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Robust finite element discretization and solvers for distributed elliptic optimal control problems

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July 9, 2022

Abstract

We consider standard tracking-type, distributed elliptic optimal control problems with L^2 regularization, and their finite element discretization. We are investigating the L^2 error between the finite element approximation $u_{\varrho h}$ of the state u_ϱ and the desired state (target) $\overline{u}$ in terms of the regularization parameter ϱ and the mesh size h that leads to the optimal choice $\varrho = h^4$. It turns out that, for this choice of the regularization parameter, we can devise simple Jacobi-like preconditioned MINRES or Bramble-Pasciak CG methods that allow us to solve the reduced discrete optimality system in asymptotically optimal complexity with respect to the arithmetical operations and memory demand. The theoretical results are confirmed by several benchmark problems with targets of various regularities including discontinuous targets.

Keywords: Elliptic optimal control problems, L^2 regularization, finite element discretization, robust error estimates, robust solvers.

2010 MSC: 49J20, 49M05, 35J05, 65M60, 65M15, 65N22

1 Introduction

Let us consider the optimal control problem: Find the optimal state u_ϱ and the optimal control z_ϱ such that the cost functional

$$\mathcal{J}(u_\varrho, z_\varrho) = \frac{1}{2} \int_\Omega [u_\varrho(x) - \overline{u}(x)]^2 \, dx + \frac{\varrho}{2} \int_\Omega [z_\varrho(x)]^2 \, dx \tag{1.1}$$

is minimized subject to the elliptic boundary value problem

$$-\Delta u_\varrho = z_\varrho \quad \text{in } \Omega, \quad \text{and} \quad u_\varrho = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\varrho > 0$ denotes the regularization parameter, $\overline{u}$ is the given desired state that is nothing but the target which we want to reach, and $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) denotes the computational domain that is assumed to be bounded with Lipschitz boundary $\partial\Omega$. This optimal control problem has a unique solution $u_\varrho \in H_0^1(\Omega)$ and $z_\varrho \in L^2(\Omega)$, where we use the usual notation for Sobolev and Lebesgue spaces; see, e.g., [11, 20].

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This solution can be determined by solving the optimality system consisting of the state (primal) problem (1.2), the gradient equation

$$p_\varrho + \varrho z_\varrho = 0 \quad \text{in } \Omega, \tag{1.3}$$

and the adjoint problem

$$-\Delta p_\varrho = u_\varrho - \overline{u} \quad \text{in } \Omega, \quad p_\varrho = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

for determining the co-state (adjoint) p_ϱ. The control z_ϱ can be eliminated by means of the gradient equation (1.3) that results in the so-called reduced optimality system which is the starting point for our analysis in Section 2. At that point we mention that there are many publications on elliptic optimal control problems such as (1.1)–(1.2), their numerical solution, and their application to practical problems, where often additional constraints, e.g., box constraints imposed on the control, are added. We refer to the monographs [11, 20] for more detailed information on optimal control problems of that kind.

In this paper, we are interested to derive regularization error estimates of the form $\|u_\varrho - \overline{u}\|_{L^2(\Omega)}$ and $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ in terms of the relaxation parameter ϱ and the finite element mesh size h. Optimal control problems without state or control constraints correspond to the solution of inverse problems with Tikhonov regularization in a Hilbert scale; see [12] for related regularization error estimates. In [13] we have given regularization error estimates for $\|u_\varrho - \overline{u}\|_{L^2(\Omega)}$ when considering both the regularization in $L^2(\Omega)$, and in the energy space $H^{-1}(\Omega)$. The order of the convergence rate in the relaxation parameter ϱ does not only depend on the regularity of the given target, but also on the choice of the regularization. In the case of the energy regularization, we have analyzed the finite element discretization of the reduced optimality system in [10]. When combining both error estimates, this results in the choice $\varrho = h^2$ to ensure optimal convergence of the approximate state $u_{\varrho h}$ to the target $\overline{u}$. This optimal choice of the regularization parameter also allows the construction of preconditioned iterative solution methods which are robust with respect to the regularization parameter $\varrho = h^2$, and the finite element mesh size h. In this work we will derive related regularization and finite element error estimates in the case of the L^2 regularization which will result in the optimal choice $\varrho = h^4$. This optimal choice of ϱ allows us to construct robust and, at the same time, very efficient iterative solvers based on diagonal, i.e. Jacobi-like preconditioners. The discretization of the reduced optimality system leads to large-scale systems of finite element equations with symmetric, but indefinite system matrices for determining the nodal vectors for the finite element approximations to the optimal state u_ϱ and optimal co-state (adjoint) p_ϱ. Iterative solvers for such kind of saddle point systems are extensively studied in the literature. We refer the reader to the survey paper [1], the monograph [4], and the more recent papers [14] and [15] for a comprehensive overview on saddle point solvers. Using an operator interpolation technique, Zulehner proposed a block-diagonal preconditioner for the symmetric and indefinite discrete reduced optimality system that is robust with respect to the regularization parameter ϱ [22]. The diagonal blocks of the preconditioner are of the form $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ and $\varrho^{-1}(\mathbf{M}_h + \varrho^{1/2}\mathbf{K})$, where $\mathbf{M}_h$ and $\mathbf{K}_h$ denote the mass and the stiffness matrices, respectively. Replacing now $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ by some ϱ - robust multigrid or multilevel preconditioner as proposed in [16] and [9], this finally leads to a robust and efficient preconditioner for the MINRES solver [17]. Surprisingly, for the optimal choice $\varrho = h^4$ of the regularization parameter, the matrix $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ is spectrally equivalent to the mass matrix $\mathbf{M}_h$, and, therefore, well-conditioned. Now, replacing $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ by some diagonal approximation of the mass matrix, we get a robust and really very cheap preconditioner for MINRES. The same observation lead to robust and asymptotically optimal preconditioners for the Bramble-Pasciak CG [2].

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The remainder of the paper is organized as follows. In Section 2, we derive the L^2 error estimate between the exact state solution u_ϱ of the optimal control problem for fixed ϱ and the desired state $\overline{u}$ in terms of the regularization parameter ϱ, whereas, in Section 3, the same estimates are derived for a finite element approximation $u_{\varrho h}$ to $\overline{u}$. Section 4 is devoted to the construction and analysis of fast and robust iterative solvers. Section 5 presents and discusses numerical results for typical benchmark problems. Finally, in Section 6, we draw some conclusions, and give an outlook on some future research topics.

2 Regularization error estimates

When using the gradient equation (1.3) to eliminate the control z_ϱ, the variational formulation of the primal Dirichlet problem (1.2) is to find $u_\varrho \in H_0^1(\Omega)$ such that

$$\frac{1}{\varrho}\langle p_\varrho, v\rangle_{L^2(\Omega)} + \langle \nabla u_\varrho, \nabla v\rangle_{L^2(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega), \qquad (2.1)$$

while the variational formulation of the adjoint problem (1.4) is to find $p_\varrho \in H_0^1(\Omega)$ such that

$$\langle \nabla p_\varrho, \nabla q\rangle_{L^2(Q)} = \langle u_\varrho - \overline{u}, q\rangle_{L^2(Q)} \quad \text{for all } q \in H_0^1(\Omega). \qquad (2.2)$$

While unique solvability of the coupled variational formulation (2.1) and (2.2) is well established, our particular interest is in estimating the regularization error $\|u_\varrho - \overline{u}\|_{L^2(\Omega)}$ which depends on the regularity of the target $\overline{u}$. As already discussed in [13], we can prove the following result:

Lemma 1. *Let $(u_\varrho, p_\varrho) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the unique solution of the coupled variational formulation (2.1) and (2.2). When assuming $\overline{u} \in L^2(\Omega)$ only, this gives*

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq \|\overline{u}\|_{L^2(\Omega)}. \qquad (2.3)$$

For $\overline{u} \in H_0^1(\Omega)$ there holds

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq \varrho^{1/4}\|\nabla\overline{u}\|_{L^2(\Omega)}. \qquad (2.4)$$

Moreover, if $\overline{u} \in H_0^1(\Omega)$ satisfies $\Delta\overline{u} \in L^2(\Omega)$, then

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq \varrho^{1/2}\|\Delta\overline{u}\|_{L^2(\Omega)}. \qquad (2.5)$$

Proof. When considering the variational formulation (2.2) for $q = u_\varrho$, and (2.1) for $v = p_\varrho$ this gives

$$\langle u_\varrho - \overline{u}, u_\varrho\rangle_{L^2(\Omega)} = \langle \nabla p_\varrho, \nabla u_\varrho\rangle_{L^2(\Omega)} = -\frac{1}{\varrho}\langle p_\varrho, p_\varrho\rangle_{L^2(\Omega)},$$

i.e., we can write

$$\frac{1}{\varrho}\|p_\varrho\|^2_{L^2(\Omega)} + \|u_\varrho - \overline{u}\|^2_{L^2(\Omega)} = \langle \overline{u} - u_\varrho, \overline{u}\rangle_{L^2(\Omega)} \leq \|u_\varrho - \overline{u}\|_{L^2(\Omega)}\|\overline{u}\|_{L^2(\Omega)},$$

and therefore, (2.3) follows.

Next we consider the case $\overline{u} \in H_0^1(\Omega)$. Then we can use (2.2) for $q = u_\varrho - \overline{u}$ and (2.1) for $v = p_\varrho$ to obtain

$$\begin{aligned}\|u_\varrho - \overline{u}\|^2_{L^2(\Omega)} &= \langle u_\varrho - \overline{u}, u_\varrho - \overline{u}\rangle_{L^2(\Omega)} = \langle \nabla p_\varrho, \nabla(u_\varrho - \overline{u})\rangle_{L^2(\Omega)} \\ &= \langle \nabla p_\varrho, \nabla u_\varrho\rangle_{L^2(\Omega)} - \langle \nabla p_\varrho, \nabla\overline{u}\rangle_{L^2(\Omega)} = -\frac{1}{\varrho}\langle p_\varrho, p_\varrho\rangle_{L^2(\Omega)} - \langle \nabla p_\varrho, \nabla\overline{u}\rangle_{L^2(\Omega)},\end{aligned}$$

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i.e.,

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 + \frac{1}{\varrho}\|p_\varrho\|_{L^2(\Omega)}^2 = -\langle \nabla p_\varrho, \nabla \overline{u}\rangle_{L^2(\Omega)} \leq \|\nabla p_\varrho\|_{L^2(\Omega)}\|\nabla \overline{u}\|_{L^2(\Omega)}. \quad (2.6)$$

Now, using (2.2) for $q = p_\varrho$ this gives

$$\|\nabla p_\varrho\|_{L^2(\Omega)}^2 = \langle \nabla p_\varrho, \nabla p_\varrho\rangle_{L^2(\Omega)} = \langle u_\varrho - \overline{u}, p_\varrho\rangle_{L^2(\Omega)} \leq \|u_\varrho - \overline{u}\|_{L^2(\Omega)}\|p_\varrho\|_{L^2(\Omega)},$$

and hence,

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 \leq \|u_\varrho - \overline{u}\|_{L^2(\Omega)}^{1/2}\|p_\varrho\|_{L^2(\Omega)}^{1/2}\|\nabla \overline{u}\|_{L^2(\Omega)}$$

follows, i.e.,

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq \|p_\varrho\|_{L^2(\Omega)}^{1/3}\|\nabla \overline{u}\|_{L^2(\Omega)}^{2/3}. \quad (2.7)$$

Moreover, (2.6) then implies

$$\begin{aligned}
\|p_\varrho\|_{L^2(\Omega)}^2 &\leq \varrho \|\nabla p_\varrho\|_{L^2(\Omega)}\|\nabla \overline{u}\|_{L^2(\Omega)} \\
&\leq \varrho \|u_\varrho - \overline{u}\|_{L^2(\Omega)}^{1/2}\|p_\varrho\|_{L^2(\Omega)}^{1/2}\|\nabla \overline{u}\|_{L^2(\Omega)} \\
&\leq \varrho \|p_\varrho\|_{L^2(\Omega)}^{2/3}\|\nabla \overline{u}\|_{L^2(\Omega)}^{4/3},
\end{aligned}$$

i.e.,

$$\|p_\varrho\|_{L^2(\Omega)} \leq \varrho^{3/4}\|\nabla \overline{u}\|_{L^2(\Omega)}.$$

Now, (2.4) follows when using (2.7).

Finally we assume $\overline{u} \in H_0^1(\Omega)$ satisfying $\Delta \overline{u} \in L^2(\Omega)$. As in the derivation of (2.6) we have

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 = \langle \nabla p_\varrho, \nabla u_\varrho\rangle_{L^2(\Omega)} - \langle \nabla p_\varrho, \nabla \overline{u}\rangle_{L^2(\Omega)}.$$

Now, using integration by parts, inserting $p_\varrho = -\varrho z_\varrho$, and using $z_\varrho = -\Delta u_\varrho$, this gives

$$\begin{aligned}
\|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 &= -\langle p_\varrho, \Delta u_\varrho\rangle_{L^2(\Omega)} + \langle p_\varrho, \Delta \overline{u}\rangle_{L^2(\Omega)} \\
&= \varrho \langle z_\varrho, \Delta u_\varrho\rangle_{L^2(\Omega)} - \varrho \langle z_\varrho, \Delta \overline{u}\rangle_{L^2(\Omega)} \\
&= -\varrho \langle \Delta u_\varrho, \Delta u_\varrho\rangle_{L^2(\Omega)} + \varrho \langle \Delta u_\varrho, \Delta \overline{u}\rangle_{L^2(\Omega)},
\end{aligned}$$

i.e.,

$$\|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 + \varrho \|\Delta u_\varrho\|_{L^2(\Omega)}^2 = \varrho \langle \Delta u_\varrho, \Delta \overline{u}\rangle_{L^2(\Omega)} \leq \varrho \|\Delta u_\varrho\|_{L^2(\Omega)}\|\Delta \overline{u}\|_{L^2(\Omega)},$$

and hence,

$$\|\Delta u_\varrho\|_{L^2(\Omega)} \leq \|\Delta \overline{u}\|_{L^2(\Omega)}, \quad \|u_\varrho - \overline{u}\|_{L^2(\Omega)}^2 \leq \varrho \|\Delta \overline{u}\|_{L^2(\Omega)}^2,$$

i.e., (2.5) follows. $\qquad \square$

Note that the regularization error estimates of Lemma 1 were already given in [13, Theorem 4.1]. But in particular the proof of (2.4) is a bit different to that of [13], resulting in an improved constant, and (2.5) is new. In addition to regularization error estimates in $L^2(\Omega)$ we also need to have related estimates in $H_0^1(\Omega)$ when assuming $\overline{u} \in H_0^1(\Omega)$.

Lemma 2. *Let $(u_\varrho, p_\varrho) \in H_0^1(\Omega) \times H_0^1(\Omega)$ be the unique solution of the coupled variational formulation (2.1) and (2.2). When assuming $\overline{u} \in H_0^1(\Omega)$, this gives*

$$\|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)} \leq \|\nabla \overline{u}\|_{L^2(\Omega)}. \quad (2.8)$$

Moreover, if $\overline{u} \in H_0^1(\Omega)$ satisfies $\Delta \overline{u} \in L^2(\Omega)$, then

$$\|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)} \leq \varrho^{1/4}\|\Delta \overline{u}\|_{L^2(\Omega)}. \quad (2.9)$$

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Proof. Due to the assumption $\overline{u} \in H_0^1(\Omega)$ we can use $v = u_\varrho - \overline{u}$ as test function in (2.1) and $q = p_\varrho$ in (2.2) to obtain

$$\langle \nabla u_\varrho, \nabla(u_\varrho - \overline{u})\rangle_{L^2(\Omega)} = -\frac{1}{\varrho}\langle p_\varrho, u_\varrho - \overline{u}\rangle_{L^2(\Omega)} = -\frac{1}{\varrho}\langle \nabla p_\varrho, \nabla_\varrho p\rangle_{L^2(\Omega)},$$

i.e., we have

$$\begin{aligned}
\frac{1}{\varrho}\|\nabla p_\varrho\|_{L^2(\Omega)}^2 + \|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)}^2 &= \langle \nabla \overline{u}, \nabla(\overline{u} - u_\varrho)\rangle_{L^2(\Omega)} \\
&\leq \|\nabla \overline{u}\|_{L^2(\Omega)}\|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)},
\end{aligned}$$

from which we conclude (2.8).

On the other hand, when assuming $\overline{u} \in H_0^1(\Omega)$ satisfying $\Delta \overline{u} \in L^2(\Omega)$, and when applying integration by parts, we also have

$$\begin{aligned}
\frac{1}{\varrho}\|\nabla p_\varrho\|_{L^2(\Omega)}^2 + \|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)}^2 &= \langle \nabla \overline{u}, \nabla(\overline{u} - u_\varrho)\rangle_{L^2(\Omega)} \\
&= \langle \Delta \overline{u}, u_\varrho - \overline{u}\rangle_{L^2(\Omega)} \leq \|\Delta \overline{u}\|_{L^2(\Omega)}\|u_\varrho - \overline{u}\|_{L^2(\Omega)}.
\end{aligned}$$

Finally, using (2.5), this gives

$$\|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)}^2 \leq \|\Delta \overline{u}\|_{L^2(\Omega)}\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq \varrho^{1/2}\|\Delta \overline{u}\|_{L^2(\Omega)}^2,$$

i.e., (2.9) follows. $\qquad\square$

3 Finite element error estimates

For the numerical solution of the coupled variational formulation (2.1) and (2.2) we first introduce the transformation $p_\varrho(x) = \sqrt{\varrho}\,\widetilde{p}_\varrho(x)$, i.e., we consider the variational formulation to find $(u_\varrho, \widetilde{p}_\varrho) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\frac{1}{\sqrt{\varrho}}\langle \widetilde{p}_\varrho, v\rangle_{L^2(\Omega)} + \langle \nabla u_\varrho, \nabla v\rangle_{L^2(\Omega)} = 0 \quad \text{for all } v \in H_0^1(\Omega), \tag{3.1}$$

and

$$-\langle \nabla \widetilde{p}_\varrho, \nabla q\rangle_{L^2(Q)} + \frac{1}{\sqrt{\varrho}}\langle u_\varrho, q\rangle_{L^2(\Omega)} = \frac{1}{\sqrt{\varrho}}\langle \overline{u}, q\rangle_{L^2(Q)} \quad \text{for all } q \in H_0^1(\Omega). \tag{3.2}$$

Let $V_h = S_h^1(\Omega) \cap H_0^1(\Omega) = \text{span}\{\varphi_{h,k}\}_{k=1}^{N_h} = \text{span}\{\varphi_k\}_{k=1}^{N_h}$ be the finite element space of piecewise linear and continuous basis functions which are defined with respect to some admissible decomposition of the computational domain Ω into shape regular and globally quasi-uniform simplicial finite elements of mesh size h. For simplicity, we omit the subindex h from the basis functions $\varphi_{h,k}$. The Galerkin variational formulation of (3.1) and (3.2) is to find $(u_{\varrho h}, \widetilde{p}_{\varrho h}) \in V_h \times V_h$ such that

$$\frac{1}{\sqrt{\varrho}}\langle \widetilde{p}_{\varrho h}, v_h\rangle_{L^2(\Omega)} + \langle \nabla u_{\varrho h}, \nabla v_h\rangle_{L^2(\Omega)} = 0 \quad \text{for all } v_h \in V_h, \tag{3.3}$$

and

$$-\langle \nabla \widetilde{p}_{\varrho h}, \nabla q_h\rangle_{L^2(Q)} + \frac{1}{\sqrt{\varrho}}\langle u_{\varrho h}, q_h\rangle_{L^2(\Omega)} = \frac{1}{\sqrt{\varrho}}\langle \overline{u}, q_h\rangle_{L^2(Q)} \quad \text{for all } q_h \in V_h. \tag{3.4}$$

The mixed finite element scheme (3.3) and (3.4) has obviously a unique solution. Indeed, choosing the test function $v_h = \widetilde{p}_{\varrho h}$ in (3.3) as well as $q_h = u_{\varrho h}$ in (3.4), and adding both equations, we see that $\widetilde{p}_{\varrho h}$ and $u_{\varrho h}$ must be zero for the homogeneous equations ($\overline{u} = 0$). Now uniqueness always yields existence in the linear finite-dimensional case. So, the corresponding system of algebraic equation also has a unique solution and vice versa; see also Section 4.

Now we are in a position to formulate the main result of this section.

Theorem 1. *Let $(u_{\varrho h}, \widetilde{p}_{\varrho h}) \in V_h \times V_h$ be the unique solution of the coupled finite element variational formulation (3.3) and (3.4). Assume that the underlying finite element mesh is globally quasi-uniform such that an inverse inequality in V_h is valid, and consider $\varrho = h^4$. For $\overline{u} \in H_0^1(\Omega)$ then there holds the error estimate*

$$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)} \leq c\, h\, |\overline{u}|_{H^1(\Omega)}. \tag{3.5}$$

For $\overline{u} \in H_0^1(\Omega) \cap H^2(\Omega)$, and if the domain Ω is either smoothly bounded or convex, then we also have

$$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)} \leq c\, h^2\, |\overline{u}|_{H^2(\Omega)}. \tag{3.6}$$

Proof. For given $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ define $(\varphi_h, \psi_h) \in V_h \times V_h$ as unique solutions satisfying the variational formulations

$$\frac{1}{\sqrt{\varrho}}\, \langle \psi_h, v_h \rangle_{L^2(\Omega)} + \langle \nabla \varphi_h, \nabla v_h \rangle_{L^2(\Omega)} = \frac{1}{\sqrt{\varrho}}\, \langle \psi, v_h \rangle_{L^2(\Omega)} + \langle \nabla \varphi, \nabla v_h \rangle_{L^2(\Omega)}, \ \forall v_h \in V_h,$$

and

$$-\langle \nabla \psi_h, \nabla q_h \rangle_{L^2(\Omega)} + \frac{1}{\sqrt{\varrho}}\, \langle \varphi_h, q_h \rangle_{L^2(\Omega)} = -\langle \nabla \psi, \nabla q_h \rangle_{L^2(\Omega)} + \frac{1}{\sqrt{\varrho}}\, \langle \varphi, q_h \rangle_{L^2(\Omega)}, q_h \in V_h.$$

When using an inverse inequality in V_h, this gives

$$\frac{1}{\sqrt{\varrho}}\, \|\varphi_h\|_{L^2(\Omega)}^2 + \|\nabla \varphi_h\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\, \|\psi_h\|_{L^2(\Omega)}^2 + \|\nabla \psi_h\|_{L^2(\Omega)}^2$$

$$\leq \left(\frac{1}{\sqrt{\varrho}} + c_I\, h^{-2}\right) \|\varphi_h\|_{L^2(\Omega)}^2 + \left(\frac{1}{\sqrt{\varrho}} + c_I\, h^{-2}\right) \|\psi_h\|_{L^2(\Omega)}^2$$

$$= \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right) \left[\frac{1}{\sqrt{\varrho}}\|\varphi_h\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\|\psi_h\|_{L^2(\Omega)}^2\right]$$

$$= \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right) \left[\frac{1}{\sqrt{\varrho}}\langle \varphi_h, \varphi_h \rangle_{L^2(\Omega)} - \langle \nabla \psi_h, \nabla \varphi_h \rangle_{L^2(\Omega)}\right.$$

$$\left. + \langle \nabla \varphi_h, \nabla \psi_h \rangle_{L^2(\Omega)} + \frac{1}{\sqrt{\varrho}}\langle \psi_h, \psi_h \rangle_{L^2(\Omega)}\right]$$

$$= \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right) \left[\frac{1}{\sqrt{\varrho}}\langle \varphi, \varphi_h \rangle_{L^2(\Omega)} - \langle \nabla \psi, \nabla \varphi_h \rangle_{L^2(\Omega)}\right.$$

$$\left. + \langle \nabla \varphi, \nabla \psi_h \rangle_{L^2(\Omega)} + \frac{1}{\sqrt{\varrho}}\langle \psi, \psi_h \rangle_{L^2(\Omega)}\right]$$

$$\leq \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right) \left[\frac{1}{\sqrt{\varrho}}\, \|\varphi\|_{L^2(\Omega)}\|\varphi_h\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)}\|\nabla \varphi_h\|_{L^2(\Omega)}^2\right.$$

$$\left. + \|\nabla \varphi\|_{L^2(\Omega)}\|\nabla \psi_h\|_{L^2(\Omega)} + \frac{1}{\sqrt{\varrho}}\, \|\psi\|_{L^2(\Omega)}\|\psi_h\|_{L^2(\Omega)}\right]$$

$$\leq \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right) \left[\frac{1}{\sqrt{\varrho}}\, \|\varphi\|_{L^2(\Omega)}^2 + \|\nabla \varphi\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\, \|\psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2\right]^{1/2}$$

$$\cdot \left[\frac{1}{\sqrt{\varrho}}\, \|\varphi_h\|_{L^2(\Omega)}^2 + \|\nabla \varphi_h\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\, \|\psi_h\|_{L^2(\Omega)}^2 + \|\nabla \psi_h\|_{L^2(\Omega)}^2\right]^{1/2}.$$

Hence,

$$\frac{1}{\sqrt{\varrho}}\, \|\varphi_h\|_{L^2(\Omega)}^2 + \|\nabla \varphi_h\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\, \|\psi_h\|_{L^2(\Omega)}^2 + \|\nabla \psi_h\|_{L^2(\Omega)}^2$$

$$\leq \left(1 + c_I\, h^{-2}\, \sqrt{\varrho}\right)^2 \left[\frac{1}{\sqrt{\varrho}}\, \|\varphi\|_{L^2(\Omega)}^2 + \|\nabla \varphi\|_{L^2(\Omega)}^2 + \frac{1}{\sqrt{\varrho}}\, \|\psi\|_{L^2(\Omega)}^2 + \|\nabla \psi\|_{L^2(\Omega)}^2\right].$$

In fact, the Galerkin projection $(\varphi, \psi) \mapsto (\varphi_h, \psi_h)$ is bounded, and in particular for $\varrho = h^4$ we therefore conclude Cea's lemma, i.e., for arbitrary $(v_h, q_h) \in V_h \times V_h$ we have

$$
h^{-2} \|u_\varrho - u_{\varrho h}\|^2_{L^2(\Omega)} + \|\nabla(u_\varrho - u_{\varrho h})\|^2_{L^2(\Omega)}
$$
$$
+ h^{-2} \|\widetilde{p}_\varrho - \widetilde{p}_{\varrho h}\|^2_{L^2(\Omega)} + \|\nabla(\widetilde{p}_\varrho - \widetilde{p}_{\varrho h})\|^2_{L^2(\Omega)}
$$
$$
\leq c \cdot \left[h^{-2} \|u_\varrho - v_h\|^2_{L^2(\Omega)} + \|\nabla(u_\varrho - v_h)\|^2_{L^2(\Omega)} \right.
$$
$$
\left. + h^{-2} \|\widetilde{p}_\varrho - q_h\|^2_{L^2(\Omega)} + \|\nabla(\widetilde{p}_\varrho - q_h)\|^2_{L^2(\Omega)} \right].
$$

For $\overline{u} \in H^1_0(\Omega)$ we can consider $v_h = \overline{u}_h$ being a suitable approximation, e.g., Scott–Zhang interpolation [3]. Then, using (2.4), this gives, recall $\varrho = h^4$,

$$
\begin{aligned}
\|u_\varrho - \overline{u}_h\|_{L^2(\Omega)} &\leq \|u_\varrho - \overline{u}\|_{L^2(\Omega)} + \|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \\
&\leq \varrho^{1/4} |\overline{u}|_{H^1(\Omega)} + c\, h\, |\overline{u}|_{H^1(\Omega)} \leq c\, h\, |\overline{u}|_{H^1(\Omega)}.
\end{aligned}
$$

Moreover, now using (2.8), we also have

$$
\|\nabla(u_\varrho - \overline{u}_h)\|_{L^2(\Omega)} \leq \|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)} + \|\nabla(\overline{u} - \overline{u}_h)\|_{L^2(\Omega)} \leq (1+c) \|\nabla \overline{u}\|_{L^2(\Omega)}.
$$

Correspondingly, let $q_h = \Pi_h \widetilde{p}_\varrho$ be some suitable approximation, e.g., again the Scott–Zhang interpolation. Then,

$$
\|\widetilde{p}_\varrho - \Pi_h \widetilde{p}_\varrho\|_{L^2(\Omega)} \leq c\, h\, \|\nabla \widetilde{p}_\varrho\|_{L^2(\Omega)} = c\, h\, \frac{1}{\sqrt{\varrho}} \|\nabla p_\varrho\|_{L^2(\Omega)}.
$$

From (2.2), and using duality we first have

$$
\|\nabla p_\varrho\|^2_{L^2(\Omega)} = \langle \nabla p_\varrho, \nabla p_\varrho \rangle_{L^2(\Omega)} = \langle u_\varrho - \overline{u}, p_\varrho \rangle_{L^2(\Omega)} \leq \|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)(\Omega)} \|\nabla p_\varrho\|_{L^2(\Omega)},
$$

and with [13, Theorem 4.1] this gives

$$
\|\nabla p_\varrho\|_{L^2(\Omega)} \leq \|u_\varrho - \overline{u}\|_{H^{-1}(\Omega)} \leq \sqrt{\varrho}\, |\overline{u}|_{H^1(\Omega)}.
$$

Hence we conclude
$$
\|\widetilde{p}_\varrho - \Pi_h \widetilde{p}_\varrho\|_{L^2(\Omega)} \leq c\, h\, |\overline{u}|_{H^1(\Omega)}.
$$

In the same way as above we also obtain

$$
\|\nabla(\widetilde{p}_\varrho - \Pi_h \widetilde{p}_\varrho)\|_{L^2(\Omega)} \leq \|\nabla \widetilde{p}_\varrho\|_{L^2(\Omega)} = \frac{1}{\sqrt{\varrho}} \|\nabla p_\varrho\|_{L^2(\Omega)} \leq |\overline{u}|_{H^1(\Omega)}.
$$

Now, summing up all contributions this gives

$$
h^{-2} \|u_\varrho - u_{\varrho h}\|^2_{L^2(\Omega)} + \|\nabla(u_\varrho - u_{\varrho h})\|^2_{L^2(\Omega)}
$$
$$
+ h^{-2} \|\widetilde{p}_\varrho - \widetilde{p}_{\varrho h}\|^2_{L^2(\Omega)} + \|\nabla(\widetilde{p}_\varrho - \widetilde{p}_{\varrho h})\|^2_{L^2(\Omega)} \leq c\, |\overline{u}|^2_{H^1(\Omega)},
$$

in particular we have

$$
\|u_\varrho - u_{\varrho h}\|^2_{L^2(\Omega)} \leq c\, h^2\, |\overline{u}|^2_{H^1(\Omega)}.
$$

Hence, (3.5) follows from, again using (2.4) and $\varrho = h^4$,

$$
\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)} \leq \|u_{\varrho h} - u_\varrho\|_{L^2(\Omega)} + \|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq c\, h\, |\overline{u}|_{H^1(\Omega)} + \varrho^{1/4} |\overline{u}|_{H^1(\Omega)}.
$$

It remains to consider $\overline{u} \in H_0^1(\Omega) \cap H^2(\Omega)$. We proceed as above, but with (2.5) we first obtain

$$
\begin{aligned}
\|u_\varrho - \overline{u}_h\|_{L^2(\Omega)} &\leq \|u_\varrho - \overline{u}\|_{L^2(\Omega)} + \|\overline{u} - \overline{u}_h\|_{L^2(\Omega)} \\
&\leq \varrho^{1/2} \|\Delta \overline{u}\|_{L^2(\Omega)} + c\,h^2 \,|\overline{u}|_{H^2(\Omega)} \leq c\,h^2 \,|\overline{u}|_{H^2(\Omega)},
\end{aligned}
$$

while with (2.9) we conclude

$$
\begin{aligned}
\|\nabla(u_\varrho - \overline{u}_h)\|_{L^2(\Omega)} &\leq \|\nabla(u_\varrho - \overline{u})\|_{L^2(\Omega)} + \|\nabla(\overline{u} - \overline{u}_h)\|_{L^2(\Omega)} \\
&\leq \varrho^{1/4} \|\Delta \overline{u}\|_{L^2(\Omega)} + c\,h\,|\overline{u}|_{H^2(\Omega)} \leq c\,h\,|\overline{u}|_{H^2(\Omega)}.
\end{aligned}
$$

Moreover, we also have, in the case of a smoothly bounded or convex domain Ω, and using (2.5),

$$
\begin{aligned}
\|\widetilde{p}_\varrho - \Pi_h \widetilde{p}_\varrho\|_{L^2(\Omega)} &\leq c\,h^2\,|\widetilde{p}_\varrho|_{H^2(\Omega)} = c\,h^2\,\frac{1}{\sqrt{\varrho}}\,|p_\varrho|_{H^2(\Omega)} \leq c\,h^2\,\frac{1}{\sqrt{\varrho}}\,\|\Delta p_\varrho\|_{L^2(\Omega)} \\
&= c\,h^2\,\frac{1}{\sqrt{\varrho}}\,\|u_\varrho - \overline{u}\|_{L^2(\Omega)} \leq c\,h^2\,\|\Delta \overline{u}\|_{L^2(\Omega)}.
\end{aligned}
$$

Finally, and following the above estimates, we have

$$
\|\nabla(\widetilde{p}_\varrho - \Pi_h \widetilde{p}_\varrho)\|_{L^2(\Omega)} \leq c\,h\,|\widetilde{p}_\varrho|_{H^2(\Omega)} = c\,h\,\frac{1}{\sqrt{\varrho}}\,|p_\varrho|_{H^2(\Omega)} \leq c\,h\,\|\Delta \overline{u}\|_{L^2(\Omega)}.
$$

Again, summing up all contributions this gives

$$
\begin{aligned}
h^{-2}\,\|u_\varrho - u_{\varrho h}\|_{L^2(\Omega)}^2 &+ \|\nabla(u_\varrho - u_{\varrho h})\|_{L^2(\Omega)}^2 \\
&+ h^{-2}\,\|\widetilde{p}_\varrho - \widetilde{p}_{\varrho h}\|_{L^2(\Omega)}^2 + \|\nabla(\widetilde{p}_\varrho - \widetilde{p}_{\varrho h})\|_{L^2(\Omega)}^2 \leq c\,h^2 |\overline{u}|_{H^2(\Omega)}^2,
\end{aligned}
$$

i.e.,

$$
\|u_\varrho - u_{\varrho h}\|_{L^2(\Omega)}^2 \leq c\,h^4\,|\overline{u}|_{H^2(\Omega)}.
$$

Now, (3.6) follows from the triangle inequality and using (2.9). $\qquad\square$

In order to derive error estimates for less regular targets $\overline{u}$ we also need the following result.

Lemma 3. *Let $(u_{\varrho h}, \widetilde{p}_{\varrho h}) \in V_h \times V_h$ be the unique solution of the coupled finite element variational formulation (3.3) and (3.4). For $\overline{u} \in L^2(\Omega)$ there holds the error estimate*

$$
\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)} \leq \|\overline{u}\|_{L^2(\Omega)}. \tag{3.7}
$$

Proof. We consider the Galerkin formulations (3.3) and (3.4) for the particular test functions $v_h = \widetilde{p}_{\varrho h}$ and $q_h = u_{\varrho h}$ to obtain

$$
\frac{1}{\sqrt{\varrho}}\,\langle \widetilde{p}_{\varrho h}, \widetilde{p}_{\varrho h}\rangle_{L^2(\Omega)} + \langle \nabla u_{\varrho h}, \nabla \widetilde{p}_{\varrho h}\rangle_{L^2(\Omega)} = 0,
$$

and

$$
-\langle \nabla \widetilde{p}_{\varrho h}, \nabla u_{\varrho h}\rangle_{L^2(Q)} + \frac{1}{\sqrt{\varrho}}\,\langle u_{\varrho h}, u_{\varrho h}\rangle_{L^2(\Omega)} = \frac{1}{\sqrt{\varrho}}\,\langle \overline{u}, u_{\varrho h}\rangle_{L^2(Q)}.
$$

When summing up both expressions this gives

$$
\langle \widetilde{p}_{\varrho h}, \widetilde{p}_{\varrho h}\rangle_{L^2(\Omega)} + \langle u_{\varrho h}, u_{\varrho h}\rangle_{L^2(\Omega)} = \langle \overline{u}, u_{\varrho h}\rangle_{L^2(\Omega)},
$$

which we can write as

$$
\langle \widetilde{p}_{\varrho h}, \widetilde{p}_{\varrho h}\rangle_{L^2(\Omega)} + \langle u_{\varrho h} - \overline{u}, u_{\varrho h} - \overline{u}\rangle_{L^2(\Omega)} = \langle \overline{u} - u_{\varrho h}, \overline{u}\rangle_{L^2(\Omega)}.
$$

From this we conclude (3.7). $\qquad\square$

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Now, using an interpolation argument, we can prove the final error estimate.

Corollary 1. *Let $(u_{\varrho h}, \widetilde{p}_{\varrho h}) \in V_h \times V_h$ be the unique solution of the coupled finite element variational formulation (3.3) and (3.4). Let $\overline{u} \in H_0^s(\Omega)$ for $s \in [0,1]$ or $\overline{u} \in H_0^1(\Omega) \cap H^s(\Omega)$ for $s \in (1,2]$, and let $\varrho = h^4$. Then,*

$$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)} \leq c\, h^s\, \|\overline{u}\|_{H^s(\Omega)}. \tag{3.8}$$

4 Robust solvers

The finite element variational formulation (3.3) and (3.4) is equivalent to a coupled linear system of algebraic equations,

$$\frac{1}{\sqrt{\varrho}}\, \mathbf{M}_h \widetilde{\mathbf{p}} + \mathbf{K}_h \mathbf{u} = \mathbf{0}, \quad \text{and} \quad -\mathbf{K}_h \widetilde{\mathbf{p}} + \frac{1}{\sqrt{\varrho}}\, \mathbf{M}_h \mathbf{u} = \frac{1}{\sqrt{\varrho}}\, \mathbf{f}, \tag{4.1}$$

where $\mathbf{K}_h$ and $\mathbf{M}_h$ are the standard finite element stiffness and mass matrices, the entries of which are given by

$$\mathbf{K}_h[\ell, k] = \int_\Omega \nabla \varphi_k \cdot \nabla \varphi_\ell \, dx \quad \text{and} \quad \mathbf{M}_h[\ell, k] = \int_\Omega \varphi_k\, \varphi_\ell \, dx \quad \text{for } k, \ell = 1, \dots, N_h,$$

respectively. The load vector $\mathbf{f} = (f_\ell)_{\ell=1,\dots,N_h} \in \mathbb{R}^{N_h}$ is given by its entries

$$f_\ell = \int_\Omega \overline{u}\, \varphi_\ell \, dx \quad \text{for } \ell = 1, \dots, N_h.$$

The stiffness matrix $\mathbf{K}_h$ and the mass matrix $\mathbf{M}_h$ are symmetric and positive definite. Furthermore, they satisfy the spectral inequalities

$$\underline{c}_{\mathrm{K}} h^d (\mathbf{v}, \mathbf{v}) \leq \lambda_{\min}(\mathbf{K})(\mathbf{v}, \mathbf{v}) \leq (\mathbf{K}\mathbf{v}, \mathbf{v}) \leq \lambda_{\max}(\mathbf{K})(\mathbf{v}, \mathbf{v}) \leq \overline{c}_{\mathrm{K}} h^{d-2}(\mathbf{v}, \mathbf{v}) \tag{4.2}$$

and

$$\underline{c}_{\mathrm{M}} h^d (\mathbf{v}, \mathbf{v}) \leq \lambda_{\min}(\mathbf{M})(\mathbf{v}, \mathbf{v}) \leq (\mathbf{M}\mathbf{v}, \mathbf{v}) \leq \lambda_{\max}(\mathbf{M})(\mathbf{v}, \mathbf{v}) \leq \overline{c}_{\mathrm{M}} h^d (\mathbf{v}, \mathbf{v}) \tag{4.3}$$

for all $\mathbf{v} \in \mathbb{R}^{N_h}$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ always denote the minimal and maximal eigenvalues of the corresponding matrices, respectively, and $(\cdot, \cdot) = (\cdot, \cdot)_{\mathbb{R}^{N_h}}$ denotes the Euclidean inner product in the Euclidean vector space $\mathbb{R}^{N_h}$. The positive constants $\underline{c}_{\mathrm{K}}$, $\overline{c}_{\mathrm{K}}$, $\underline{c}_{\mathrm{M}}$, and $\overline{c}_{\mathrm{M}}$ are independent of the mesh-size h; see, e.g., [5, 18].

Since the mass matrix $\mathbf{M}_h$ is invertible, we can eliminate the modified adjoint $\widetilde{\mathbf{p}}$, and hence we can rewrite (4.1) as Schur complement system

$$\left[\varrho\, \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h + \mathbf{M}_h \right] \mathbf{u} = \mathbf{f}. \tag{4.4}$$

The Schur complement $\mathbf{S}_h = \varrho\, \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h + \mathbf{M}_h$ is obviously symmetric and positive definite, and hence invertible. Therefore, the coupled system (4.1) also has a unique solution $\widetilde{\mathbf{p}} = (\widetilde{p}_k)_{k=1,\dots,N_h}$, $\mathbf{u} = (u_k)_{k=1,\dots,N_h} \in \mathbb{R}^{N_h}$ which delivers the nodal parameters for the unique solution $\widetilde{p}_{\varrho h} = \sum_{k=1}^{N_h} \widetilde{p}_k \varphi_k$ and $u_{\varrho h} = \sum_{k=1}^{N_h} u_k \varphi_k$ of the mixed finite element scheme (3.3) and (3.4) as we have already mentioned in Section 3. Let $0 < \lambda_{\min}(\mathbf{M}_h^{-1} \mathbf{K}_h) = \lambda_1 \leq \dots \leq \lambda_{N_h} = \lambda_{\max}(\mathbf{M}_h^{-1} \mathbf{K}_h)$ be the eigenvalues, and $\mathbf{e}_1, \dots, \mathbf{e}_{N_h} \in \mathbb{R}^{N_h}$ the corresponding eigenvectors of the generalized eigenvalue problem

$$\mathbf{K}_h \mathbf{e} = \lambda\, \mathbf{M}_h \mathbf{e}, \tag{4.5}$$

and let us suppose that the eigenvectors $\mathbf{e}_1, \dots, \mathbf{e}_{N_h}$ are orthonormal with respect to the $\mathbf{M}_h$-energy inner product $(\mathbf{M}_h \cdot, \cdot)$, i.e., $(\mathbf{M}_h \mathbf{e}_i, \mathbf{e}_j) = \delta_{i,j}$ for all $i, j = 1, \dots, N_h$.

It is well known, see also (4.2) and (4.3), that there exists positive constants $\underline{c}_{\mathrm{MK}}$ and $\bar{c}_{\mathrm{MK}}$, which are independent of h, such that

$$\underline{c}_{\mathrm{MK}} \leq \lambda_{\min}(\mathbf{M}_h^{-1}\mathbf{K}_h) \quad \text{and} \quad \lambda_{\max}(\mathbf{M}_h^{-1}\mathbf{K}_h) \leq \bar{c}_{\mathrm{MK}}h^{-2}. \tag{4.6}$$

These bounds are sharp with respect to h.

Lemma 4. *If $\varrho = h^4$, then the Schur complement $\mathbf{S}_h$ is spectrally equivalent to the mass matrix $\mathbf{M}_h$. More precisely, there hold the spectral equivalence inequalities*

$$\underline{c}_{MS}(\mathbf{M}_h\mathbf{v}, \mathbf{v}) \leq (\mathbf{S}_h\mathbf{v}, \mathbf{v}) \leq \bar{c}_{MS}(\mathbf{M}_h\mathbf{v}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbb{R}^{N_h}, \tag{4.7}$$

where $\underline{c}_{MS} = 1$ and $\bar{c}_{MS} = \bar{c}_{MK}^2 + 1$, with $\bar{c}_{MK}$ from (4.6).

Proof. Let $\mathbf{v} \in \mathbb{R}^{N_h}$ be arbitrary, and let us expand $\mathbf{v} = \sum_{i=1}^{N_h} v_i^e \mathbf{e}_i$ in the orthonormal eigenvector basis, where $v_i^e = (\mathbf{M}_h\mathbf{v}, \mathbf{e}_i)$. Then we can represent $(\mathbf{S}_h\mathbf{v}, \mathbf{v})$ in the form

$$\begin{aligned}
(\mathbf{S}_h\mathbf{v}, \mathbf{v}) &= \sum_{i,j=1}^{N_h} v_i^e v_j^e (\varrho \mathbf{K}_h\mathbf{M}_h^{-1}\mathbf{K}_h\mathbf{e}_i + \mathbf{M}_h\mathbf{e}_i, \mathbf{e}_j) \\
&= \sum_{i,j=1}^{N_h} v_i^e v_j^e (\varrho\lambda_i^2 + 1)(\mathbf{M}_h\mathbf{e}_i, \mathbf{e}_j) = \sum_{i=1}^{N_h}(v_i^e)^2(\varrho\lambda_i^2 + 1).
\end{aligned}$$

Using now the identity $(\mathbf{M}_h\mathbf{v}, \mathbf{v}) = \sum_{i=1}^{N_h}(v_i^e)^2$, and the bounds (4.6), we immediately get the estimates

$$(\mathbf{S}_h\mathbf{v}, \mathbf{v}) \geq (\varrho\lambda_{\min}(\mathbf{M}_h^{-1}\mathbf{K}_h)^2 + 1)\sum_{i=1}^{N_h}(v_i^e)^2 \geq (\varrho\underline{c}_{\mathrm{MK}}^2 + 1)(\mathbf{M}_h\mathbf{v}, \mathbf{v}) \geq (\mathbf{M}_h\mathbf{v}, \mathbf{v})$$

and

$$(\mathbf{S}_h\mathbf{v}, \mathbf{v}) \leq (\varrho\lambda_{\max}(\mathbf{M}_h^{-1}\mathbf{K}_h)^2 + 1)\sum_{i=1}^{N_h}(v_i^e)^2 \leq (\varrho\bar{c}_{\mathrm{MK}}^2 h^{-4} + 1)(\mathbf{M}_h\mathbf{v}, \mathbf{v}),$$

from which the spectral equivalence inequalities (4.7) follow for $\varrho = h^4$. $\qquad\square$

Since $\mathbf{S}_h = \varrho\mathbf{K}_h\mathbf{M}_h^{-1}\mathbf{K}_h + \mathbf{M}_h$ is spectrally equivalent to the mass matrix $\mathbf{M}_h$, we can efficiently solve the symmetric and positive definite Schur complement system (4.4) by means of the Preconditioned Conjugate Gradient (PCG) method with the symmetric and positive definite preconditioner $\mathbf{C}_h = \mathbf{M}_h$. Using the well-known convergence estimate for the PCG method (see, e.g., [18, Chapter 13]), and the spectral equivalence inequalities (4.7) from Lemma 4, we arrive at estimates of the iteration error in the $L^2(\Omega)$-norm for the corresponding finite element functions:

$$\begin{aligned}
\|u_{\varrho h} - u_{\varrho h}^n\|_{L^2(\Omega)} &= \|\mathbf{u} - \mathbf{u}^n\|_{\mathbf{M}_h} := (\mathbf{M}_h(\mathbf{u} - \mathbf{u}^n), \mathbf{u} - \mathbf{u}^n)^{1/2} \\
&\leq \underline{c}_{\mathrm{MS}}^{-1/2}(\mathbf{S}_h(\mathbf{u} - \mathbf{u}^n), \mathbf{u} - \mathbf{u}^n)^{1/2} \\
&= \underline{c}_{\mathrm{MS}}^{-1/2}\|\mathbf{u} - \mathbf{u}^n\|_{\mathbf{S}_h} \leq \underline{c}_{\mathrm{MS}}^{-1/2} 2q^n \|\mathbf{u} - \mathbf{u}^0\|_{\mathbf{S}_h} \\
&\leq \left(\frac{\bar{c}_{\mathrm{MS}}}{\underline{c}_{\mathrm{MS}}}\right)^{1/2} 2q^n \|\mathbf{u} - \mathbf{u}^0\|_{\mathbf{M}_h} \\
&= \left(\frac{\bar{c}_{\mathrm{MS}}}{\underline{c}_{\mathrm{MS}}}\right)^{1/2} 2q^n \|u_{\varrho h} - u_{\varrho h}^0\|_{L^2(\Omega)},
\end{aligned}$$

where

$$q = \frac{\sqrt{\mathrm{cond}_2(\mathbf{M}_h^{-1}\mathbf{S}_h)} - 1}{\sqrt{\mathrm{cond}_2(\mathbf{M}_h^{-1}\mathbf{S}_h)} + 1}, \quad \text{with } \mathrm{cond}_2(\mathbf{M}_h^{-1}\mathbf{S}_h) = \frac{\lambda_{\max}(\mathbf{M}_h^{-1}\mathbf{S}_h)}{\lambda_{\min}(\mathbf{M}_h^{-1}\mathbf{S}_h)} \le \frac{\bar{c}_{\mathrm{MS}}}{\underline{c}_{\mathrm{MS}}}.$$

Since we can take $\underline{c}_{\mathrm{MS}} = 1$, we finally arrive at the $L^2(\Omega)$ iteration error estimate

$$\|u_{\varrho h} - u_{\varrho h}^n\|_{L^2(\Omega)} \le 2\,(\bar{c}_{\mathrm{MS}})^{1/2}\,\bar{q}^n\,\|u_{\varrho h} - u_{\varrho h}^0\|_{L^2(\Omega)}, \qquad (4.8)$$

where

$$\bar{q} = \frac{\sqrt{\bar{c}_{\mathrm{MS}}} - 1}{\sqrt{\bar{c}_{\mathrm{MS}}} + 1} = \frac{\sqrt{\bar{c}_{\mathrm{MK}}^2 + 1} - 1}{\sqrt{\bar{c}_{\mathrm{MK}}^2 + 1} + 1} < 1$$

is independent of h, using $\bar{c}_{\mathrm{MK}}$ from (4.6). Here the finite element functions $u_{\varrho h}(x) = \sum_{k=1}^{N_h} u_k \varphi_k(x)$ and $u_{\varrho h}^n(x) = \sum_{k=1}^{N_h} u_k^n \varphi_k(x)$ correspond to the solution $\mathbf{u} = (u_k)_{k=1,\dots,N_h} \in \mathbb{R}^{N_h}$ of the Schur complement system (4.4) and the n-th PCG iterate $\mathbf{u}^n = (u_k^n)_{k=1,\dots,N_h} \in \mathbb{R}^{N_h}$, respectively.

Now, using the triangle inequality, the $L^2(\Omega)$-norm discretization error estimate (3.8), and the $L^2(\Omega)$-norm iteration error estimate (4.8), we finally get

$$
\begin{aligned}
\|\bar{u} - u_{\varrho h}^n\|_{L^2(\Omega)} &\le \|\bar{u} - u_{\varrho h}\|_{L^2(\Omega)} + \|u_{\varrho h} - u_{\varrho h}^n\|_{L^2(\Omega)} \\
&\le c\,h^s\,\|\bar{u}\|_{H^s(\Omega)} + 2\,(\bar{c}_{\mathrm{MS}})^{1/2}\,\bar{q}^n\,\|u_{\varrho h} - u_{\varrho h}^0\|_{L^2(\Omega)} \\
&\le h^s\Big(c\,\|\bar{u}\|_{H^s(\Omega)} + 2\,(\bar{c}_{\mathrm{MS}})^{1/2}\,\|u_{\varrho h} - u_{\varrho h}^0\|_{L^2(\Omega)}\Big)
\end{aligned}
$$

provided that $\bar{q}^n \le h^s$ that yields $n \ge \ln h^{-s}/\ln \bar{q}^{-1}$. Since $\bar{q}$ is independent of h, the number of PCG iterations only logarithmically grows with respect to h in order to obtain the total error in the order $O(h^s)$ of the discretization error. This logarithmical growth can be avoided in a nested iteration setting on a sequence of grids; see, e.g, [6].

However, each Schur Complement PCG iteration step requires the solution of two systems of algebraic equations with the symmetric and positive definite well-conditioned mass matrix $\mathbf{M}_h$ as system matrix, namely,

1. in the matrix-vector multiplication $\mathbf{S}_h \mathbf{u}^n = \varrho\,\mathbf{K}_h\mathbf{M}_h^{-1}\mathbf{K}_h\mathbf{u}^n + \mathbf{M}_h\mathbf{u}^n$,

2. and in the preconditioning step $\mathbf{M}_h\mathbf{w}^n = \mathbf{r}^n$.

Thus, in a preprocessing step, we can factorize $\mathbf{M}_h$, e.g., by means of the $\mathbf{L}_h\mathbf{D}_h\mathbf{L}_h^T$ or the Cholesky factorization, and then use fast forward and backward substitutions at each iteration step. In the preconditioning step, we can avoid the solution of the preconditioning system $\mathbf{C}_h\mathbf{w}^n = \mathbf{r}^n$ with $\mathbf{C}_h = \mathbf{M}_h$ by replacing $\mathbf{C}_h$ with a spectrally equivalent preconditioner such as $\mathbf{C}_h = \mathrm{diag}(\mathbf{M}_h)$, $\mathbf{C}_h = \mathrm{lump}(\mathbf{M}_h)$, $\mathbf{C}_h = \mathrm{area}(\mathbf{M}_h)$, or even $\mathbf{C}_h = h^d\mathbf{I}_h$; cf. (4.3). Here, $\mathrm{diag}(\mathbf{M}_h)$ is the diagonal matrix with the diagonal entries from $\mathbf{M}_h$, $\mathrm{lump}(\mathbf{M}_h)$ is the lumped mass matrix that is diagonal; see, e.g., [19], $\mathbf{C}_h = \mathrm{area}(\mathbf{M}_h)$ denotes the diagonal matrix with the k-th diagonal entry which coincides with the area of the support of the basis function φ_k; see [18, Lemma 9.7] that also provides the spectral equivalence constants, and $\mathbf{I}_h$ is the identity matrix. If we replace $\mathbf{M}_h^{-1}$ in the Schur complement system (4.4) by the diagonal matrix $(\mathrm{lump}(\mathbf{M}_h))^{-1}$, then the solution of the corresponding inexact Schur complement system

$$\Big[\varrho\,\mathbf{K}_h(\mathrm{lump}(\mathbf{M}_h))^{-1}\mathbf{K}_h + \mathbf{M}_h\Big]\tilde{\mathbf{u}} = \mathbf{f} \qquad (4.9)$$

by means of the PCG preconditioned by $\mathrm{diag}(\mathbf{M}_h)$, $\mathrm{lump}(\mathbf{M}_h)$ or $\mathbf{C}_h = \mathrm{area}(\mathbf{M}_h)$, is obviously of asymptotically optimal complexity $\mathcal{O}(N_h \ln \varepsilon^{-1})$ for some fixed relative

accuracy $\varepsilon \in (0, 1)$. However, the use of mass lumping in the discretization needs an additional analysis that goes beyond the scope of this paper. Instead, in Section 5, we compare the discretization error and the efficiency of this inexact Schur comlement PCG, that we call inexSCPCG, with the discretization error without mass lumping corresponding to the solution of (4.1) and the efficiency of the other solvers proposed below.

The solution of a mass matrix system in the matrix-vector multiplication $\mathbf{S}_h \mathbf{u}^n$ can also be avoided by a mixed reformulation as a system of double size with a symmetric, but indefinite system matrix that reads as follows: Find $(\mathbf{u}, \widehat{\mathbf{p}}) \in \mathbb{R}^{N_h} \times \mathbb{R}^{N_h}$ such that

$$\begin{bmatrix} \mathbf{M}_h & \mathbf{K}_h \\ \mathbf{K}_h & -\varrho^{-1}\mathbf{M}_h \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \widehat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \tag{4.10}$$

where $\widehat{\mathbf{p}} = -\sqrt{\varrho}\, \widetilde{\mathbf{p}} = -\mathbf{p}$. The symmetric and indefinite system (4.10) is obviously equivalent to the non-symmetric and positive definite system (4.1), and, therefore, to the Schur complement system (4.4).

The block-diagonal matrix

$$\mathcal{P}_h = \begin{bmatrix} \mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h & \mathbf{0} \\ \mathbf{0} & \varrho^{-1}(\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h) \end{bmatrix} \tag{4.11}$$

provides a preconditioner for the MINRES solver that is robust with respect to the mesh size h and the regularization parameter ϱ. This result is proven in [22], where also the corresponding convergence rate estimates are given. In order to obtain an efficient, but at the same time robust preconditioner, we have to replace the block-entry $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ by a spectrally equivalent preconditioner $\mathbf{C}_h$ such that the spectral equivalence constants do not depend neither on h nor on ϱ. Symmetric and positive definite multigrid preconditioners based on symmetric V- or W-cycles are certainly suitable candidates. We came back to this choice later.

We recall that the optimal choice of the regularization parameter is $\varrho = h^4$. Using again the eigenvector expansion $\mathbf{v} = \sum_{i=1}^{N_h} v_i^e \mathbf{e}_i$, we get

$$((\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h)\mathbf{v}, \mathbf{v}) = \sum_{i=1}^{N_h} (1 + \varrho^{1/2}\lambda_i)(v_i^e)^2 = \sum_{i=1}^{N_h} (1 + h^2\lambda_i)(v_i^e)^2 \tag{4.12}$$

for all $\mathbf{v} \in \mathbb{R}^{N_h}$. This representation and the eigenvalue estimates (4.6) immediately yield the spectral equivalence inequalities

$$(1 + h^2\underline{c}_{\mathrm{MK}})\,(\mathbf{M}_h\mathbf{v}, \mathbf{v}) \leq ((\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h)\mathbf{v}, \mathbf{v}) \leq (1 + \overline{c}_{\mathrm{MK}})\,(\mathbf{M}_h\mathbf{v}, \mathbf{v}), \; \forall \mathbf{v} \in \mathbb{R}^{N_h},$$

which we now rewrite in short form as

$$\underline{c}_{\mathrm{MA}}\,\mathbf{M}_h \leq \mathbf{A_h} \leq \overline{c}_{\mathrm{MA}}\,\mathbf{M}_h, \tag{4.13}$$

where $\mathbf{A_h} = \mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h = \mathbf{M}_h + h^2\mathbf{K}_h$, $\underline{c}_{\mathrm{MA}} = 1 + h^2\underline{c}_{\mathrm{MK}} \geq 1$, and $\overline{c}_{\mathrm{MA}} = 1 + \overline{c}_{\mathrm{MK}}$. Replacing now the mass matrix $\mathbf{M}_h$ by the diagonal preconditioner $\mathbf{C}_h = \mathrm{diag}(\mathbf{M}_h)$, we see that

$$\underline{c}_{\mathrm{CA}}\,\mathbf{C}_h \leq \mathbf{A_h} \leq \overline{c}_{\mathrm{CA}}\,\mathbf{C}_h, \tag{4.14}$$

with h-independent, positive constants $\underline{c}_{\mathrm{CA}}$ and $\overline{c}_{\mathrm{CA}}$. These spectral equivalence inequalities easily follow from (4.13) and the spectral equivalence of $\mathbf{M}_h$ to $\mathrm{diag}(\mathbf{M}_h)$. Therefore, the diagonal preconditioner

$$\mathcal{P}_{\mathrm{diag},h} = \begin{bmatrix} \mathbf{C}_h & \mathbf{0} \\ \mathbf{0} & h^{-4}\mathbf{C}_h \end{bmatrix} = \begin{bmatrix} \mathrm{diag}(\mathbf{M}_h) & \mathbf{0} \\ \mathbf{0} & h^{-4}\mathrm{diag}(\mathbf{M}_h) \end{bmatrix} \tag{4.15}$$

is spectrally equivalent to $\mathcal{P}_h$. Thus, the MINRES preconditioned by $\mathcal{P}_{\mathrm{diag},h}$ converges with a rate that is independent of h, and with an asymptotically optimal complexity $\mathcal{O}(N_h \ln \varepsilon^{-1})$ for some fixed relative accuracy $\varepsilon \in (0,1)$. We call this solver $\mathcal{P}_{\mathrm{diag}}$MINRES. Numerical tests underpinning our theoretical results are presented in Section 5. If we replace $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ by $\mathbf{C}_h = \mathrm{lump}(\mathbf{M}_h)$, $\mathbf{C}_h = \mathrm{area}(\mathbf{M}_h)$, or $\mathbf{C}_h = \mathrm{diag}(\mathbf{M}_h + h^2\mathbf{K}_h)$, we also get diagonal preconditioners with the same properties as $\mathrm{diag}(\mathbf{M}_h)$.

At first glance, it does not seem necessary to replace $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ by a symetric and positive definite multigrid preconditioner $\mathbf{C}_h = (\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h)(\mathbf{I}_h - \mathbf{MG}_h^k)^{-1}$ as it was proposed in [22] for fixed positive ϱ, since, for $\varrho = h^4$, the block $\mathbf{M}_h + \varrho^{1/2}\mathbf{K}_h$ is spectrally equivalent to $\mathbf{M}_h$ and, therefore, well conditioned. Here, $\mathbf{I}_h$ is the identity, $\mathbf{MG}$ denotes the multigrid iteration matrix, and k is the number of multigrid iterations per preconditioning step (usually, $k = 1$). However, we can improve the spectral equivalent constants in comparison to $\mathbf{C}_h = \mathrm{diag}(\mathbf{M}_h)$ for the price of higher arithmetical, but still asymptotically optimal cost per preconditioning step. More precisely, for the multigrid preconditioner $\mathbf{C}_h = \mathbf{A}_h(\mathbf{I}_h - \mathbf{MG}_h^k)^{-1}$, we get the spectral equivalence inequalities (4.14) with the spectral constants $\underline{c}_{\mathrm{CA}} = 1 - \eta^k$ and $\overline{c}_{\mathrm{CA}} = 1 + \eta^k$, where η is an upper bound for the $\mathbf{A}_h$-energy norm of the multigrid iteration matrix $\mathbf{MG}_h$ that is nothing but the multigrid convergence rate in the $\mathbf{A}_h$-energy norm. We mention that $\overline{c}_{\mathrm{CA}}$ is even equal to 1 if k is even or if the multigrid iteration matrix $\mathbf{MG}_h$ is non-negative with respect to the $\mathbf{A}_h$-energy inner product. We refer the reader to [7, 8] for details on multigrid preconditioners. Therefore, the multigrid preconditioner

$$\mathcal{P}_{\mathrm{mg},h} = \begin{bmatrix} \mathbf{C}_h & \mathbf{0} \\ \mathbf{0} & h^{-4}\mathbf{C}_h \end{bmatrix} = \begin{bmatrix} \mathbf{A}_h(\mathbf{I}_h - \mathbf{MG}_h^k)^{-1} & \mathbf{0} \\ \mathbf{0} & h^{-4}\mathbf{A}_h(\mathbf{I}_h - \mathbf{MG}_h^k)^{-1} \end{bmatrix} \quad (4.16)$$

is spectrally equivalent to $\mathcal{P}_h$. Thus, the MINRES preconditioned by $\mathcal{P}_{\mathrm{mg},h}$ also converges with a rate that is independent of h, and with asymptotically optimal complexity $\mathcal{O}(N_h \ln \varepsilon^{-1})$. We call this solver $\mathcal{P}_{\mathrm{mg}}$MINRES. Numerical tests illustrating the behavior of this multigrid preconditioner and comparing it with the diagonal preconditioner $\mathcal{P}_{\mathrm{diag},h}$ are reported in Section 5.

Instead of the MINRES, we can also use Bramble-Pasciak PCG (BP-PCG) as solver; see [2]. In order to apply BP-PCG, we reformulate the coupled system (4.1) in the equivalent form

$$\begin{bmatrix} \mathbf{M}_h & \sqrt{\varrho}\,\mathbf{K}_h \\ \sqrt{\varrho}\,\mathbf{K}_h & -\mathbf{M}_h \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{p}} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\mathbf{f} \end{bmatrix} \quad (4.17)$$

Applying the Bramble-Pasciak transformation

$$\mathcal{T}_h = \begin{bmatrix} \mathbf{M}_h \mathbf{C}_{M_h}^{-1} - \mathbf{I} & \mathbf{0} \\ \sqrt{\varrho}\,\mathbf{K}_h \mathbf{C}_{M_h}^{-1} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} (\mathbf{M}_h - \mathbf{C}_{M_h})\mathbf{C}_{M_h}^{-1}\mathbf{I} & \mathbf{0} \\ \sqrt{\varrho}\,\mathbf{K}_h \mathbf{C}_{M_h}^{-1} & -\mathbf{I} \end{bmatrix}$$

to the symmetric, but indefinite system (4.17), we arrive at the symetric and positive definite system

$$\mathcal{K}_h \begin{bmatrix} \widetilde{\mathbf{p}} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} \equiv \mathcal{T}_h \begin{bmatrix} \mathbf{0} \\ -\mathbf{f} \end{bmatrix}, \quad (4.18)$$

with the system matrix

$$\begin{aligned}
\mathcal{K}_h &= \begin{bmatrix} \mathbf{M}_h \mathbf{C}_{M_h}^{-1} - \mathbf{I} & \mathbf{0} \\ \sqrt{\varrho}\,\mathbf{K}_h \mathbf{C}_{M_h}^{-1} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{M}_h & \sqrt{\varrho}\,\mathbf{K}_h \\ \sqrt{\varrho}\,\mathbf{K}_h & -\mathbf{M}_h \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{M}_h \mathbf{C}_{M_h}^{-1}\mathbf{M}_h - \mathbf{M}_h & \sqrt{\varrho}(\mathbf{M}_h \mathbf{C}_{M_h}^{-1} - \mathbf{I})\mathbf{K}_h \\ \sqrt{\varrho}\mathbf{K}_h(\mathbf{C}_{M_h}^{-1}\mathbf{M}_h - \mathbf{I}) & \varrho\mathbf{K}_h \mathbf{C}_{M_h}^{-1}\mathbf{K}_h + \mathbf{M}_h \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{M}_h - \mathbf{C}_{M_h})\mathbf{C}_{M_h}^{-1}\mathbf{M}_h & \sqrt{\varrho}(\mathbf{M}_h - \mathbf{C}_{M_h})\mathbf{C}_{M_h}^{-1}\mathbf{K}_h \\ \sqrt{\varrho}\mathbf{K}_h(\mathbf{C}_{M_h}^{-1}\mathbf{M}_h - \mathbf{I}) & \varrho\mathbf{K}_h \mathbf{C}_{M_h}^{-1}\mathbf{K}_h + \mathbf{M}_h \end{bmatrix},
\end{aligned}$$

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where $\mathbf{C}_{M_h}$ is symmetric and positive definite, spectrally equivalent to the mass matrix $\mathbf{M}_h$, and

$$\mathbf{C}_{M_h} < \mathbf{M}_h. \qquad (4.19)$$

Therefore, we can choose

$$\mathbf{C}_{M_h} = 0.25 \operatorname{diag}(\mathbf{M}_h) \qquad (4.20)$$

that is spectrally equivalent to $\mathbf{M}_h$, and that ensures (4.19); see, e.g., [4]. More precisely, there are spectral equivalence constants $1 < \underline{c}_{\mathrm{CM}} \leq \bar{c}_{\mathrm{CM}}$ such that

$$\mathbf{C}_{M_h} < \underline{c}_{\mathrm{CM}} \mathbf{C}_{M_h} \leq \mathbf{M}_h \leq \bar{c}_{\mathrm{CM}} \mathbf{C}_{M_h}$$

Then the exact BP preconditioner

$$\mathcal{P}_{BP,h} = \begin{bmatrix} \mathbf{M}_h - \mathbf{C}_{M_h} & \mathbf{0} \\ \mathbf{0} & \varrho \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h + \mathbf{M}_h \end{bmatrix} \qquad (4.21)$$

is spectrally equivalent to $\mathcal{K}_h$. More precisely, the spectral equivalence inequalities

$$\underline{c}_{\mathrm{PK}} \leq \mathcal{P}_{BP,h} \leq \mathcal{K}_h \leq \bar{c}_{\mathrm{PK}} \mathcal{P}_{BP,h} \qquad (4.22)$$

hold with the spectral equivalence constants

$$\underline{c}_{\mathrm{PK}} = \frac{1 - \sqrt{\alpha}}{1 - \alpha} \quad \text{and} \quad \bar{c}_{\mathrm{PK}} = \frac{1 + \sqrt{\alpha}}{1 - \alpha}, \qquad (4.23)$$

where $\alpha = 1 - (1/\underline{c}_{\mathrm{CM}})$. The lower constant $\underline{c}_{\mathrm{PK}}$ was derived in [2], wheras the upper constant can be found in [21]. Now, replacing the Schur complement $\mathbf{S}_h = \varrho \mathbf{K}_h \mathbf{M}_h^{-1} \mathbf{K}_h + \mathbf{M}_h$ in the exact BP preconditioner (4.21) by $\operatorname{diag}(\mathbf{M}_h)$ that is spectrally equivalent to $\mathbf{S}_h$, we arrive at the inexact BP preconditioner

$$\widetilde{\mathcal{P}}_{BP,h} = \begin{bmatrix} \mathbf{M}_h - \mathbf{C}_{M_h} & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\mathbf{M}_h) \end{bmatrix} = \begin{bmatrix} \mathbf{M}_h - 0.25 \operatorname{diag}(\mathbf{M}_h) & \mathbf{0} \\ \mathbf{0} & \operatorname{diag}(\mathbf{M}_h) \end{bmatrix}$$

that is spectrally equivalent to $\mathcal{K}_h$ as well. Thus, the BP-PCG, that is here nothing but the PCG preconditioned by $\widetilde{\mathcal{P}}_{BP,h}$ applied to the symmetric and positive definite system (4.17), converges with a h-independent rate in asymptotically optimal complexity $\mathcal{O}(N_h \ln \varepsilon^{-1})$. In the next section, we numerically compare exactly this BP-PCG with $\mathcal{P}_{\mathrm{diag}}$MINRES and $\mathcal{P}_{\mathrm{mg}}$MINRES as well as with the inexact Schur complement PCG inexactSCPCG where we use mass lumping in the discretization in order to make the multiplications with the Schur complement effficient.

5 Numerical results

In our numerical examples, we consider the computational domain $\Omega = (0,1)^3$, that is decomposed into uniformly refined tetrahedral elements. The starting mesh contains 384 tetrahedral elements and 125 vertices, i.e., 5 vertices in each direction, which leads to an initial mesh size $h = 2^{-2}$. The tests are performed on 8 uniformly refined mesh levels L_i, $i = 1, ..., 8$. The number of vertices, the mesh size h, and the corresponding regularization parameter $\varrho = h^4$ are given in Table 1.

To confirm the convergence rate as given in (3.6) of the finite element solution $u_{\varrho h}$ to a given target $\bar{u}$, we have considered the following four representative targets with different regularities, similar to [13]:

Target 1: A smooth target $\bar{u} = \sin(\pi x) \sin(\pi y) \sin(\pi z)$, $\bar{u} \in H_0^1(\Omega) \cap H^2(\Omega)$;

Target 2: A piecewise linear continuous target $\bar{u}$ being one in the mid point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and zero in all corner points of Ω, $\bar{u} \in H_0^1(\Omega) \cap H^s(\Omega)$, $s < \frac{3}{2}$;

Level	Number of vertices	h	$\varrho \ (= h^4)$
L_1	125	2^{-2}	2^{-8}
L_2	729	2^{-3}	2^{-12}
L_3	4,913	2^{-4}	2^{-16}
L_4	35,937	2^{-5}	2^{-20}
L_5	274,625	2^{-6}	2^{-24}
L_6	2,146,689	2^{-7}	2^{-28}
L_7	16,974,593	2^{-8}	2^{-32}
L_8	135,005,697	2^{-9}	2^{-36}

Table 1: The number of vertices, the mesh size h, and the related regularization parameter $\varrho = h^4$ on 8 uniformly refined mesh levels.

Target 3: A piecewise constant discontinuous function $\overline{u}$ being one in the inscribed cube $(\frac{1}{4}, \frac{3}{4})^3$ and zero elsewhere, $\overline{u} \in H^s(\Omega)$, $s < \frac{1}{2}$;

Target 4: A smooth target $\overline{u} = 1 + \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$ that violates the homogeneous Dirichlet boundary conditions, $\overline{u} \in H^s(\Omega)$, $s < \frac{1}{2}$.

We will further report robustness and computational cost of four preconditioned Krylov subspace solvers for the large scale linear system of algebraic equations that are arising from the finite element discretization of the optimality system with the choice of the regularization parameter $\varrho = h^4$. More precisely, we study the numerical performance of the following four Krylov subspace solvers described in Section 4:

1. $\mathcal{P}_{\mathrm{mg}}$MINRES: multigrid-preconditioned MINRES for solving (4.10),

2. $\mathcal{P}_{\mathrm{diag}}$MINRES: diagonal-preconditioned MINRES for solving (4.10),

3. BP-PCG: Bramble-Pasciak PCG for solving (4.17),

4. inexSCPCG: inexact Schur complement PCG solving (4.9).

The multigrid preconditioner (4.16) with $k = 1$, which is applied in $\mathcal{P}_{\mathrm{mg}}$MINRES, is based on a W-cycle that starts with a zero initial guess, uses 2 forward Gauss–Seidel presmoothing and 2 backward Gauss–Seidel postsmoothing steps, and canonical transfer operators such that the multigrid preconditioner is symmetric and positive definite; see [7, 8]. We note that the first three solvers solve mixed systems with $2N_h$ degrees of freedoms, whereas the last method solves the inexact Schur complement system that only has N_h degrees of freedom. We recall that the systems (4.1), (4.10), (4.17) and (4.4) are equivalent. But since we use the inexact Schur complement instead of the exact Schur complement in (4.9), we compute a perturbed solution $\widetilde{u}_{\varrho h}$ with some additional error. Hence we also compare all discretization errors. In all of these approaches, the solvers stop the iterations as soon as the preconditioned residual is reduced by a factor 10^{11}. Since the residual of the inexact Schur complement system (4.9) is computed for the primal unknown $\mathbf{u}$ only, the resulting L^2 error of $\widetilde{u}_{\varrho h}$ is different compared to the system (4.1), where also the residual of the adjoint $\mathbf{p}$ is involved. We finally mention that the preconditioned residual norm here reproduces the L^2 norm in which we are primarily interested.

5.1 Convergence studies

The errors $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ between the finite element solution $u_{\varrho h}$ and the given target $\overline{u}$ are computed by means of the first three methods that solve the equivalent mixed formulations (4.10) or (4.17), whereas the errors $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ are computed

by solving the inexact Schur complement equation (4.9) using inexSCPCG. These errors are given in Tables 2–5 for Targets 1–4, respectively. For all of these cases, we have used the required scaling $\varrho = h^4$, which leads to optimal convergence with respect to the mesh size h, depending on the corresponding regularity of the given target $\overline{u}$. This is observed as the experimental order of convergence (eoc) in Tables 2–5. Further, the solution $\widetilde{u}_{\varrho h}$ from solving the inexact Schur complement equation (Approach 4) does not deteriorate with respect to the accuracy and convergence rate. This is confirmed by comparison with the solution from the exact Schur complement equation that is equivalent to solving the mixed fromulations (4.10) or (4.17) as we did in the first three approaches.

Level	Approaches 1 - 3		Approach 4	
	$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc	$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc
L_1	3.04904e−1	−	3.03162e−1	−
L_2	7.14457e−2	2.09	6.92534e−2	2.13
L_3	5.35113e−3	3.74	5.29228e−3	3.71
L_4	6.22449e−4	3.10	6.19849e−4	3.09
L_5	1.34331e−4	2.21	1.33758e−4	2.21
L_6	3.27079e−5	2.03	3.25740e−5	2.04
L_7	8.07438e−6	2.02	8.04282e−6	2.02
L_8	2.00173e−6	2.01	1.99422e−6	2.01
Theory:		2		2

Table 2: Comparison of error $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approaches 1 - 3) and $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approach 4) for **Target 1**.

Level	Approaches 1 - 3		Approach 4	
	$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc	$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc
L_1	2.72445e−1	−	2.71300e−1	−
L_2	8.50409e−2	1.68	8.41925e−2	1.69
L_3	2.99226e−2	1.51	2.90354e−2	1.54
L_4	1.04906e−2	1.51	1.00864e−2	1.53
L_5	3.70527e−3	1.50	3.54103e−3	1.51
L_6	1.30970e−3	1.50	1.24752e−3	1.51
L_7	4.63061e−4	1.50	4.40293e−4	1.50
L_8	1.63735e−4	1.50	1.55529e−4	1.50
Theory:		1.5		1.5

Table 3: Comparison of error $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approaches 1 - 3) and $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approach 4) for **Target 2**.

5.2 Solver performance

We recall that all solvers stop the iterations as soon as the preconditioned residual is reduced by a factor 10^{11}. A comparison of the number of iterations (#Its) and the required solving time in seconds (s) using these four solvers are provided in Tables 6– 9 for the given targets **Target 1−4**, respectively. $\mathcal{P}_{\mathrm{mg}}$MINRES requires the fewest iteration numbers among all these solvers. The solver inexSCPCG outperforms the other three solvers regarding the solving time. This is mainly due to the fact that the inexact Schur complement equation only needs half of degrees of freedom in comparison with the mixed formulations. Finally, we observe that all of the preconditioned Krylov subspace methods show their robustness with respect to the

Level	Approaches 1 - 3		Approach 4	
	$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc	$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc
L_1	3.28255e−1	−	3.26425e−1	−
L_2	2.30561e−1	0.51	2.25595e−1	0.53
L_3	1.63827e−1	0.49	1.59922e−1	0.50
L_4	1.15682e−1	0.50	1.12852e−1	0.50
L_5	8.16986e−2	0.50	7.96806e−2	0.50
L_6	5.77276e−2	0.50	5.62946e−2	0.50
L_7	4.08035e−2	0.50	3.97882e−2	0.50
L_8	2.88466e−2	0.50	2.81281e−2	0.50
Theory:		0.5		0.5

Table 4: Comparison of error $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approaches 1 - 3) and $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approach 4) for **Target 3**.

Level	Approaches 1 - 3		Approach 4	
	$\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc	$\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$	eoc
L_1	1.15861e−0	−	1.15659e−0	−
L_2	6.72524e−1	0.78	6.73325e−1	0.78
L_3	4.63819e−1	0.54	4.62241e−1	0.54
L_4	3.27310e−1	0.50	3.25524e−1	0.51
L_5	2.31129e−1	0.50	2.29647e−1	0.50
L_6	1.63305e−1	0.50	1.62176e−1	0.50
L_7	1.15426e−1	0.50	1.14599e−1	0.50
L_8	8.16011e−2	0.50	8.10057e−2	0.50
Theory:		0.5		0.5

Table 5: Comparison of error $\|u_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approaches 1 - 3) and $\|\widetilde{u}_{\varrho h} - \overline{u}\|_{L^2(\Omega)}$ (Approach 4) for **Target 4**.

mesh size h, using the particular choice for the regularization parameter $\varrho = h^4$. Furthermore, solvers $\mathcal{P}_{\mathrm{diag}}$MINRES, BP-PCG, and inexSCPCG are relatively easy to parallelize due to the fact that each iteration of these approaches only requires matrix-vector multiplications, and the preconditioning step only requires a vector scaling operation by simply using the diagonal of the corresponding matrix as a preconditioner thanks to the spectral equivalence inequalities.

6 Conclusions and outlook

We have derived robust estimates of the derivation of the finite element approximation $u_{\varrho h}$ of the state u_ϱ from the target (desired state) $\overline{u}$ in the $L^2(\Omega)$ norm, and robust, asymptotically optimal solvers for distributed elliptic optimal control problems with L^2-regularization. Due to the optimal choice $\varrho = h^4$ of the regularization parameter, Jacobi-like preconditioners are sufficient to construct MINRES or Bramble-Pasciak CG solvers of asymptotically optimal complexity with respect to arithmetical operations and memory demand. The parallelization of these iterative methods is straightforward, and will lead to very scalable implementations since, in contrast to multigrid preconditioners, diagonal preconditioners are trivial to parallelize. The numerical results yield that the multigrid preconditioned MINRES solver is slightly more efficient in a single processor implementation.

Our numerical experiments show that the inexact Schur Complement PCG (inexactSCPCG) seems to be the most promising iterative solver, in particular, in its parallel version, but also the single processor implementation is the most efficient

Level	$\mathcal{P}_{\text{mg}}$MINRES		$\mathcal{P}_{\text{diag}}$MINRES		BP-PCG		inexSCPCG	
	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)
L_1	19	2.9e−2	21	2.9e−3	24	4.9e−2	10	6.6e−4
L_2	21	2.7e−1	172	1.4e−1	180	4.5e−1	88	2.5e−2
L_3	20	1.8e−0	234	2.3e−0	254	1.0e−0	126	2.6e−1
L_4	20	1.4e+1	231	3.8e+1	247	1.3e+1	132	3.3e−0
L_5	18	1.0e+2	225	4.2e+2	242	2.0e+2	130	2.0e+1
L_6	18	8.1e+2	220	2.1e+3	235	2.6e+3	128	1.8e+2
L_7	18	7.9e+3	213	2.4e+4	229	9.5e+3	124	4.1e+3
L_8	18	7.1e+4	205	2.1e+5	223	1.5e+5	120	3.6e+4

Table 6: Comparison of the number of iterations (#Its) and the solving time in seconds (s) for **Target 1**.

Level	$\mathcal{P}_{\text{mg}}$MINRES		$\mathcal{P}_{\text{diag}}$MINRES		BP-PCG		inexSCPCG	
	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)
L_1	19	2.8e−2	21	2.9e−3	24	2.5e−3	10	6.4e−4
L_2	23	2.9e−1	185	1.5e−1	184	3.1e−1	94	2.7e−2
L_3	23	2.0e−0	258	1.5e−0	265	3.0e−0	133	2.8e−1
L_4	23	1.6e+1	256	5.2e+1	275	4.1e+1	138	3.5e−0
L_5	24	1.3e+2	248	1.1e+2	257	2.4e+2	137	3.9e+1
L_6	24	1.2e+3	240	1.9e+3	249	2.5e+3	134	6.4e+2
L_7	22	1.0e+4	230	2.9e+4	241	1.3e+4	129	3.3e+3
L_8	20	7.9e+4	220	2.2e+5	234	1.7e+5	123	2.6e+4

Table 7: Comparison of the number of iterations (#Its) and the solving time in seconds (s) for **Target 2**.

one in comparison with the MINRES and Bramble-Pasciak CG solvers. The numerical experiments also show that the use of the inexact Schur complement, where the inverse of the mass matrix is replaced by the inverse of the lumped mass matrix, does not affect the accuracy. Here a rigoros numerical analysis is still needed. Moreover, the development of a nested iteration framework with an a posteriori control of the discretization error and its parallel implementation is a future research topic. An adaptive mesh refinement will probably require variable regularization functions $\varrho(x)$ adapted to the mesh density function rather than a fixed choice as we did in this paper where we have investigated uniform mesh refinement.

This approach is not only restricted to the simple model problem of the Poisson equation as constraint, extensions to more complicated elliptic equations, but also to parabolic, e.g., the heat equation, and hyperbolic, e.g., the wave equation, can be done in a similar way, and will be reported elsewhere. Moreover, the consideration of control constraints requires the efficient solution of a sequence of linear algebraic systems as considered in this paper.

Acknowledgments

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[1] https://www3.risc.jku.at/projects/mach2/
[2] https://www.oeaw.ac.at/ricam/hpc

Level	$\mathcal{P}_{\mathrm{mg}}$MINRES		$\mathcal{P}_{\mathrm{diag}}$MINRES		BP-PCG		inexSCPCG	
	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)
L_1	21	3.2e−2	21	7.4e−2	25	2.5e−3	10	5.9e−4
L_2	25	3.1e−1	191	1.5e−1	183	1.8e−1	97	2.8e−2
L_3	25	2.2e−0	268	1.6e−0	272	3.4e−0	136	3.0e−1
L_4	25	1.8e+1	276	1.8e+1	285	3.1e+1	149	4.3e−0
L_5	26	1.5e+2	274	1.8e+2	284	1.1e+2	149	5.0e+1
L_6	26	1.5e+3	276	3.7e+3	279	2.6e+3	149	7.0e+2
L_7	26	1.2e+4	274	3.3e+4	266	1.7e+4	145	3.7e+3
L_8	26	1.5e+5	271	2.4e+5	237	1.6e+5	141	4.2e+4

Table 8: Comparison of the number of iterations (#Its) and the solving time in seconds (s) for **Target 3**.

Level	$\mathcal{P}_{\mathrm{mg}}$MINRES		$\mathcal{P}_{\mathrm{diag}}$MINRES		BP-PCG		inexSCPCG	
	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)	#Its	Time (s)
L_1	21	3.5e−2	21	2.9e−3	24	2.1e−3	10	6.1e−4
L_2	23	3.2e−1	182	1.5e−1	185	9.7e−2	96	2.8e−2
L_3	25	2.3e−0	264	1.6e−0	269	3.2e−0	137	2.8e−1
L_4	24	1.7e+1	270	1.4e+1	268	3.4e+1	147	2.5e−0
L_5	26	1.7e+2	268	3.3e+2	269	1.6e+2	148	2.3e+1
L_6	26	1.6e+3	271	4.2e+3	267	4.2e+3	150	4.9e+2
L_7	26	1.3e+4	268	4.0e+4	266	2.4e+4	149	2.8e+3
L_8	24	1.1e+5	265	2.7e+5	263	2.4e+5	147	3.6e+4

Table 9: Comparison of the number of iterations (#Its) and the solving time in seconds (s) for **Target 4**.

References

[1] M. Benzi, G. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.

[2] J. H. Bramble and J. E. Pasciak. A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. *Math. Comp.*, 50(181):1–17, 1988.

[3] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.

[4] H. C. Elman, D. J. Silvester, and A. J. Wathen. *Finite elements and fast iterative solvers: With applications in incompressible fluid dynamics*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 2005.

[5] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*. Springer-Verlag, New York, 2004.

[6] W. Hackbusch. *Iterative Solution of Large Sparse Systems of Equations*, volume 95 of *Applied Mathematical Sciences*. Springer, Cham, second edition, 2016.

[7] M. Jung and U. Langer. Application of multilevel methods to practical problems. *Surveys on Mathematics for Industry*, 1:217–257, 1991.

[8] M. Jung, U. Langer, A. Meyer, W. Queck, and M. Schneider. Multigrid preconditioners and their applications. In G. Telschow, editor, *Third Multigrid Seminar, Biesenthal 1988*, number Report R–MATH–03/89, pages 11–52, Berlin, 1989. Karl–Weierstrass–Institut.

[9] J. Kraus and M. Wolfmayr. On the robustness and optimality of algebraic multilevel methods for reaction–diffusion type problems. *Computing and Visualization in Science*, 16(1):15–32, 2014.

[10] U. Langer, O. Steinbach, and H. Yang. Robust discretization and solvers for elliptic optimal control problems with energy regularization. *Comput. Meth. Appl. Math.*, 22(1):97–111, 2022.

[11] J.-L. Lions. *Optimal control of systems governed by partial differential equations*. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York-Berlin, 1971. Translated from the French by S. K. Mitter.

[12] F. Natterer. Error bounds for Tikhonov regularization in Hilbert scales. *Applicable Anal.*, 18(1-2):29–37, 1984.

[13] M. Neumüller and O. Steinbach. Regularization error estimates for distributed control problems in energy spaces. *Math. Methods Appl. Sci.*, 44:4176–4191, 2021.

[14] Y. Notay. A new analysis of block preconditioners for saddle point problems. *SIAM J. Matrix Anal. Appl.*, 35(1):143–173, 2014.

[15] Y. Notay. Convergence of some iterative methods for symmetric saddle point linear systems. *SIAM J. Matrix Anal. Appl.*, 40(1):122–146, 2019.

[16] M. Olahanskii and A. Reusken. On the convergence of a multigrid method for linear reaction-diffusion problems. *Computing*, 65:193–202, 2000.

[17] C. Paige and M. Saunders. Solution of sparse indefinite systems of linear equations. *SIAM J. Numer. Anal.*, 12:617–629, 1975.

[18] O. Steinbach. *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements*. Springer, 2008.

[19] V. Thomée. *Galerkin Finite Element Methods for Parabolic Problems*, volume 25 of *Springer Series in Computational Mathematics*. Springer, Berlin, 1997.

[20] F. Tröltzsch. *Optimal control of partial differential equations*, volume 112 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Theory, methods and applications, Translated from the 2005 German original by Jürgen Sprekels.

[21] W. Zulehner. Analysis of iterative methods for saddle point problems: a unified approach. *Math. Comp.*, 71(238):479–505, 2002.

[22] W. Zulehner. Nonstandard norms and robust estimates for saddle point problems. *SIAM J. Matrix Anal. Appl.*, 32(2):536–560, 2011.