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Abstract

In this paper boundary integral formulations for a time-harmonic acoustic scattering-resonance problem are analyzed. The eigenvalues of eigenvalue problems resulting from boundary integral formulations for scattering-resonance problems split in general into two parts. One part consists of scattering-resonances and the other one corresponds to eigenvalues of some Laplacian eigenvalue problem for the interior of the scatterer. The proposed combined boundary integral formulations enable a better separation of the unwanted spectrum from the scattering-resonances which allows in practical computations a reliable and simple identification of the scattering-resonances in particular for non-convex domains. The convergence of conforming Galerkin boundary element approximations for the combined boundary integral formulations of the resonance problem is shown in canonical trace spaces. Numerical experiments confirm the theoretical results.

1 Introduction

We consider a time-harmonic acoustic scattering-resonance problem with Neumann boundary conditions in the exterior $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega^-}$ of a bounded Lipschitz domain $\Omega^- \subset \mathbb{R}^3$. For the exterior domain Ω^+ we assume that it is simply connected. The resonance problem is formulated in terms of the Helmholtz equation in the following way: Find resonances $k \in \mathbb{C}$ and corresponding resonance functions $u \in H^1_{\rm loc}(\Omega^+) \setminus \{0\}$ such that

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega^+, \tag{1a}$$

$$\frac{\partial}{\partial n} u = 0 \quad \text{on } \Gamma := \partial\Omega^-, \tag{1b}$$

$$u \text{ satisfies a radiation condition.} \tag{1c}$$

As radiation condition in (1c) we impose that u has outside of any ball $B_{r_0} := \{x : |x| < r_0\}$ which contains Ω^- an expansion in terms of the spherical Hankel functions of the first kind $h_n^{(1)}$ of the form

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} h_n^{(1)}(kr) Y_n^m \left(\frac{x}{|x|} \right) \quad \text{for } r = |x| > r_0, \tag{2}$$

where Y_n^m are the spherical harmonics, see [18]. This radiation condition describes outgoing time-harmonic functions of the form $P(x,t) = \mathrm{Re}\left(e^{-ikct} u(x)\right)$, where c is the speed of sound in Ω^+. For real and positive k the radiation condition (2) coincides with the classical Sommerfeld radiation condition [12, Remark 2.1]. Equivalent characterizations of the radiation condition via surface potentials [20, Chapt. VIII] and via a representation formula for the solution of (1a) [16, 21] will be given in the appendix of this paper. It is well known that the resonances of (1) have a negative imaginary part [17, Chapt. 9] which describes the damping of the resonance function $P(x,t)$ in time.

In this paper we analyze different boundary integral formulations for the resonance problem (1) where our focus is on appropriate formulations for Galerkin boundary element approximations and on some practical aspects of the computation of resonances. Boundary integral formulations of the resonance problem (1) yield nonlinear eigenvalue problems with respect to the eigenvalue k due to the nonlinear dependence of k in the fundamental solution of the Helmholtz equation. The considered boundary integral formulations in this paper are eigenvalue problem formulations for holomorphic Fredholm operator-valued functions where the standard convergence results for a Galerkin approximation can be applied [10, 11, 25, 28]. The eigenvalues of the eigenvalue problems resulting from boundary integral formulations for resonance problems split in general into two parts. One part coincides with the resonances and the other part corresponds to the eigenvalues of some eigenvalue problem for $-\Delta$ for the interior domain of the scatterer. We use so-called combined boundary integral equations as suggested in [4, 5, 8, 24, 31] for boundary value problems also for resonance problems since they separate the unwanted spectrum from the resonances such that in practical computations both spectra can be easily identified. This is, in particular, an advantage for domains with open cavities where resonances are very close to the real axis. Moreover, a clear separation of the unwanted spectrum from the resonances is useful when the contour integral method [2, 3] is applied to the discretized eigenvalue problem, since then the unwanted spectrum is even not computed. The contour integral method is a reliable method for finding all eigenvalues and corresponding eigenvectors which lie inside of a given contour in the complex plane by reducing the nonlinear eigenvalue problem to an equivalent linear one which has the same eigenvalues as the nonlinear eigenvalue problem inside the contour.

Combined boundary integral equations for scattering-resonance problems are also analyzed in [27, Sect. 9.7]. However, the analysis there is restricted to domains with C^2-boundaries. Moreover, the approximation of resonance pairs and their practical computations were not considered there. A numerical analysis of standard boundary integral formulations for scattering-resonance problems has been presented in [19, 25]. Boundary

element methods are also used for several other kinds of eigenvalue problems for partial differential operators. For some recent works we refer to [1, 13, 30, 33].

Alternative numerical approaches for scattering-resonance problems are the perfectly matched layer (PML) method [12] and the Hardy space infinite element method [9]. Both approaches are based on the finite element method. They differ in the truncation of the infinite domain and the incorporation of the radiation condition for the resonance function. In both approaches spurious resonances occur and a main computational challenge is to fit the parameter of the discretization such that the spurious resonances can be distinguished by the actual ones.

The rest of the paper is organized as follows: In Section 2 a survey on standard boundary integral formulations for the resonance problem (1) is given. In addition, a brief introduction into the concept of eigenvalue problems for holomorphic Fredholm operator-value functions is provided which is the basis of the spectral analysis of the eigenvalue problems resulting from boundary integral formulations of the resonance problem. In Section 3 combined boundary integral formulations for the resonance problem are analyzed. The convergence of a conforming Galerkin approximation of the proposed boundary integral formulations is addressed in Section 4. Numerical experiments are presented in Section 5 which show that the combined boundary integral equations separates the resonances and the unwanted spectrum in such a way that resonances can be easily identified. In the appendix different equivalent formulations of the radiation condition (2) for solutions of the Helmholtz equation are analyzed.

2 Boundary integral characterizations of resonance pairs

In this section we first introduce the notations and basic properties of boundary integral operators for the Helmholtz equation. Our main references are the textbooks [17, 23]. For $s \geq 0$ we define

$$H^s_{\mathrm{loc}}(\Omega^+) := \{u \in \mathcal{D}^*(\Omega^+) : u_{|\Omega^+_\rho} \in H^s(\Omega^+_\rho) \text{ for each finite } \rho > 0 \text{ such that } \overline{\Omega^-} \subseteq B_\rho\},$$

where $\mathcal{D}^*(\Omega^+)$ is the space of distributions, B_ρ is the ball with radius ρ and center 0, and $\Omega^+_\rho := B_\rho \cap \Omega^+$. By γ_0^- and γ_0^+ we denote the standard one-sided Dirichlet trace operators

$$\gamma_0^- : H^1(\Omega^-) \to H^{1/2}(\Gamma), \qquad \gamma_0^+ : H^1_{\mathrm{loc}}(\Omega^+) \to H^{1/2}(\Gamma),$$

which are the extensions of the pointwise traces of smooth functions. The one-sided Neumann trace operators γ_1^- and γ_1^+ are defined for smooth functions $u^\pm \in \mathbb{C}^\infty(\overline{\Omega^\pm})$ by

$$\gamma_1^\pm u^\pm(x) = \nabla u^\pm(x) \cdot n(x), \quad x \in \Gamma,$$

where n is the unit normal vector pointing from Ω^- into Ω^+. γ_1^- and γ_1^+ have unique extensions to

$$\gamma_1^- : H^1(\Delta, \Omega) \to H^{-1/2}(\Gamma), \qquad \gamma_1^+ : H^1_{\mathrm{loc}}(\Delta, \Omega) \to H^{-1/2}(\Gamma),$$

where $H^1(\Delta,\Omega) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$ and $H^1_{\mathrm{loc}}(\Delta,\Omega) := \{u \in H^1_{\mathrm{loc}}(\Omega) : \Delta u \in L^2_{\mathrm{loc}}(\Omega)\}$, see [17, Lemma 4.3]. We consider $H^{-1/2}(\Gamma)$ as realization of the dual space to $H^{1/2}(\Gamma)$ with $L^2(\Gamma)$ as pivot space. The corresponding duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$.

For a given wavenumber $k \in \mathbb{C}$ the single layer and double layer potential corresponding to the Helmholtz equation are defined by

$$(\mathrm{SL}(k)\psi)(x) := \int_\Gamma \Phi_k(x,y)\psi(y)ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$
$$(\mathrm{DL}(k)\varphi)(x) := \int_\Gamma \partial_{n,y}\Phi_k(x,y)\varphi(y)ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$

where

$$\Phi_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

is the fundamental solution. The potentials give rise to continuous mappings [17, Thm. 6.12]:

$$\mathrm{SL}(k) : H^{-1/2}(\Gamma) \to H^1(\Delta,\Omega^-) \times H^1_{\mathrm{loc}}(\Delta,\Omega^+),$$
$$\mathrm{DL}(k) : H^{1/2}(\Gamma) \to H^1(\Delta,\Omega^-) \times H^1_{\mathrm{loc}}(\Delta,\Omega^+).$$

Moreover, $\mathrm{SL}(k)\psi$ for $\psi \in H^{-1/2}(\Gamma)$ and $\mathrm{DL}(k)\varphi$ for $\varphi \in H^{1/2}(\Gamma)$ provide solutions of the Helmholtz equation with wavenumber $k \in \mathbb{C}$ in Ω^- as well as in Ω^+ and fulfill for $k \neq 0$ the radiation condition (2), see Appendix, Thm. 6.4. The potentials satisfy the following jump relations [17, Thm. 6.11]:

$$\gamma_0^+\mathrm{SL}(k)\psi - \gamma_0^-\mathrm{SL}(k)\psi = 0, \qquad \gamma_1^+\mathrm{SL}(k)\psi - \gamma_1^-\mathrm{SL}(k)\psi = -\psi, \tag{3}$$
$$\gamma_0^+\mathrm{DL}(k)\varphi - \gamma_0^-\mathrm{DL}(k)\varphi = \varphi, \qquad \gamma_1^+\mathrm{DL}(k)\varphi - \gamma_1^-\mathrm{DL}(k)\varphi = 0. \tag{4}$$

We will consider the four standard boundary integral operators defined by

$$\tfrac{1}{2}[\gamma_0^+\mathrm{SL}(k) + \gamma_0^-\mathrm{SL}(k)] =: V(k) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$
$$\tfrac{1}{2}[\gamma_1^+\mathrm{SL}(k) + \gamma_1^-\mathrm{SL}(k)] =: K'(k) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$
$$\tfrac{1}{2}[\gamma_0^+\mathrm{DL}(k) + \gamma_0^-\mathrm{DL}(k)] =: K(k) : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),$$
$$\tfrac{1}{2}[\gamma_1^+\mathrm{DL}(k) + \gamma_1^-\mathrm{DL}(k)] =: -D(k) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

where $V(k)$ is the single layer operator, $K(k)$ the double layer operator, $K'(k)$ the adjoint double layer operator, and $D(k)$ the hypersingular operator. These operators are continuous mappings of the above indicated function spaces [17, Thm. 6.11]. The following expressions for the Dirichlet and Neumann trace of the single layer and the double layer potential follow from the definitions of the boundary integral operators and from the jump relations (3) and (4):

$$\gamma_0^\pm\mathrm{SL}(k)\psi = V(k)\psi, \qquad \gamma_1^\pm\mathrm{SL}(k)\psi = \left[\mp\tfrac{1}{2}I + K'(k)\right]\psi, \tag{5}$$
$$\gamma_0^\pm\mathrm{DL}(k)\varphi = \left[\pm\tfrac{1}{2}I + K(k)\right]\varphi, \qquad \gamma_1^\pm\mathrm{DL}(k)\varphi = -D(k)\varphi. \tag{6}$$

It is well known [17, Thm. 6.10] that any solution $u \in H^1(\Omega^-)$ of the Helmholtz equation in Ω^- with wavenumber $k \in \mathbb{C}$ can be represented by

$$u = \mathrm{SL}(k)\gamma_0^- u - \mathrm{DL}(k)\gamma_1^- u \quad \text{in } \Omega^-.$$

By application of the interior Dirichlet and Neumann trace we get the following Calderón identity:

$$\begin{pmatrix} \gamma_0^- u \\ \gamma_1^- u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K(k) & V(k) \\ D(k) & \frac{1}{2}I + K'(k) \end{pmatrix} \begin{pmatrix} \gamma_0^- u \\ \gamma_1^- u \end{pmatrix}. \tag{7}$$

If $u \in H^1_{\mathrm{loc}}(\Omega^+)$ is a solution of the Helmholtz equation in the exterior domain Ω^+ with wavenumber k and if it fulfills the radiation condition (2), then we have

$$u = -\mathrm{SL}(k)\gamma_1^+ u + \mathrm{DL}(k)\gamma_0^+ u \quad \text{in } \Omega^+, \tag{8}$$

see Appendix, Cor. 6.5. From this we get the system of boundary integral equations

$$\begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K(k) & -V(k) \\ -D(k) & \frac{1}{2}I - K'(k) \end{pmatrix} \begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix}. \tag{9}$$

2.1 Boundary integral characterizations of resonance pairs based on the representation formula

A resonance function $u \in H^1_{\mathrm{loc}}(\Omega^+)$ corresponding to a resonance $k \in \mathbb{C}$ of the resonance problem (1) can be represented by the representation formula (8) in the following way:

$$u = \mathrm{DL}(k)\gamma_0^+ u \quad \text{in } \Omega^+.$$

The application of the exterior Dirichlet and Neumann trace, respectively, gives by (6) the boundary integral equations

$$\left[\tfrac{1}{2}I + K(k)\right]\varphi = \varphi, \tag{10}$$

$$D(k)\varphi = 0, \tag{11}$$

where $\varphi = \gamma_0^+ u$. The following theorem shows that equations (10) and (11) provide for wavenumbers k with $\mathrm{Im}(k) < 0$ equivalent characterizations of the resonances of the resonance problem (1).

Theorem 2.1. *Let $k \in \mathbb{C}$ with $\mathrm{Im}(k) < 0$ and let $\varphi \in H^{1/2}(\Gamma) \setminus \{0\}$. Then the following assertions are equivalent:*

i) k is a resonance and $u = (\mathrm{DL}(k)\varphi)_{|\Omega^+}$ is a corresponding resonance function of the resonance problem (1) with $\varphi = \gamma_0^+ u$.

ii) $\left[\frac{1}{2}I - K(k)\right]\varphi = 0$.

iii) $D(k)\varphi = 0$.

If one of the above assertions is satisfied, then we have

$$\mathrm{DL}(k)\varphi = 0 \text{ in } \Omega^- \quad \text{and} \quad \gamma_0^+ \mathrm{DL}(k)\varphi = \varphi. \tag{12}$$

Proof. If $(k, (\mathrm{DL}(k)\varphi)_{|\Omega^+})$ is a Neumann resonance pair, then the assertions ii) and iii) follow immediately from (10) and (11), respectively.

Suppose now that $\left[\frac{1}{2}I - K(k)\right]\varphi = 0$. Let us define $v := \mathrm{DL}(k)\varphi$ on $\Omega^- \cup \Omega^+$. Then we get $\gamma_0^+ v = \left[\frac{1}{2}I + K(k)\right]\varphi = \varphi$ by (6). The jump relation (4), $\gamma_0^+ v - \gamma_0^- v = \varphi$, yields $\gamma_0^- v = 0$. This implies that $v = 0$ in Ω^- because otherwise k^2 would be a non-real eigenvalue of $-\Delta$ with Dirichlet boundary conditions for the domain Ω^-. From this we obtain from the jump relation (4) of the double layer potential that $0 = \gamma_1^- v = \gamma_1^+ v$. Further, $v \neq 0$ in Ω^+, since otherwise we would have $0 = \gamma_0^+ v = \varphi$. Hence, $(k, (\mathrm{DL}(k)\varphi)_{|\Omega^+})$ is a Neumann resonance pair.

Let now $D(k)\varphi = 0$. We show that assertion i) holds. First, obviously $\gamma_1^+ \mathrm{DL}(k)\varphi = 0$. It remains to show that $\mathrm{DL}(k)\varphi \neq 0$ in Ω^+. Let us define $w := \mathrm{DL}(k)\varphi$ in $\Omega^- \cup \Omega^+$. From the jump relation (4) of the double layer potential we get $\gamma_1^+ w = \gamma_1^- w = 0$ which implies as above that $w = 0$ in Ω^-. With (4) we obtain $\gamma_0^+ w = \varphi \neq 0$. Hence, $w \neq 0$ in Ω^+.

Suppose now that $(k, (\mathrm{DL}(k)\varphi)_{|\Omega^+})$ is a Neumann resonance pair. We show by contradiction that the first assertion in (12) holds. Let us therefore assume that $\mathrm{DL}(k)\varphi \neq 0$ in Ω^-. Then $(k^2, \mathrm{DL}(k)\varphi)_{|\Omega^-})$ is an eigenpair of the eigenvalue problem for $-\Delta$ with Neumann boundary conditions for the domain Ω^- because $\gamma_1^- \mathrm{DL}(k)\varphi = \gamma_1^+ \mathrm{DL}(k)\varphi = 0$. Hence $k \in \mathbb{R}$, which gives the contradiction. The second assertion in (12) follows now from $\gamma_0^- \mathrm{DL}(k)\varphi = 0$ and the jump relation (4) of the double layer potential. □

In the case that k is real, the boundary integral equations (10) and (11) have also nontrivial solutions $\varphi \in H^{1/2}(\Gamma)$ which are traces of some eigenfunctions of $-\Delta$ for the domain Ω^-:

Proposition 2.2. *Let $k \in \mathbb{C}$ with $\mathrm{Im}(k) \geq 0$ and let $\varphi \in H^{1/2}(\Gamma) \setminus \{0\}$. Then we have:*

a) If $\left[\frac{1}{2}I - K(k)\right]\varphi = 0$, then $k \in \mathbb{R}$ and k^2 is an eigenvalue of $-\Delta$ with Dirichlet boundary conditions for the domain Ω^- and $(\mathrm{DL}(k)\varphi)_{|\Omega^-}$ is a corresponding eigenfunction.

b) If $D(k)\varphi = 0$, then $k \in \mathbb{R}$ and k^2 is an eigenvalue of $-\Delta$ with Neumann boundary conditions for the domain Ω^- and $(\mathrm{DL}(k)\varphi)_{|\Omega^-}$ is a corresponding eigenfunction. Further we have $\mathrm{DL}(k)\varphi = 0$ in Ω^+.

Proof. a) Let us define $v = \mathrm{DL}(k)\varphi$ in $\Omega^- \cup \Omega^+$. Then from $\gamma_0^- v = \left[\frac{1}{2}I - K(k)\right]\varphi = 0$ and by the jump relation (4) of the double layer potential we get $\gamma_0^+ v = \varphi$.

Suppose for a contradiction that $\mathrm{Im}(k) > 0$. From $\gamma_0^- v = 0$ it follows that $v = 0$ in Ω^- since the eigenvalue problem for $-\Delta$ with Dirichlet boundary conditions in bounded domains has only positive eigenvalues. From $0 = \gamma_1^- v = \gamma_1^+ v$ we get $v = 0$ in Ω^+ because the boundary value problem of the Helmholtz equation in Ω^+ has a unique outgoing solution for wavenumbers with non-negative imaginary part [17, Thm. 9.10]. Hence, $0 = \gamma_0^+ v = \varphi$, which is a contradiction.

If k is real, then we conclude from $\gamma_0^+ v \neq 0$ that $v \neq 0$ in Ω^+. Because of the unique solvability of the Helmholtz equation in Ω^+, we obtain $\gamma_1^+ v \neq 0$. By the jump property (4) of the double layer potential we have $\gamma_1^- v = \gamma_1^+ v$ and therefore $v \neq 0$ in Ω^-. Hence $(k^2, v_{|\Omega^-})$ is an eigenpair of $-\Delta$ with Dirichlet boundary conditions for the domain Ω^-.

b) Let us define $u = \mathrm{DL}(k)\varphi$ in $\Omega^- \cup \Omega^+$. Then we have $\gamma_1^+ u = \gamma_1^- u = D(k)\varphi = 0$ by (4). From $\gamma_1^+ u = 0$ it follows that $u = 0$ in Ω^+ since the Neumann boundary value problem for the Helmholtz equation has a unique outgoing solution in Ω^+ for wavenumbers with non-negative imaginary part. By the jump relation (4) of the double layer potential we have $-\gamma_0^+ u = \varphi$ and hence $u \neq 0$ in Ω^-. This implies that (k^2, u) is an eigenpair of $-\Delta$ with Neumann boundary conditions for the domain Ω^-. □

2.2 Boundary integral characterizations of resonance pairs based on the single layer potential ansatz

In this subsection we consider the representation of a resonance function of the resonance problem (1) in terms of the single layer potential

$$u = \mathrm{SL}(k)\psi \quad \text{in } \Omega^+, \tag{13}$$

where $\psi \in H^{-1/2}(\Gamma) \setminus \{0\}$ is a density function. Note that the representation of a resonance function in terms of the double layer potential coincides with the representation formula which we have discussed in the last section. Applying the exterior Neumann trace to the single layer potential representation of u in (13) yields by (5) the following boundary integral equations for the density function ψ:

$$\left[\tfrac{1}{2} I - K'(k)\right] \psi = 0.$$

In the following proposition we characterize the values $k \in \mathbb{C}$ for which the operator $\frac{1}{2} I - K'(k) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is not injective.

Proposition 2.3. *Let $k \in \mathbb{C}$ and $\psi \in H^{1/2}(\Gamma) \setminus \{0\}$. Suppose that*

$$\left[\tfrac{1}{2} I - K'(k)\right] \psi = 0.$$

a) If $k \in \mathbb{R}$, then k^2 is an eigenvalue of $-\Delta$ with Dirichlet boundary conditions for the domain Ω^- and $(\mathrm{SL}(k)\psi)_{|\Omega^-}$ is a corresponding eigenfunction.

b) If $k \notin \mathbb{R}$, then k is a resonance of the resonance problem (1) and $(\mathrm{SL}(k)\psi)_{|\Omega^+}$ defines a corresponding resonance function in Ω^+.

Proof. Suppose that $\left[\frac{1}{2} I - K'(k)\right] \psi = 0$. Define $v = \mathrm{SL}(k)\psi$ in $\Omega^- \cup \Omega^+$. Then $\gamma_1^+ v = 0$ and by the jump relation (3) of the single layer potential we have $\gamma_1^- v = \psi$.

a) If k is real, then it follows that $v = 0$ in Ω^+ and we conclude by the jump relation (3) of the single layer potential that $\gamma_0^- v = 0$. Because of $\gamma_1^- v = \psi \neq 0$, we have $v \neq 0$ in Ω^-.

Hence, (k^2, v) is an eigenpair of $-\Delta$ with Dirichlet boundary conditions for the domain Ω^-.

b) Suppose that k is non-real. Then we have $v \neq 0$ in Ω^+ because otherwise it would follow as in a) that k^2 is a non-positive eigenvalue of $-\Delta$ in the domain Ω^-. Thus, $(k, (\mathrm{SL}(k)\psi)_{|\Omega^+})$ is a resonance pair of (1). □

2.3 Notation and properties of eigenvalue problems for holomorphic Fredholm operator-valued functions

In this subsection we introduce notions and properties of eigenvalue problems for holomorphic Fredholm operator-valued functions where we follow [14, Appendix]. Let X, Y be Hilbert spaces and let $\Lambda \subset \mathbb{C}$ be open and connected. We assume that $A : \Lambda \to \mathcal{L}(X, Y)$ is a holomorphic operator-valued function and that $A(\lambda) : X \to Y$ is Fredholm with index zero for all $\lambda \in \Lambda$. The set

$$\rho(A) := \{\lambda \in \Lambda : \exists A(\lambda)^{-1} \in \mathcal{L}(Y, X)\}$$

is called the resolvent set of A. In the following we will assume that the resolvent set of A is not empty. The complement of the resolvent set $\rho(A)$ in Λ is called spectrum $\sigma(A)$. A number $\lambda_0 \in \Lambda$ is an eigenvalue of A if there exists a non-trivial $x_0 \in X \setminus \{0\}$ such that

$$A(\lambda_0)x_0 = 0.$$

x_0 is called an eigenelement of A corresponding to the eigenvalue λ_0. The spectrum $\sigma(A)$ has no cluster points in Λ [7, Corollary IV.8.4] and each $\lambda \in \sigma(A)$ is an eigenvalue of A which follows from the Fredholm alternative. The dimension of the nullspace $\ker A(\lambda_0)$ of an eigenvalue λ_0 is called the geometric multiplicity. An ordered collection of elements $x_0, x_1, \ldots, x_{m-1}$ in X is called a Jordan chain of λ_0 if x_0 is an eigenelement corresponding to λ_0 and if

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0)x_{n-j} = 0 \quad \text{for all } n = 0, 1, \ldots, m-1 \tag{14}$$

is satisfied, where $A^{(j)}$ denotes the jth derivative. The length of any Jordan chain of an eigenvalue is finite [14, Lemma A.8.3]. The maximal length of a Jordan chain of the eigenvalue λ_0 is denoted by $\varkappa(A, \lambda_0)$. Elements of any Jordan chain of an eigenvalue λ_0 are called generalized eigenelements of λ_0. The closed linear hull of all generalized eigenelements of an eigenvalue λ_0 is called generalized eigenspace of λ_0 and is denoted by $G(A, \lambda_0)$. The dimension of the generalized eigenspace $G(A, \lambda_0)$ is finite [14, Prop. A.8.4] and it is referred to as algebraic multiplicity of λ_0.

Finally we cite a perturbation result [10, Thm. 5] which we need for the analysis of the combined boundary integral formulations of the resonance problem (1).

Theorem 2.4. *Let $\Lambda \subset \mathbb{C}$ be open and connected with a simple rectifiable boundary. Let $A : \overline{\Lambda} \to \mathcal{L}(X, Y)$ be holomorphic on Λ and continuous on $\overline{\Lambda}$, and let $\partial\Lambda \subset \rho(A)$. Then*

there exists a $\delta > 0$ such that for each function $B : \overline{\Lambda} \to \mathcal{L}(X, Y)$ which is holomorphic on Λ and continuous on $\overline{\Lambda}$, and which satisfies

$$\max_{\lambda \in \partial\Lambda} \|B(\lambda) - A(\lambda)\|_{\mathcal{L}(X,Y)} < \delta,$$

it follows that the sum of the algebraic multiplicities of the eigenvalues of A and B in $\overline{\Lambda}$ coincides.

3 Combined boundary integral formulations of the resonance problem

In the previous section we have seen that the resonances of the resonance problem (1) coincide with the non-real wavenumbers k for which the boundary integral operators $D(k)$, $\frac{1}{2}I - K(k)$ and $\frac{1}{2}I - K'(k)$ are not injective. Hence, the resonances of (1) are the non-real eigenvalues of the operator-valued functions D, $\frac{1}{2}I - K(\cdot)$ and $\frac{1}{2}I - K'(\cdot)$. The real eigenvalues of these operator-valued functions correspond to eigenvalues of some eigenvalue problems of $-\Delta$ for the domain Ω^-. For non-convex domains, in particular for domains with cavities, the resonances can lie very close to the real axis. When using a boundary element method for an approximation of these eigenvalue problems a distinction between the resonances and the real eigenvalues is hard since the approximated real eigenvalues have a small imaginary part.

In this section we analyze combined boundary integral formulations of the resonance problem (1). The spectra of the resulting eigenvalue problems exhibit besides the resonances also an additional spectrum but this lies, as we will see, in the upper complex half plane.

3.1 Combined direct boundary integral formulations

As a combined direct boundary integral formulation for the resonance problem (1) we use a formulation which was introduced in [31] for the solution of the exterior Neumann boundary value problem of the Helmholtz equation for positive wavenumbers k. The formulation there is based on the Calderon's formula (9) of the traces of a solution u of the Helmholtz equation

$$\begin{pmatrix} D(k) & -\frac{1}{2}I + K'(k) \\ \frac{1}{2}I - K(k) & V(k) \end{pmatrix} \begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix} = \begin{pmatrix} -\gamma_1^+ u \\ 0 \end{pmatrix}. \tag{15}$$

For real wavenumbers k the block operator in (15) is injective up to eigenvalues of V [31, Lemma 5.28]. Therefore in [31] on both sides of the second equation in (15) the term $i\mu V(0)\gamma_1^+ u$, $\mu \in \mathbb{R} \setminus \{0\}$, is added which yields

$$\mathcal{A}_\mu(k) \begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix} := \begin{pmatrix} D(k) & -\frac{1}{2}I + K'(k) \\ \frac{1}{2}I - K(k) & V(k) + i\mu V(0) \end{pmatrix} \begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix} = \begin{pmatrix} -\gamma_1^+ u \\ i\mu V(0)\gamma_1^+ u \end{pmatrix}. \tag{16}$$

The operator $\mathcal{A}_\mu(k) : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ is injective for real wavenumbers k and satisfies a Gårding's inequality [31]. Obviously, if k is a resonance of the resonance problem (1) and if $u \in H^1_{\rm loc}(\Omega^+)$ is a corresponding resonance function, then $\mathcal{A}_\mu(k) \begin{pmatrix} \gamma_0^+ u \\ \gamma_1^+ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The following theorem addresses the question of injectivity of $\mathcal{A}_\mu(k)$ for wavenumbers in the right complex half-plane.

Theorem 3.1. *Let $\mu < 0$, $k \in \mathbb{C}$ with $\mathrm{Re}(k) \geq 0$, and let $(\varphi, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ with $(\varphi, t) \neq (0, 0)$. Suppose that $(k, (\varphi, t))$ satisfies*

$$\mathcal{A}_\mu(k) \begin{pmatrix} \varphi \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Define

$$u := \mathrm{DL}(k)\varphi - \mathrm{SL}(k)t \quad \text{in } \Omega^- \cup \Omega^+. \tag{17}$$

Then we have:

a) $\gamma_1^+ u = 0$, $\gamma_1^- u = -t$ *and* $\gamma_0^- u + i\mu V(0)\gamma_1^- u = 0$.

b) If $\mathrm{Im}(k) \leq 0$, *then* $t = 0$ *and* $(k, u_{|\Omega^+})$ *is an eigenpair of the resonance problem* (1) *with* $\varphi = \gamma_0^+ u$. *Moreover we have* $u_{|\Omega^-} = 0$.

c) If $\mathrm{Im}(k) > 0$, *then* $u_{|\Omega^+} = 0$ *and* $u_{|\Omega^-}$ *is a non-trivial solution of the boundary value problem*

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega^-, \qquad \gamma_0^- u + i\mu V(0)\gamma_1^- u = 0 \quad \text{on } \Gamma, \tag{18}$$

where $(\gamma_0^- u, \gamma_1^- u) = (-\varphi, -t)$.

Proof. a) From the first equation of

$$\mathcal{A}_\mu(k) \begin{pmatrix} \varphi \\ t \end{pmatrix} = \begin{pmatrix} D(k) & -\frac{1}{2}I + K'(k) \\ \frac{1}{2}I - K(k) & V(k) + i\mu V(0) \end{pmatrix} \begin{pmatrix} \varphi \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

it follows that $\gamma_1^+ u = 0$ by (5) and (6). The application of the interior Neumann trace to u gives using again the first equation of (19):

$$\gamma_1^- u = -D(k)\varphi - \left[\tfrac{1}{2}I + K'(k)\right] t = -D(k)\varphi + \left[\tfrac{1}{2}I - K'(k)\right] t - t = -t. \tag{20}$$

The jump relations (3) and (4) imply $\gamma_0^+ u - \gamma_0^- u = \varphi$. Using $\gamma_0^+ u = \left[\frac{1}{2}I + K(k)\right]\varphi - V(k)t$ and the second equation of (19), we get $\gamma_0^- u = i\mu V(0)t$. Finally, we obtain $\gamma_0^- u = -i\mu V(0)\gamma_1^- u$ by (20).

b) Let $\mathrm{Im}(k) \leq 0$. Green's first formula gives with the assertions in a)

$$\int_\Omega (|\nabla u|^2 - k^2|u|^2)dx = \langle \gamma_1^- u, \overline{\gamma_0^- u} \rangle_\Gamma = \langle -t, \overline{i\mu V(0)t} \rangle_\Gamma = i\mu \langle t, \overline{V(0)t} \rangle_\Gamma.$$

Taking the imaginary part yields

$$-2\,\mathrm{Re}(k)\,\mathrm{Im}(k)\|u\|^2_{L^2(\Omega^-)} = \mu\langle t, \overline{V(0)t}\rangle_\Gamma.$$

The left hand side is not negative and the right hand side is not positive since $\mu < 0$ and $V(0)$ is $H^{-1/2}(\Gamma)$-elliptic. Therefore we get $t = 0$ on Γ. From assertion a) it follows that $u = 0$ in Ω^-. This implies $u = \mathrm{DL}(k)\varphi$ and $D(k)\varphi = 0$. If $\mathrm{Im}(k) < 0$, then, by Theorem 2.1, k is a resonance of the resonance problem (1) and $\varphi = \gamma_0^+ u$.

By contradiction we show now that k is not real. If k is real, then as already shown, we have $t = 0$ and $\gamma_0^- u = i\mu V(0)t = 0$. From $D(k)\varphi = 0$ it follows by Proposition 2.2 b) that $\varphi = \gamma_0^- u$, thus $\varphi = 0$. But by assumption we have $(\varphi, t) \neq (0, 0)$.

c) Let $\mathrm{Im}(k) > 0$, then k is not a resonance and therefore we get from $\gamma_1^+ u = 0$ that $u_{|\Omega^+} = 0$ and $\gamma_0^+ u = 0$. In the proof of assertion a) we have shown that $\varphi = \gamma_0^+ u - \gamma_0^- u$, hence $\varphi = -\gamma_0^- u$. From $t = -\gamma_1^- u$ it follows with $(\varphi, t) \neq (0, 0)$ that $u_{|\Omega^-} \neq 0$. □

Lemma 3.2. *Let $\mu \in \mathbb{R}$. Then $\mathcal{A}_\mu(k)$ is coercive for all $k \in \mathbb{C}$, i. e., there exist a compact operator $\mathcal{C}(k) : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and a constant $\alpha > 0$ such that*

$$\mathrm{Re}\left\langle (\mathcal{A}_\mu(k) - \mathcal{C}(k)) \begin{pmatrix} \varphi \\ t \end{pmatrix}, \overline{\begin{pmatrix} \varphi \\ t \end{pmatrix}} \right\rangle \geq \alpha \left(\|\varphi\|^2_{H^{1/2}(\Gamma)} + \|t\|^2_{H^{-1/2}(\Gamma)} \right)$$

for all $(\varphi, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

Proof. We split

$$\mathcal{A}_\mu(k) = \begin{pmatrix} \widetilde{D}(0) & -\frac{1}{2}I + K'(0) \\ \frac{1}{2}I - K(0) & V(0) + i\mu V(0) \end{pmatrix} + \mathcal{C}(k),$$

where

$$\mathcal{C}(k) = \begin{pmatrix} D(k) - \widetilde{D}(0) & K'(k) - K'(0) \\ -K(k) + K(0) & V(k) - V(0), \end{pmatrix},$$

and where $\tilde{D}(0)$ is the modified hypersingular operator of the Laplace equation [23, p. 176]. The operator $\mathcal{C}(k)$ is compact since each component is compact, see [8, Lemma 3.2]. Since $\widetilde{D}(0)$ is $H^{1/2}(\Gamma)$-elliptic [23, p. 176] and $V(0)$ is $H^{-1/2}(\Gamma)$-elliptic [17, Cor. 8.13] there exists a constant $\alpha > 0$ such that

$$\begin{aligned} &\mathrm{Re}\left\langle (\mathcal{A}_\mu(k) - \mathcal{C}(k)) \begin{pmatrix} \varphi \\ t \end{pmatrix}, \overline{\begin{pmatrix} \varphi \\ t \end{pmatrix}} \right\rangle \\ &\quad = \langle \widetilde{D}(0)\varphi, \overline{\varphi}\rangle_\Gamma - \mathrm{Re}\langle [\tfrac{1}{2}I - K'(0)]t, \overline{\varphi}\rangle_\Gamma + \mathrm{Re}\langle [\tfrac{1}{2}I - K(0)]\varphi, \overline{t}\rangle_\Gamma + \langle V(0)t, \overline{t}\rangle_\Gamma \\ &\qquad\qquad \geq \alpha \left(\|\varphi\|^2_{H^{1/2}(\Gamma)} + \|t\|^2_{H^{-1/2}(\Gamma)} \right), \end{aligned}$$

where we have used that $\langle K'(0)t, \overline{\varphi}\rangle_\Gamma = \overline{\langle \varphi, \overline{K'(0)t}\rangle_\Gamma} = \overline{\langle K(0)\varphi, \overline{t}\rangle_\Gamma}$. □

From the coercivity of $\mathcal{A}_\mu(k)$ it follows that $\mathcal{A}_\mu(k)$ is a Fredholm operator of index 0 [17, Thm. 2.34]. Since the dependence of k is holomorphic in $\mathcal{A}_\mu(k)$, see e. g. [13, Thm. 2.3] , we can apply the results of theory of eigenvalue problems for holomorphic Fredholm operator-valued functions of Section 2.3 to A_μ for its spectral analysis.

Theorem 3.1 and the discussion at the beginning of this subsection show that for $\mu < 0$ the eigenvalues of $\mathcal{A}_\mu$ in the lower complex half plane coincide with the resonances of the resonance problem (1). The existence of eigenvalues of $\mathcal{A}_\mu$ in the upper right complex half plane was not addressed so far. Theorem 3.1 only states that if eigenvalues in the upper right complex half plane exit, then they are eigenvalues of the eigenvalue problem for $-\Delta$ in Ω^- with the Robin-type boundary conditions (18). A possible approach to the question of the existence of such eigenvalues is to utilize the perturbation result of Thm. 2.4 and interpret the eigenvalues of $\mathcal{A}_\mu$ as perturbed eigenvalues of $\mathcal{A}_0$. The following lemma shows that the eigenvalues of $\mathcal{A}_0$ are either resonances of (1) or corresponds to Dirichlet eigenvalues of $-\Delta$ in Ω^-.

Lemma 3.3. *Let $k \in \sigma(\mathcal{A}_0)$. Then $\operatorname{Im}(k) \leq 0$ and we have:*

a) If $\operatorname{Im}(k) < 0$, then k is a resonance of the resonance problem (1).

b) If $k \in \mathbb{R}$, then k^2 is an eigenvalue of $-\Delta$ with Dirichlet boundary condition for the domain Ω^-.

Proof. Let $(k, (\varphi, t)^\top)$ be an eigenpair of $\mathcal{A}_0$ and define u as in Thm. 3.1 by

$$u := \mathrm{DL}(k)\varphi - \mathrm{SL}(k)t \quad \text{in } \Omega^- \cup \Omega^+.$$

Then we may conclude as in the proof of part a) of Thm. 3.1 that

$$\gamma_1^+ u = 0, \quad \gamma_1^- u = -t, \quad \gamma_0^- u = 0.$$

First we show that $u \neq 0$ in Ω^+ if $\operatorname{Im}(k) \neq 0$. From this it follows that $\operatorname{Im}(k) \leq 0$ and assertion a). Suppose for a contradiction that $u = 0$ in Ω^+. From

$$\gamma_0^+ u = (\tfrac{1}{2} I + K(k))\varphi - V(k)t \quad \text{and} \quad (\tfrac{1}{2} I - K(k))\varphi + V(k)t = 0$$

we get $\varphi = \gamma_0^+ u = 0$. Because $u = 0$ in Ω^+, it follows $\varphi = 0$. Since k is assumed not to be real, $\gamma_0^- u = 0$ implies that $u = 0$ in Ω^- because otherwise k^2 would be a non-positive Dirichlet eigenvalue of $-\Delta$ for the domain Ω^-. From $\gamma_1^- u = -t$ we get $t = 0$, which is a contradiction to $(\varphi, t) \neq (0, 0)$.

Let now $k \in \mathbb{R}$. Since $\gamma_0^- u = 0$ it only remains to show that $u \neq 0$ in Ω^-. Because k is real and u is an outgoing solution of the Helmholtz equation with $\gamma_1^+ u = 0$ we know that $u = 0$ in Ω^+ [17, Chapt. 9]. This implies $\gamma_0^+ u = 0$. With $\varphi = \gamma_0^+ u - \gamma_0^- u$ we get $\varphi = -\gamma_0^- u$. From $t = -\gamma_1^- u$ it follows with $(\varphi, t) \neq (0, 0)$ that $u \neq 0$ in Ω^-. □

The above lemma and the perturbation result of Thm. 2.4 show that the eigenvalues of $\mathcal{A}_\mu$ in the upper left quadrant of the complex plane are the perturbed real eigenvalues of $\mathcal{A}_0$ which correspond to the Dirichlet eigenvalues of $-\Delta$ in Ω^-.

3.2 Combined indirect formulations

In this subsection we consider the classical indirect combined field integral equation approach [4] as basis for a boundary integral formulation of the resonance problem (1). The following ansatz for the representation of a resonance function u is used,

$$u = \mathrm{DL}(k)\psi + i\eta \mathrm{SL}(k)\psi \quad \text{in } \Omega^+,\ \eta > 0, \tag{21}$$

where $\psi \in H^{1/2}(\Gamma)$ is the density function which has to be determined. For an analysis of this approach for scattering problems we refer to [8]. The boundary integral equation for the resonance problem (1) derived from (21) is given by

$$\gamma_1^+ u = [-D(k) + i\eta(K'(k) - \tfrac{1}{2}I)]\psi = 0.$$

In [8, Lem. 4.2] it is shown that the underlying operator

$$B_\eta(k) := -D(k) + i\eta(K'(k) - \tfrac{1}{2}I) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \tag{22}$$

is coercive for $\eta \in \mathbb{R}$. Hence, $B_\eta(k)$ is a Fredholm operator of index 0. In addition, the dependence of k is holomorphic in $B_\eta(k)$.

Theorem 3.4. *Let $k \in \mathbb{C}$ with $\mathrm{Re}(k) \geq 0$, $\psi \in H^{1/2}(\Gamma) \setminus \{0\}$, and $\eta > 0$. Suppose that*

$$B_\eta(k)\psi = 0. \tag{23}$$

Define

$$u := [\mathrm{DL}(k) + i\eta \mathrm{SL}(k)]\psi. \tag{24}$$

Then we have:

a) $k \notin \mathbb{R}$.

b) If $\mathrm{Im}(k) > 0$, then u is a solution of

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega, \qquad \gamma_1^- u + i\eta\gamma_0^- u = 0 \quad \text{on } \Gamma. \tag{25}$$

c) If $\mathrm{Im}(k) < 0$, then k is a resonance of the resonance problem (1) and $u_{|\Omega^+}$ is a corresponding resonance function.

Proof. Let (k, ψ), $\psi \in H^{1/2}(\Gamma) \setminus \{0\}$, be a solution of (23). Then u as given in (24) is a solution of the Helmholtz equation in Ω^- and Ω^+ with $\gamma_1^+ u = 0$.

We first consider the case that k is not a resonance of the resonance problem (1). Then $u = 0$ in Ω^+ and hence $\gamma_0^+ u = 0$. From the jump relations (4) and (3) we obtain

$$\begin{aligned}
\gamma_0^- u &= \gamma_0^- u - \gamma_0^+ u = \gamma_0^- \mathrm{DL}(k)\psi - \gamma_0^+ \mathrm{DL}(k)\psi = -\psi, \\
\gamma_1^- u &= \gamma_1^- u - \gamma_1^+ u = i\eta \left(\gamma_1^- \mathrm{SL}(k)\psi - \gamma_1^+ \mathrm{SL}(k)\psi\right) = i\eta\psi,
\end{aligned}$$

thus $\gamma_1^- u + i\eta\gamma_0^- u = 0$. This shows that b) holds. Still assuming that k is not a Neumann resonance, we consider now the case that $\mathrm{Im}(k) \leq 0$. By Green's first formula we have

$$\int_\Omega (|\nabla u|^2 - k^2|u|^2)dx = \langle \gamma_1^- u, \overline{\gamma_0^- u}\rangle = \langle i\eta\psi, \overline{-\psi}\rangle = -i\eta\|\psi\|^2_{L^2(\Gamma)}.$$

Taking the imaginary part yields

$$2\,\mathrm{Re}(k)\,\mathrm{Im}(k)\|u\|^2_{L^2(\Omega)} = \eta\|\psi\|^2_{L^2(\Gamma)}.$$

The left hand side is not positive and the right hand side is positive. This implies that if $\mathrm{Im}(k) \leq 0$, then k is a resonance of the resonance problem (1). This shows in particular that k cannot be real. □

Corollary 3.5. *Let $\eta > 0$, and let $k \in \mathbb{C}$ with $\mathrm{Re}(k) \geq 0$ and $\mathrm{Im}(k) \leq 0$. Then k is a resonance of the resonance problem (1) if and only if there exists a $\psi \in H^{1/2}(\Gamma)$ such that $[D(k) - i\eta(K'(k) - \frac{1}{2}I)]\psi = 0$.*

Proof. It only remains to show that if k is a resonance of the resonance problem (1) that then there exists a $\psi \in H^{1/2}(\Gamma) \setminus \{0\}$ such that $B_\eta(k)\psi = 0$. If this would not to be the case, then from the Fredholm alternative it would follow that the Neumann boundary value problem of the Helmholtz equation in Ω^+ would be solvable for any given Neumann data. □

The eigenvalues of B_η in the upper right quadrant of the complex plane can be interpreted by the perturbation result of Thm. 2.4 as the perturbed real eigenvalues of $B_0 = -D$ which coincide with the square roots of the Neumann eigenvalues of $-\Delta$ for the domain Ω^-.

4 Galerkin approximation

The operator-valued functions $\mathcal{A}_\mu$ and B_η of the combined boundary integral formulations for the resonance problem (1) are holomorphic coercive operator-valued functions. For such kind of eigenvalue problems convergence of the Galerkin approximation in conforming ansatz spaces is guaranteed, see [10, 29, 28, 25] for general results and [25] for the convergence for a conforming boundary element method for the Dirichlet resonance problem. The error of the approximations of an eigenvalue and of an eigenfunction depends on the regularity of the eigenfunction, on the approximation quality of the ansatz space and on the maximal length of the Jordan chain of the eigenvalue. For comprehensive convergence results for eigenvalue problems for holomorphic coercive operator-valued functions we refer to [10, 11, 25].

5 Numerical examples

In the following numerical examples we use conforming Galerkin approximations of the eigenvalue problems for $\mathcal{A}_\mu$ and B_η to compute approximations of resonances of the resonance problem (1). In particular, we study the separation of the resonances from the rest

of the spectrum for different parameters μ and η. For the computations the open-source boundary element library BEM++ [22] is used.

The Galerkin approximations of the eigenvalue problems for the operator-valued functions $\mathcal{A}_\mu$ and B_η result in holomorphic matrix eigenvalue problems. For the numerical solution of these eigenvalue problems we use the contour integral method as given in [3]. For other variants of the contour integral method we refer to [2, 32]. We also want to mention [15] where a first variant of the contour integral method has been introduced and which is not known in the literature on eigenvalue problems.

In the following we describe the basic ideas and the main steps of the used algorithm for the numerical solution of the discretized eigenvalue problems. Let $T : \mathbb{C} \to \mathbb{C}^{n\times n}$ denote the matrix-valued function resulting from a Galerkin approximation of the operator-valued functions $\mathcal{A}_\mu$ or B_η. The contour integral method is suitable for the approximation of all eigenvalues of the matrix-valued function T (and all related eigenvectors) which lie inside of a given contour C in the complex plane. In order to keep the presentation simple we assume that the eigenvalues of T inside of C are simple and that the corresponding eigenvectors are linearly independent. The general case can be treated similarly with almost the same computational cost, see [3, Sect. 3.2] for the case of multiple eigenvalues and [3, Sect. 5] for linearly dependent eigenvectors. Let $\lambda_1, \ldots, \lambda_k$ be the pairwise distinct eigenvalues of T inside of C and let $v_1, \ldots, v_k$ be the corresponding eigenvectors. The basic principle of the contour integral method is that

$$V := \begin{pmatrix} v_1 \cdots v_k \end{pmatrix} \quad \text{and} \quad \Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$$

can be represented by the following integrals of the resolvent of T over the contour C,

$$\frac{1}{2\pi i} \int_C T(z)^{-1} dz = VW^H \quad \text{and} \quad \frac{1}{2\pi i} \int_C zT(z)^{-1} dz = V\Lambda W^H, \tag{26}$$

where $W = (w_1, \ldots, w_k) \in \mathbb{C}^{n\times k}$ and $w_1, \ldots, w_k$ are some eigenvectors of the adjoint eigenvalue problem for T [3, Thm. 2.9]. Let $\hat{V} \in C^{n\times \ell}$ be a randomly chosen matrix such that $\operatorname{rank}(W^H\hat{V}) = k$. This condition on the rank can be expected to hold in a generic sense if $\ell \geq k$. Define

$$A_0 := \frac{1}{2\pi i} \int_C T(z)^{-1}\hat{V} dz \quad \text{and} \quad A_1 := \frac{1}{2\pi i} \int_C zT(z)^{-1}\hat{V} dz, \tag{27}$$

then $A_0 = VW^H\hat{V}$ and $A_1 = V\Lambda W^H\hat{V}$ because of (26). Some simple calculations show [3, Thm. 3.1] that the eigenvalues of the matrix

$$B := V_0^H A_1 W_0 \Sigma_0^{-1} \in \mathbb{C}^{k\times k}$$

coincide with $\lambda_1, \ldots, \lambda_k$, where V_0, Σ_0, W_0 are the factors of a reduced singular value decomposition of $A_0 = V_0\Sigma_0 W_0^H$ with $\Sigma_0 = \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$. Note that by assumption on the rank of $W^H\hat{V}$ we have $\sigma_{k+1} = \ldots = \sigma_\ell = 0$. The eigenvectors of T can be obtained from the eigenvectors $s_1, \ldots, s_k$ of B by $v_j = V_0 s_j$, $j = 1, \ldots, k$.

For the numerical realization of the described procedure the integrals A_0 and A_1 in (27) are approximated by the composite trapezoidal rule. For a contour C with a 2π-periodic parametrization ϕ and N equidistant nodes, the approximations $A_{0,N}$ and $A_{1,N}$ of A_0 and A_1, respectively, have the form

$$A_{r,N} = \frac{1}{iN} \sum_{j=0}^{N-1} \Phi(t_j)^r T(\Phi(t_j))^{-1} \hat{V}, \quad r \in \{0, 1\}, \tag{28}$$

where the nodes are given by $t_j = \frac{2j\pi}{N}$, $j = 0, \ldots, N-1$. The error $\|A_{r,N} - A_r\|$ converges exponentially to zero as $N \to \infty$ [3, Thm. 4.7]. For sufficiently large N this allows to detect the correct rank of A_0 and further to compute approximations of the eigenvalues $\lambda_1, \ldots, \lambda_k$. A pseudo-code of the algorithm is given below [3, Integral algorithm 1]. For details of the implementation we refer to [3, Sect. 3-5].

Contour integral algorithm for the approximation of the eigenvalues of T inside the contour C and of corresponding eigenvectors

Input: Tolerance tol_{rank} for the determination of the rank of A_0 by the singular values of $A_{0,N}$

1: Choose an index $\ell \leq n$ and a matrix $\hat{V} \in \mathbb{C}^{n \times \ell}$ at random.
2: Compute $A_{0,N}$ and $A_{1,N}$ as given in (28).
3: Compute a SVD $A_{0,N} = V_N \Sigma_N W_N^H$, where $V_N \in \mathbb{C}^{n \times \ell}$, $W_N \in \mathbb{C}^{\ell \times \ell}$ such that $V_N^H V_N = W_N^H W_N = I_\ell$ and $\Sigma_N = \text{diag}(\sigma_1, \ldots, \sigma_\ell)$.
4: Perform a rank test for Σ_N, i. e., find $0 < k \leq \ell$ such that $\sigma_1 \geq \ldots \geq \sigma_k > \text{tol}_{\text{rank}} > \sigma_{k+1}$. If $k = \ell$, then increase ℓ and go to step 1. Else let $V_{0,N} = V_N(1:n, 1:k)$, $W_{0,N} = W(1:\ell, 1:k)$ and $\Sigma_{0,N} = \text{diag}(\sigma_1, \ldots, \sigma_k)$.
5: Compute $B_N = V_{0,N}^H A_{1,N} W_{0,N} \Sigma_{0,N}^{-1}$.
6: Compute the eigenvalues $\lambda_{1,N}, \ldots, \lambda_{k,N}$ and corresponding eigenvectors $s_{1,N}, \ldots, s_{k,N}$ of B_N.
7: Compute $v_{j,N} = V_{0,N} s_{j,N}$.

Output: Approximations of eigenpairs of T: $(\lambda_{j,N}, v_{j,N})$, $j = 1, \ldots, k$.

The main computational cost of the presented contour integral algorithm consists in the computation of $A_{0,N}$ and $A_{1,N}$ in step 2 for which $T(\Phi(t_j))^{-1}\hat{V}$ for $\hat{V} \in \mathbb{C}^{n \times \ell}$ and $j = 0, \ldots, N-1$ has to be computed. This requires the solution of N linear systems each with ℓ different right hand sides.

5.1 Cavity resonances

As first example we consider the cavity-domain

$$\Omega^- = (-1.2, 1.2)^3 \setminus ([-0.8, 0.8]^3 \cup (0.8, 1.2) \times (-0.2, 0.2)^2),$$

see Figure 3. The eigenvalues of $\mathcal{A}_\mu$ for different μ are plotted in Figure 1. The spectrum of $\mathcal{A}_\mu$ consists of the resonances and of the square-roots of the eigenvalues of $-\Delta$ for the

domain Ω^- with the boundary condition $\gamma_0^+ u + i\mu V(0)\gamma_1^- u = 0$, see (18). The parameter μ is the factor of the perturbation of the Dirichlet boundary condition. The square root of the smallest Dirichlet eigenvalue of $-\Delta$ for the domain Ω^- is about 7.4. For the chosen parameters of $\mu < 0$ a significant shift of the real spectrum to the upper complex half-plane is visible. Numerical experiments show that the same holds for smaller chosen μ.

Figure 2 shows the eigenvalues of B_η for different η. The eigenvalues of B_η consists of the resonances and of the square-roots of the eigenvalues of $-\Delta$ for the domain Ω^- with the boundary condition $\gamma_1^- u + i\eta\gamma_0^- u = 0$, see (25). The parameter η describes the perturbation of the Neumann boundary condition. In Figure 2 a clear separation of the computed resonances from the rest of the spectrum is observable for $\eta \geq 0.5$. In Figure 3

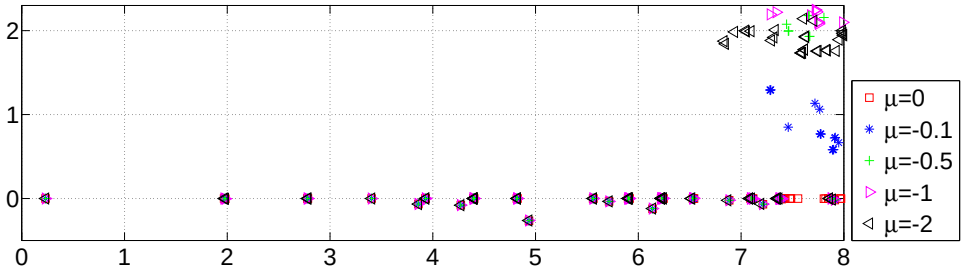

Figure 1: Computed eigenvalues of $\mathcal{A}_\mu$ for different μ for the cavity-domain.

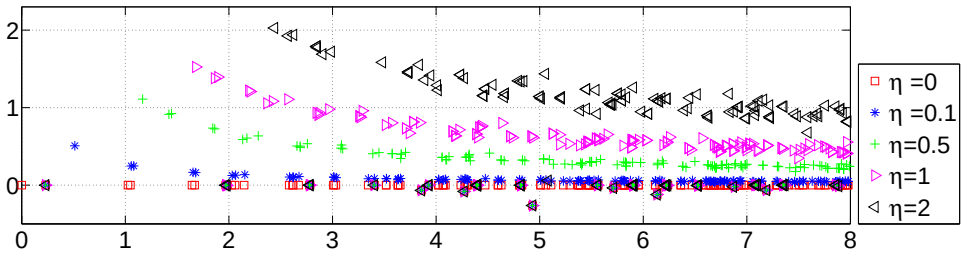

Figure 2: Computed eigenvalues of B_η for different η for the cavity-domain.

the resonance functions for the resonances which have the two smallest positive real parts are plotted.

5.2 Resonances of a pipe

In this example we have chosen as domain for Ω^- a cylindrical pipe with length 5, outside diameter 1 and wall-thickness 0.25. In Figure 4 and Figure 5 the eigenvalues of $\mathcal{A}_\mu$ and B_η for different μ and η are shown. The same observations concerning the separation of the resonances form the rest of the spectrum as for the cavity-domain can be made for the

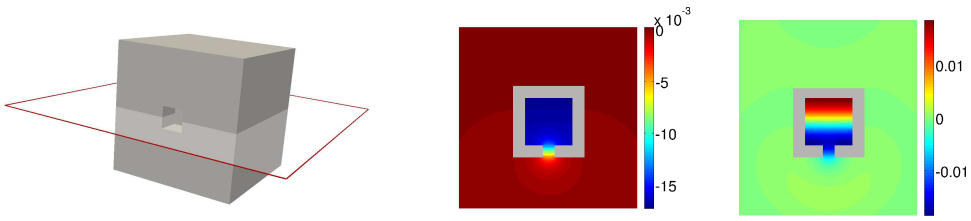

Figure 3: *Left:* Geometry of the cavity-domain and plane for the evaluation of the resonance functions. *Middle and right:* Resonance functions of the cavity domain corresponding to the resonances with the two smallest real parts.

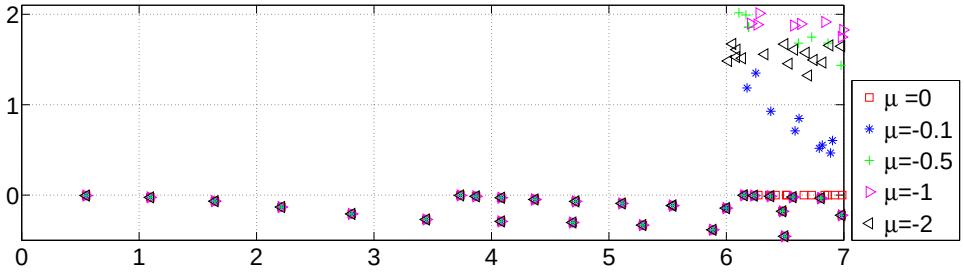

Figure 4: Computed eigenvalues of $\mathcal{A}_\mu$ for different μ for the pipe.

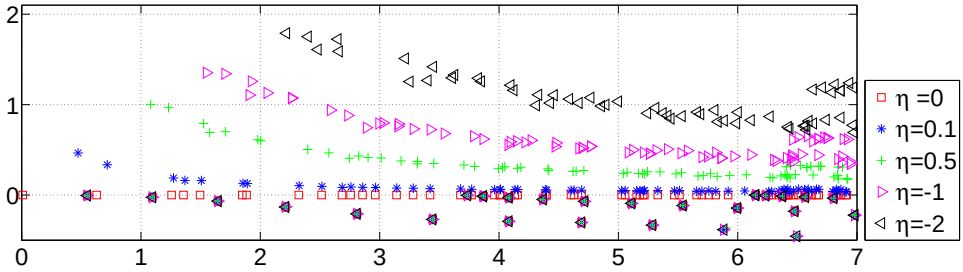

Figure 5: Computed eigenvalues of B_η for different η for the pipe.

pipe. In Figure 6 the resonance functions for the resonances which have the two smallest positive real parts are plotted.

6 Conclusions

In this paper we have analyzed different boundary integral formulations for the time-harmonic acoustic scattering-resonance problem with Neumann boundary conditions. The

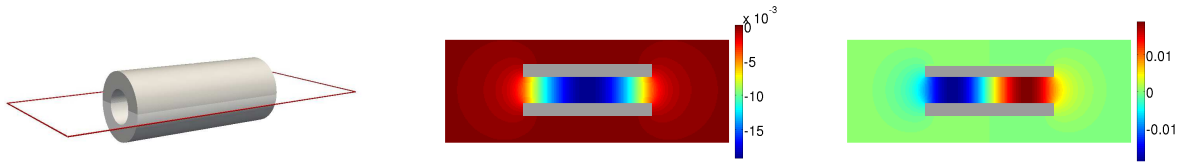

Figure 6: *Left:* Geometry of the pipe and plane for the evaluation of the resonance functions. *Middle and right:* Resonance functions of the pipe corresponding to the resonances with the two smallest real parts.

eigenvalues of the eigenvalue problems resulting from boundary integral formulations split in the one hand into resonances and in the other hand into eigenvalues which correspond to eigenvalues of some Laplacian eigenvalue problem for the interior domain. We have proposed a direct and an indirect combined boundary integral formulation for the resonance problem and we have shown that a better separation of the resonances from the other eigenvalues is achieved than for standard boundary integral formulations. Numerical experiments confirm that combined boundary integral formulations allow a reliable and simple identification of the resonances also for domains with open cavities. For conforming Galerkin boundary element approximations of the considered combined boundary integral formulations standard convergence results in the canonical trace spaces can be applied.

A similar approach and analysis can be carried over to the acoustic scattering-resonance problem with Dirichlet boundary conditions. Combined boundary integral formulations which are used for the Dirichlet boundary value problem as in [5, 26] are suitable as formulations for the resonance problem.

Appendix

In this appendix we present different equivalent characterizations of the radiation condition (2) for the solution of the Helmholtz equation (1a). Crucial for this is the representation of the fundamental solution of the Helmholtz equation

$$\Phi_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

in terms of the spherical harmonics Y_n^m [18, Sect. 14.30] and the spherical Bessel functions $h_n^{(1)} = j_n + iy_n$ [18, Sect. 10.47].

Theorem 6.1. *Let $x, y \in \mathbb{R}^3$ with $|x| > |y| > 0$ and $k \in \mathbb{C} \setminus \{0\}$. Then we have*

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} = ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) j_n(k|y|) \overline{Y_n^m\left(\frac{y}{|y|}\right)}. \tag{29}$$

The series and its term by term first derivatives with respect to $|x|$ and $|y|$ are absolutely and uniform convergent on compact subsets of $[|y|, |x|]$, i. e., there exist constants $C_1, C_2, C_3 > 0$ such that for all $z_1, z_2 \in \mathbb{R}^3$ with $|z_1|, |z_2| \in [|y|, |x|]$ it holds

$$\sum_{m=-n}^{n} \left| h_n^{(1)}(k|z_1|) Y_n^m\left(\frac{z_1}{|z_1|}\right) j_n(k|z_2|) \overline{Y_n^m\left(\frac{z_2}{|z_2|}\right)} \right| \leq C_1 \frac{|z_2|^n}{|z_1|^n}, \tag{30a}$$

$$\sum_{m=-n}^{n} \left| h_n^{(1)\prime}(k|z_1|) Y_n^m\left(\frac{z_1}{|z_1|}\right) j_n(k|z_2|) \overline{Y_n^m\left(\frac{z_2}{|z_2|}\right)} \right| \leq C_2 \frac{|z_2|^n}{|z_1|^n}, \tag{30b}$$

$$\sum_{m=-n}^{n} \left| h_n^{(1)}(k|z_1|) Y_n^m\left(\frac{z_1}{|z_1|}\right) j_n'(k|z_2|) \overline{Y_n^m\left(\frac{z_2}{|z_2|}\right)} \right| \leq C_3 \frac{|z_2|^n}{|z_1|^n} \tag{30c}$$

for all $n \in \mathbb{N}_0$.

Proof. The series representation (29) follows immediately from [18, Equations 10.60.1, 10.60.2],

$$\frac{\cos(k|x-y|)}{|x-y|} = ik \sum_{n=0}^{\infty} (2n+1) j_n(k|y|) i y_n(k|x|) P_n(\cos\alpha),$$
$$i\frac{\sin(k|x-y|)}{|x-y|} = ik \sum_{n=0}^{\infty} (2n+1) j_n(k|y|) j_n(k|x|) P_n(\cos\alpha),$$

the relation [6, Thm. 2.9],

$$\sum_{m=-n}^{n} Y_n^m\left(\frac{x}{|x|}\right) \overline{Y_n^m\left(\frac{y}{|y|}\right)} = \frac{2n+1}{4\pi} P_n(\cos\alpha), \tag{31}$$

and from $h_n^{(1)} = j_n + i y_n$. Here P_n is the Legendre polynomial of order n and α the angle between x and y.

To prove the estimate (30a) we use (31) to get

$$\sum_{m=-n}^{n} \left| h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) j_n(k|y|) Y_n^m\left(\frac{y}{|y|^n}\right) \right|^n \leq \frac{2n+1}{4\pi} \left| h_n^{(1)}(k|x|) j_n(k|y|) \right|.$$

The series representations of h_n and j_n [18, Sect. 10.53] imply

$$j_n(kt) = \frac{(kt)^n}{1 \cdot 3 \cdots (2n+1)} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \quad h_n^{(1)}(kt) = \frac{1 \cdot 3 \cdots (2n-1)}{i(kt)^{n+1}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$$

uniformly for t on compact subsets of $(0, \infty)$ as $n \to \infty$. From this the estimate (30a) follows. The estimates (30b) and (30c) can be proven analogously using the differentiation formula for the spherical Hankel functions [18, Equations 10.51.1] □

Proposition 6.2. *Let $u \in H^1_{\mathrm{loc}}(\Omega^c)$ be a solution of the Helmholtz equation with wavenumber $k \in \mathbb{C} \setminus \{0\}$. Assume that $\Omega^- \subset B_r(0)$ and $x \in \mathbb{R}^3$ with $r < |x| < R$. Then we have*

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{n,m} h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{n,m} j_n(k|x|) Y_n^m\left(\frac{x}{|x|}\right),$$

with

$$A_{\ell,m} = ik \int_{\partial B_r(0)} \left[-j_\ell(k|y|)\partial_n u(y) + \partial_n j_\ell(k|y|) u(y)\right] \overline{Y_\ell^m\left(\frac{y}{|y|}\right)} ds_y,$$

$$B_{\ell,m} = ik \int_{\partial B_R(0)} \left[h_\ell^{(1)}(k|y|)\partial_n u(y) - \partial_n h_\ell^{(1)}(k|y|) u(y)\right] \overline{Y_\ell^m\left(\frac{y}{|y|}\right)} ds_y.$$

Proof. The representation formula for the annulus $B_R(0) \cap \overline{B_r(0)}$ reads

$$u(x) = \int_{\partial B_{r(0)}} \left[-\Phi_k(x,y)\partial_n u(y) + \partial_{n_y}\Phi_k(x,y) u(y)\right] ds_y + \int_{\partial B_R(0)} \left[\Phi_k(x,y)\partial_n u(y) - \partial_{n_y}\Phi_k(x,y) u(y)\right] ds_y.$$

Inserting the series representation (29) of the fundamental solution and interchanging the order of integration and summation, which is possible because of the inequalities in (30), we get the representation of u. □

Theorem 6.3. *Let $u \in H^1_{\mathrm{loc}}(\Omega^c)$ be a solution of the Helmholtz equation for $k \in \mathbb{C} \setminus \{0\}$. Then the following statements are equivalent:*

a) u satisfies the radiation condition (2).

b) For any $R > 0$ such that $\overline{\Omega^-} \subset B_R(0)$ it holds

$$\int_{\partial B_R(0)} \left[\Phi_k(x,y)\partial_n u(y) - \partial_{n_y}\Phi_k(x,y) u(y)\right] ds_y = 0 \tag{32}$$

for all $x \in \Omega^+ \cap B_R$.

Proof. Let us first assume that u satisfies the radiation condition (2), i. e., for any $r_0 > 0$ such that $\Omega^- \subset B_{r_0}$ we have

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{n,m} h_n^1(kr) Y_n^m\left(\frac{x}{|x|}\right) \quad \text{for } r = |x| > r_0.$$

On the other hand, by Proposition 6.2 $u(x)$ has an expansion of the form

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{n,m} h_n^{(1)}(k|x|) Y_n^m\left(\frac{x}{|x|}\right) + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{n,m} j_n(k|x|) Y_n^m\left(\frac{x}{|x|}\right).$$

From these both representations of u it follows for any $\tilde{r} \in (r, R)$ that

$$0 = \int_{\mathbb{S}^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left[(a_{n,m} - A_{n,m}) h_n^{(1)}(k\tilde{r}) + B_{n,m} j_n(k\tilde{r}) \right] Y_n^m(\omega)\, d\omega.$$

Since the spherical harmonics are linearly independent in $L^2(\mathbb{S}^2)$ it follows that

$$0 = (a_{n,m} - A_{n,m}) h_n^{(1)}(k\tilde{r}) + B_{n,m} j_n(k\tilde{r}) \qquad \text{for all } \tilde{r} \in (r, R).$$

The linear independence of $h_n^{(1)}$ and j_n implies that $B_{n,m} = 0$. Hence, we have

$$0 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{n,m} j_n(k|x|) Y_n^m\left(\frac{x}{|x|}\right) = \int_{\partial B_R} \left[\Phi_k(x,y) \partial_n u(y) - \partial_{n_y} \Phi_k(x,y) u(y) \right] ds_y.$$

Let us now assume that u satisfies b). Let $r > 0$ such that $\Omega^- \subset B_r$. For $x \in \mathbb{R}^3$ with $|x| > r$ choose $R > 0$ such that $r < |x| < R$. Then, by the representation formula for $u(x)$ in the annulus $B_R(0) \cap \overline{B_r(0)}$ we get with (32) that

$$u(x) = \int_{\partial B_{r(0)}} \left[-\Phi_k(x,y) \partial_n u(y) + \partial_{n_y} \Phi_k(x,y) u(y) \right] ds_y.$$

Inserting the series representation (29) of the fundamental solution and integrating term by term yields the series representation of u in the form of (2). □

Theorem 6.4. *Let $k \in \mathbb{C} \setminus \{0\}$, $\varphi \in H^{-1/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$, then*

$$u(x) = \frac{1}{4\pi} \int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} \varphi(y) ds_y \qquad \text{and} \qquad v(x) = \frac{1}{4\pi} \int_\Gamma \partial n_y \left(\frac{e^{ik|x-y|}}{|x-y|} \right) \psi(y) ds_y$$

satisfy the radiation condition (2).

Proof. The assertions follow by using the representation (29) of the fundamental solution and by interchanging the order of integration and summation, which is possible because of the inequalities in (30). □

Corollary 6.5. *Let $k \in \mathbb{C} \setminus \{0\}$ and let $u \in H^1_{\mathrm{loc}}(\Omega^+)$ be a solution of the Helmholtz equation in Ω^+ with wavenumber k. Then, u fulfills the radiation condition (2) if and only if u has in Ω^+ a representation of the form*

$$u(x) = \int_\Gamma \left[-\Phi_k(x,y) \partial_n u(y) + \partial_{n_y} \Phi_k(x,y) u(y) \right] ds_y. \tag{33}$$

Proof. If u has a representation as in (33), then u is obviously a solution of the Helmholtz equation in Ω^+ and by Theorem 6.4 it fulfills the radiation condition (2). On the other hand, if u is solution of the Helmholtz equation in Ω^+ and satisfies (2), then the representation formula for u in $\Omega^+ \cap B_R$ for sufficiently large R implies with (32) the representation of u as in (33). □

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