1} f(x) v(x)\, dx \tag{5.9}$$

$$\langle V t_0, \tau \rangle_{\Gamma_0} - \langle (\frac{1}{2} I + K) u_{1|\Gamma_C}, \tau \rangle_{\Gamma_0} = 0 \tag{5.10}$$

is satisfied for all $(v, \tau) \in H^1_0(\Gamma_1, \Gamma_1 \cap \Gamma) \times H^{-1/2}(\Gamma)$, where $t_0 := \frac{\partial}{\partial n_0} u_0$. Note that we consider the extension of $u_{1|\Gamma_C} \in \widetilde{H}^{1/2}(\Gamma_C)$ to Γ_0 by zero whenever needed. Due to the fact that both subproblems involve a Dirichlet boundary condition, we do not need to impose any additional constraints. The associated bilinear form reads as

$$a(u, t; v, \tau) := \langle A\nabla u, \nabla v \rangle_{L_2(\Omega_1)} + \langle t, v \rangle_{\Gamma_C} + \langle V t, \tau \rangle_{\Gamma_0} - \langle (\frac{1}{2} I + K) u_{|\Gamma_C}, \tau \rangle_{\Gamma_0}, \tag{5.11}$$

and to ensure ellipticity we need to consider the spectral equivalence inequality

$$\mu_{\min} \langle S_0 v, v \rangle_{\Gamma_C} \le \langle S_1 v, v \rangle_{\Gamma_C} \quad \text{for all } v \in \widetilde{H}^{1/2}(\Gamma_C), \tag{5.12}$$

where the Steklov–Poincaré operators S_i are defined with respect to the solution of related Dirichlet boundary value problems in the subdomains Ω_i. As in the proof of the three–dimensional case in Theorem 2.2 we can now prove the ellipticity estimate

$$a(v, \tau; v, \tau) \ge c_1^A \Big[\|\nabla v\|^2_{L_2(\Omega_1)} + \|\tau\|^2_V \Big] \quad \text{for all } (v, \tau) \in H^1_0(\Omega_1, \Gamma_1 \cap \Gamma) \times H^{-1/2}(\Gamma_0)$$

with some positive constant $c_1^A > 0$ if the condition

$$\lambda_{\min} > \frac{1}{4} \frac{c_K}{\mu_{\min}}$$

is satisfied.

6 Conclusions

In this paper we have presented a stability analysis for the Johnson–Nédélec coupling of finite and boundary element methods in the case of boundary value problems. While the sufficient and necessary conditions to ensure ellipticity are comparable to the conditions obtained in the case of transmission boundary value problems, they involve minimal eigenvalues of related Steklov–Poincaré operator eigenvalue problems in addition. Although these eigenvalues can be reformulated as eigenvalue problems for underlying partial differential equations, it seems that not so much is known on the analysis and numerics of these eigenvalue problems, and hence, this may require further considerations. Moreover, these results have to be extended to systems of partial differential equations, and to nonlinear equations.

As a general result we can state, that the one–equation or Johnson–Nédélec coupling of finite and boundary element methods is stable for almost arbitrary choices of basis functions, as long as some analytical conditions on the underlying partial differential equations, and on the shape of the interface and subdomain boundaries are satisfied.

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