Technische Universität Graz

Robust iterative solvers for algebraic systems arising from elliptic optimal control problems

U. Langer, R. Löscher, O. Steinbach, H. Yang

Berichte aus dem Institut für Angewandte Mathematik

Bericht 2023/2

Technische Universität Graz

Robust iterative solvers for algebraic systems arising from elliptic optimal control problems

U. Langer, R. Löscher, O. Steinbach, H. Yang

Berichte aus dem Institut für Angewandte Mathematik

Bericht 2023/2

Technische Universität Graz
Institut für Angewandte Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.applied.math.tugraz.at`

Robust Iterative Solvers for Algebraic Systems Arising from Elliptic Optimal Control Problems

Ulrich Langer[1], Richard Löscher[2], Olaf Steinbach[2], and Huidong Yang[3]

[1] Institute for Computational Mathematics, Johannes Kepler University Linz, Altenbergerstr. 69, 4040 Linz, Austria
ulanger@numa.uni-linz.ac.at

[2] Institut für Angewandte Mathematik, Technische Universität Graz, Steyrergasse 30, 8010 Graz, Austria
loescher@math.tugraz.at, o.steinbach@tugraz.at

[3] Faculty of Mathematics, and Doppler Laboratory MaMS, University of Vienna, Oskar–Morgenstern–Platz 1, 1090 Wien, Austria
huidong.yang@univie.ac.at

Abstract. We consider tracking-type, distributed elliptic optimal control problems with standard L_2 and more general energy regularizations. We propose, analyze, and test new robust preconditioned iterative solvers for systems of linear algebraic equations arising from the finite element discretization of the reduced optimality systems defining the optimal solution in the case of the optimal choice of the regularization parameter. In particular, we study variable regularization parameters adapted to the local behavior of the mesh-size that can heavily change in the case of adaptive mesh refinements as required for discontinuous target functions.

Keywords: Elliptic optimal control problems · L_2 regularization · energy regularization · finite element discretization · iterative solvers.

1 Introduction

In this paper, we consider elliptic optimal control problems of the form: find the state $y_\varrho \in Y = H^1_0(\Omega)$ and the control $u_\varrho \in U$ minimizing the cost functional

$$J(y_\varrho, u_\varrho) := \frac{1}{2}\,\|y_\varrho - y_d\|^2_{L_2(\Omega)} + \frac{1}{2}\,\|\sqrt{\varrho}\,u_\varrho\|^2_U, \tag{1}$$

subject to the elliptic boundary value problem

$$By_\varrho = u_\varrho \quad \text{in } U \subset Y^* = H^{-1}(\Omega), \tag{2}$$

where the computational domain $\Omega \subset \mathbb{R}^d$, with $d \in \{1, 2, 3\}$, is assumed to be bounded and Lipschitz. We use the standard notations for Lebesgue and Sobolev spaces. We emphasize that the regularization "parameter" $\varrho \in L_\infty(\Omega)$ can be a uniformly positive function. If $\varrho = \text{constant} > 0$, then $\frac{1}{2}\|\sqrt{\varrho}\,u_\varrho\|^2_U$ turns into the usual Tikhonov-like regularization term $\frac{\varrho}{2}\|u_\varrho\|^2_U$. Besides the standard

L_2 regularization where $U = L_2(\Omega)$, we also consider the energy regularization given by $U = Y^*$; see [4] for more details. We take the Laplace operator $-\Delta$ as model for the elliptic operator $B : Y \to Y^*$ defined by the variational identity

$$\langle By, z\rangle = (\nabla y, \nabla z)_{L_2(\Omega)} := \int_\Omega \nabla y \cdot \nabla z \, dx \quad \forall y, z \in Y, \tag{3}$$

where $\langle \cdot, \cdot \rangle : Y^* \times Y \to \mathbb{R}$ is the duality pairing. The unique solvability of the optimal control problem (1)-(2) follows from standard arguments; see, e.g., [7].

The optimal solution to (1)-(2) can be found by solving the first-order optimality system that is nothing but a system of coupled PDEs. The finite element (fe) discretization of the reduced (after elimination of the control u_ϱ) optimality system leads to a symmetric, but indefinite system of the form: find the fe nodal vectors $(\mathbf{p}_h, \mathbf{y}_h) \in \mathbb{R}^{N_h = n_h + n_h}$, representing the fe approximations to the adjoint p_ϱ and the state y_ϱ, such that

$$\begin{bmatrix} A_{\varrho h} & K_h \\ K_h^\top & -M_h \end{bmatrix} \begin{bmatrix} \mathbf{p}_h \\ \mathbf{y}_h \end{bmatrix} = \begin{bmatrix} \mathbf{0}_h \\ -\mathbf{y}_{dh} \end{bmatrix}, \tag{4}$$

where the stiffness matrix K_h, the mass matrix M_h, and the regularization matrix $A_{\varrho h}$ are symmetric and positive definite (spd), and $\mathbf{y}_{dh} \in \mathbb{R}^{n_h}$ is the fe load vector representing the desired state y_d. In the standard case of a constant positive regularization parameter ϱ, the regularization matrix $A_{\varrho h}$ equals $\varrho^{-1} M_h$ for the L_2 regularization and $\varrho^{-1} K_h$ for the energy regularization. In the case of a variable regularization parameter $\varrho(x)$, we have $A_{\varrho h} = M_{\varrho h}$ (L_2 regularization) and $A_{\varrho h} = K_{\varrho h}$ (energy regularization) defined by the variational identities

$$(M_{\varrho h}\mathbf{v}_h, \mathbf{w}_h) = (\varrho^{-1} v_h, w_h)_{L_2(\Omega)} \text{ and } (K_{\varrho h}\mathbf{v}_h, \mathbf{w}_h) = (\varrho^{-1}\nabla v_h, \nabla w_h)_{L_2(\Omega)} \tag{5}$$

for all $\mathbf{v}_h$ and $\mathbf{w}_h \in \mathbb{R}^{n_h}$, and the corresponding fe functions v_h and w_h from the fe space $P_h = Y_h = V_h \subset P = Y = V = H_0^1(\Omega)$ spanned by the fe basis functions $\varphi_{h1}, \ldots, \varphi_{hn_h}$, where $(\cdot, \cdot)$ denotes the Euclidean inner product. Here we only consider continuous, piecewise affine-linear basis functions on a shape-regular triangulation $\mathcal{T}_h$ of Ω, where h is a suitably chosen discretization parameter such that $n_h = O(h^{-d})$. We mention that the system (4) turns into an equivalent positive definite, but nonsymmetric (block-antisymmetric) system when the second block-row of (4) is multiplied by (-1). Further, eliminating the fe adjoint state $\mathbf{p}_h$ from (4), we arrive at the Schur-complement system: find the fe state $\mathbf{y}_h \in \mathbb{R}^{n_h}$ such that

$$S_{\varrho h}\mathbf{y}_h = \mathbf{y}_{dh} \tag{6}$$

with the spd Schur-complement matrix $S_{\varrho h} = K_h A_{\varrho h}^{-1} K_h + M_h$. Therefore, system (6) can be solved by the preconditioned conjugate gradient (PCG) method provided that a good preconditioner is available. The matrix-by-vector multiplication $S_{\varrho h} * \mathbf{y}_h$, which is the basic operation in the PCG, always requires the action of $A_{\varrho h}^{-1}$, i.e. the solution of a system with the matrix $A_{\varrho h}$ with high accuracy. This is a principle drawback of the Schur-complement approach. However, in the case of the constant energy regularization, $A_{\varrho h}^{-1} = \varrho K_h^{-1}$, and, therefore,

$S_{\varrho h} = \varrho K_h + M_h$. A good preconditioner for $\varrho K_h + M_h$ will turn the PCG into a perfect solver. In general, when we want to avoid the action of $A_{\varrho h}^{-1}$, we can directly solve the saddle-point problem (4) by some Krylov subspace iteration method. There we only need preconditioners for $S_{\varrho h}$ and not the application of $S_{\varrho h}$. There is a huge amount of literature on efficient preconditioned solvers for saddle-point systems like (4) in general; see, e.g., the survey papers [2, 8]. Special h and ϱ robust preconditioned iterative solvers for discrete (reduced) optimality systems of the kind (4) in the case of the standard L_2 regularization with a constant regularization parameter ϱ were investigated, e.g., in [10, 13]; see also [11] and [1] for handling control and state constraints, and the references therein.

In this paper, we consider the case of the optimal choice of the regularization parameter or function ϱ with respect to the best approximation of the desired state y_d by the computed fe state $y_{\varrho h}$ in the $L_2(\Omega)$ norm. The first step towards such estimates was made in [9] where $\|y_\varrho - y_d\|_{L_2(\Omega)}$ was estimated in terms of ϱ and the regularity of y_d without any discretization. Then these estimates have been used in [5] and [4, 6] to show that the choices $\varrho = h^4$ and $\varrho = h^2$ lead to asymptotically optimal estimates of $\|y_{\varrho h} - y_d\|_{L_2(\Omega)}$ for the L_2 regularization and the energy regularization, respectively. These choices of the regularization parameter imply that the mass matrix M_h is spectrally equivalent to the Schur complement $S_{\varrho h}$ in both cases; see [5]. Therefore, its diagonal replacement $\mathrm{diag}(M_h)$ or the lumped version $\mathrm{lump}(M_h)$ can be used as simple preconditioners for $S_{\varrho h}$. The adaption of the regularization parameter ϱ to the local (element) mesh size h_τ for $\tau \in \mathcal{T}_h$ was studied in [4]. It was shown that $\varrho(x) = h_\tau^2$ for $x \in \tau$ again leads to best-balanced estimates of $\|y_{\varrho h} - y_d\|_{L^2(\Omega)}$ in the case of the energy regularization. In the same paper numerical experiments supported that the scaled mass matrix $M_{\varrho h}$ is an efficient preconditioner for the Schur complement $S_{\varrho h}$, but without any rigorous analysis. We mention that the spectral analysis used in the case of a constant regularization parameter does not work here. In the next section, we will provide a new rigorous analysis of the spectral equivalence of the scaled mass matrix $M_{\varrho h}$ and the Schur complement $S_{\varrho h}$ in the case of the popular L_2 regularization. This result is the basis for the construction of efficient solvers for (4) or even (6) when $A_{\varrho h} = M_{\varrho h}$.

2 Solvers in the Case of Optimal L_2 Regularizations

Let us first recall the case of the L_2 regularization with constant $\varrho = h^4$ that is the optimal choice when aiming at the best approximation of the desired state y_d by the computed fe state $y_{\varrho h}$ with respect to the $L_2(\Omega)$ norm. Then $A_{\varrho h} = M_{\varrho h} = \varrho^{-1} M_h$, and, therefore, $S_{\varrho h} = \varrho K_h M_h^{-1} K_h + M_h$. Expanding vectors $\mathbf{v}_h = \sum_{i=1}^{n_h} v_i^e \mathbf{e}_{hi}$ into the M_h-orthonormal eigenvector basis $\mathbf{e}_{h1}, \ldots, \mathbf{e}_{hn_h}$ provided by the generalized eigenvalue problem $K_h \mathbf{e}_{hi} = \lambda_{hi} M_h \mathbf{e}_{hi}$ with $(M_h \mathbf{e}_{hi}, \mathbf{e}_{hj}) = \delta_{ij}$ and eigenvalues $0 < \underline{c}_{\mathrm{KM}} \le \lambda_{h1} \le \cdots \le \lambda_{hn_h} \le \overline{c}_{\mathrm{KM}} h^{-2}$, we get

$$(S_{\varrho h}\mathbf{v}_h, \mathbf{v}_h) = ((\varrho K_h M_h^{-1} K_h + M_h)\mathbf{v}_h, \mathbf{v}_h) = \sum_{i=1}^{n_h} (\varrho \lambda_{hi}^2 + 1)(v_i^e)^2,$$

from which the spectral equivalence inequalities

$$\underline{c}_{\mathrm{SM}} M_h \leq S_{\varrho h} \leq \overline{c}_{\mathrm{SM}} M_h, \tag{7}$$

with $\underline{c}_{\mathrm{SM}} = 1$ and $\overline{c}_{\mathrm{SM}} = \overline{c}^2_{\mathrm{KM}} + 1$, immediately follow; see [5] for details.

Now let us consider the more interesting case of variable regularization parameters defined by

$$\varrho(x) = h_\tau^4 \quad \text{for all } x \in \tau \text{ and for all } \tau \in \mathcal{T}_h. \tag{8}$$

Then the regularization matrix $A_{\varrho h} = M_{\varrho h}$ is defined by (5). It is well-known that the mass matrix M_h is spectrally equivalent to $D_h = \mathrm{diag}(M_h)$ that is nothing but the diagonal matrix with the same diagonal elements as M_h. The same is true for the scaled mass matrix $M_{\varrho h}$.

Theorem 1. *There are positive, and mesh-independent constants $\underline{c}_{MD}$ and $\overline{c}_{MD}$ such that the spectral inequalities*

$$\underline{c}_{MD} D_{\varrho h} \leq M_{\varrho h} \leq \overline{c}_{MD} D_{\varrho h} \tag{9}$$

hold, where $D_{\varrho h} = diag(d_1, \ldots, d_{n_h}) = diag(M_{\varrho h})$.

Proof. Here we skip the proof. It is even easy to compute the constants on the basis of the corresponding element matrices. □

Due to the locally scaled mass matrix $M_{\varrho h}$, we cannot use the same approach for estimating the Schur complement $S_{\varrho h}$ by M_h from above as in the case $\varrho = h^4$. Fortunately, we can prove the following spectral equivalence theorem by means of another technique that allows localization.

Theorem 2. *Let us assume that $\varrho = \varrho(x)$ is chosen as in* (8)*. Then the spectral equivalence inequalities* (7) *hold with $\underline{c}_{SM} = 1$ and $\overline{c}_{SM} = c + 1$, where c is a generic constant that does not depend on the mesh refinement but only on the shape regularity parameters.*

Proof. It is obviously sufficient to estimate $(K_h M_{\varrho h}^{-1} K_h \mathbf{v}_h, \mathbf{v}_h)$ from above. Using (9) and the fact that $D_{\varrho h}$ is spd, we get the estimate

$$(K_h M_{\varrho h}^{-1} K_h \mathbf{v}_h, \mathbf{v}_h) \leq \underline{c}_{\mathrm{MD}}^{-1} \, (D_{\varrho h}^{-1} K_h \mathbf{v}_h, K_h \mathbf{v}_h) = \underline{c}_{\mathrm{MD}}^{-1} \, \|D_{\varrho h}^{-1/2} K_h \mathbf{v}_h\|^2. \tag{10}$$

Now we proceed to represent $\|D_{\varrho h}^{-1/2} K_h \mathbf{v}_h\|$ as follows:

$$\begin{aligned}
\|D_{\varrho h}^{-1/2} K_h \mathbf{v}_h\| &= \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{(D_{\varrho h}^{-1/2} K_h \mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{w}_h\|} = \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{(K_h \mathbf{v}_h, D_{\varrho h}^{-1/2} \mathbf{w}_h)}{\|\mathbf{w}_h\|} \\
&= \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{(\nabla v_h, \nabla \widetilde{w}_h)_{L_2(\Omega)}}{\|\mathbf{w}_h\|} = \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{\sum_{\tau \in \mathcal{T}_h} (\nabla v_h, \nabla \widetilde{w}_h)_{L_2(\tau)}}{\|\mathbf{w}_h\|} \\
&= \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{\sum_{\tau \in \mathcal{T}_h} (K_\tau \mathbf{v}_\tau, D_\tau^{-1/2} \mathbf{w}_\tau)}{\|\mathbf{w}_h\|} \\
&= \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{\sum_{\tau \in \mathcal{T}_h} (D_\tau^{-1/2} K_\tau \mathbf{v}_\tau, \mathbf{w}_\tau)}{\|\mathbf{w}_h\|},
\end{aligned} \tag{11}$$

where the fe function $\widetilde{w}_h = \sum_{i=1}^{n_h} d_i^{-1/2} w_i \varphi_{hi} \in V_h$ corresponds to the fe vector $D_{\varrho h}^{-1/2}\mathbf{w}_h \in \mathbb{R}^{n_h}$ via the fe isomorphism, $\mathbf{v}_\tau, \mathbf{w}_\tau \in \mathbb{R}^{|\tau|}$ are the corresponding local fe vectors, K_τ is the $|\tau| \times |\tau|$ element stiffness matrix, and D_τ is the $|\tau| \times |\tau|$ diagonal matrix with diagonal entries from $D_{\varrho h}$ corresponding to $\tau \in \mathcal{T}_h$. We mention that the supremum over $\mathbf{w}_h \in \mathbb{R}^{n_h}$ always means $\mathbf{w}_h \in \mathbb{R}^{n_h} \setminus \{\mathbf{0}_h\}$. Using the Cauchy-Schwarz inequality twice and $\left(\sum_{\tau\in\mathcal{T}_h} \|\mathbf{w}_\tau\|^2\right)^{1/2} \leq c_m \|\mathbf{w}_h\|$ that follows from the shape regularity of the mesh (maximal number of simplicies around the vertices) for estimating (11), we arrive at the estimate

$$\begin{aligned}\|D_{\varrho h}^{-1/2} K_h \mathbf{v}_h\| &\leq \sup_{\mathbf{w}_h \in \mathbb{R}^{n_h}} \frac{\left(\sum_{\tau\in\mathcal{T}_h} \|D_\tau^{-1/2} K_\tau \mathbf{v}_\tau\|^2\right)^{1/2} \left(\sum_{\tau\in\mathcal{T}_h} \|\mathbf{w}_\tau\|^2\right)^{1/2}}{\|\mathbf{w}_h\|}, \\ &\leq c_m \left(\sum_{\tau\in\mathcal{T}_h} \|D_\tau^{-1/2} K_\tau \mathbf{v}_\tau\|^2\right)^{1/2}. \end{aligned} \tag{12}$$

Using $\|D_\tau^{-1}\| \leq c_{\mathrm{D}} h_\tau^4 h_\tau^{-d}$, $\|K_\tau\| \leq c_{\mathrm{K}} h_\tau^{d-2}$, and $c_{\mathrm{M}}^{-1} h_\tau^d (\mathbf{v}_\tau, \mathbf{v}_\tau) \leq (M_\tau \mathbf{v}_\tau, \mathbf{v}_\tau)$, we arrive at the estimates

$$\begin{aligned}\sum_{\tau\in\mathcal{T}_h} \|D_\tau^{-1/2} K_\tau \mathbf{v}_\tau\|^2 &= \sum_{\tau\in\mathcal{T}_h} (D_\tau^{-1} K_\tau \mathbf{v}_\tau, K_\tau \mathbf{v}_\tau) \\ &\leq \sum_{\tau\in\mathcal{T}_h} \|D_\tau^{-1}\| \|K_\tau\|^2 \|\mathbf{v}_\tau\|^2 \\ &\leq c_{\mathrm{D}} c_{\mathrm{K}}^2 \sum_{\tau\in\mathcal{T}_h} h_\tau^4 h_\tau^{-d} (h_\tau^{d-2})^2 \|\mathbf{v}_\tau\|^2 \\ &= c_{\mathrm{D}} c_{\mathrm{K}}^2 \sum_{\tau\in\mathcal{T}_h} h_\tau^d \|\mathbf{v}_\tau\|^2 \leq c_{\mathrm{D}} c_{\mathrm{K}}^2 c_{\mathrm{M}} \sum_{\tau\in\mathcal{T}_h} (M_\tau \mathbf{v}_\tau, \mathbf{v}_\tau) \\ &= c_{\mathrm{D}} c_{\mathrm{K}}^2 c_{\mathrm{M}} (M_h \mathbf{v}_h, \mathbf{v}_h). \end{aligned} \tag{13}$$

We note that the positive constants c_{M}, c_{D}, and c_{K} can be chosen globally due to the shape regularity assumption. Combining (10), (11), (12), and (13), we get the upper bound $\overline{c}_{\mathrm{SM}} = c + 1$ with $c = \underline{c}_{\mathrm{MD}}^{-1} c_m^2 c_{\mathrm{D}} c_{\mathrm{K}}^2 c_{\mathrm{M}}$. The lower bound $\underline{c}_{\mathrm{SM}} = 1$ is trivial. This concludes the proof of the theorem. □

Remark 1. Theorem 2 even remains valid for nonsymmetric stiffness matrices K_h arising from the fe discretization of more general elliptic equations like diffusion-convection-reaction equations. Indeed, we have to estimate $(K_h^\top M_{\varrho h}^{-1} K_h \mathbf{v}_h, \mathbf{v}_h) = (M_{\varrho h}^{-1} K_h \mathbf{v}_h, K_h \mathbf{v}_h)$ that can be done as in the proof of Theorem 2.

It is now clear from the spectral equivalence Theorems 1 and 2 that, in (7), the mass matrix M_h can be replaced by a suitable diagonal approximation D_h such as $\mathrm{diag}(M_h)$ or the lumped mass matrix $\mathrm{lump}(M_h)$.

Using these spectral equivalence results, we can now construct robust and efficient preconditioned iterative Krylov subspace solvers like the preconditioned MINRES or Bramble-Pasciak's Preconditioned Conjugate Gradient (BP-PCG)

tailored to symmetric, but indefinite systems like (4). In this paper, we focus on the BP-PCG that was proposed in [3]; see also [12] for improved convergence rate estimates. Thus, we consider a properly scaled diagonal approximation

$$D_{\varrho h} = \delta \operatorname{diag}(M_{\varrho h}) \quad \text{or} \quad D_{\varrho h} = \delta \operatorname{lump}(M_{\varrho h})$$

such that (9) is valid with $1 < \underline{c}_{\mathrm{MD}} \leq \overline{c}_{\mathrm{MD}}$, i.e. $D_{\varrho h} < M_{\varrho h}$ as requested in BP-PCG. We note that the positive scaling parameter δ can be easily calculated on element level. Furthermore, we consider

$$D_h = \operatorname{diag}(M_h) \quad \text{or} \quad D_h = \operatorname{lump}(M_h) \tag{14}$$

that are spectrally equivalent to $S_{\varrho h}$ under the assumptions of Theorem 2. Then the BP-PCG is nothing but the PCG applied to the spd system

$$\mathcal{K}_h \begin{bmatrix} \mathbf{p}_h \\ \mathbf{y}_h \end{bmatrix} = \begin{bmatrix} \mathbf{0}_h \\ -\mathbf{y}_{dh} \end{bmatrix}, \text{ with } \mathcal{K}_h = \begin{bmatrix} M_{\varrho h} D_{\varrho h}^{-1} - I & 0_h \\ K_h D_{\varrho h}^{-1} & -I \end{bmatrix} \begin{bmatrix} M_{\varrho h} & K_h \\ K_h^\top & -M_h \end{bmatrix}, \tag{15}$$

which is equivalent to (4), with the preconditioner

$$\mathcal{P}_h = \begin{bmatrix} M_{\varrho h} - D_{\varrho h} & 0_h \\ 0_h & D_h \end{bmatrix}. \tag{16}$$

The BP-PCG converges with an h-independent rate in asymptotically optimal complexity $O(n_h \ln(\varepsilon^{-1}))$, where $\varepsilon \in (0,1)$ denotes a fixed relative accuracy with respect to the preconditioned residual norm; see [3] and [12].

Alternatively, we can solve the spd Schur complement system (6), arising from (4) when the term $(\varrho^{-1} v, w)_{L_2(\Omega)}$ is discretized by the lumped mass techniques leading to $A_{\varrho h} = \operatorname{lump}(M_{\varrho h})$, by means of the PCG method preconditioned by D_h as defined in (14). This Schur complement PCG solver converges in optimal complexity; see [5] for numerical results in the case of constant $\varrho = h^4$.

3 Numerical Results

We here focus on the three-dimensional ($d = 3$) case with the discontinuous desired state $y_d(x)$ that is equal to 1 for $x \in (0.25, 0.75)^3$ and 0 elsewhere, where the computational domain $\Omega = (0,1)^3$. Thus, the desired state y_d is not contained in the state space $Y = H_0^1(\Omega)$ and has a rather low regularity. More precisely, $y_d \in H^{1/2-\varepsilon}(\Omega)$ for any $\varepsilon > 0$. The same example was already numerically studied in [4] where the variable energy regularization was considered. Here we focus on the variable L_2 regularization in connection with the same simple adaptive procedure as in [4]. This adaptive procedure will considerably improve the accuracy in comparison with the uniform refinement. We expect that the preconditioners presented in Section 2 lead to asymptotically optimal and robust iterative solvers.

The domain $\Omega = (0,1)^3$ is decomposed into uniformly refined tetrahedral elements. The starting mesh contains 384 tetrahedral elements and 125 vertices, leading to an initial mesh size $h = 2^{-2}$. Our numerical tests are running on 8 uniformly refined mesh levels L_i, $i = 1, ..., 8$. On the finest level,

we have $135,005,697$ vertices, $270,011,394$ degrees of freedom (#Dofs), $h = 2^{-9} = 1.9531\text{e}{-3}$, and $\varrho = h^4 = 2^{-36} = 1.4552\text{e}{-11}$. For the tests performed on the adaptive meshes, we have employed the conventional red-green refinement of tetrahedral elements, and we have chosen the local regularization parameter $\varrho_\tau = h_\tau^4$ on each tetrahedral element τ. The comparison of convergence on both uniform and adaptive refinements is illustrated in Figure 1, from which we easily see that the adaptive refinement leads to a much better convergence rate $h^{0.75}$ than the uniform one $h^{0.5}$. In all numerical tests, we run the BP-PCG iterations until the preconditioned residual is reduced by a factor 10^6. A comparison of the number of iterations (Its) on both adaptive and uniform refinements is given in Table 1. It is clearly observed that our preconditioner is robust with respect to the mesh size and the local adaptivity under the choice $\varrho_\tau = h_\tau^4$.

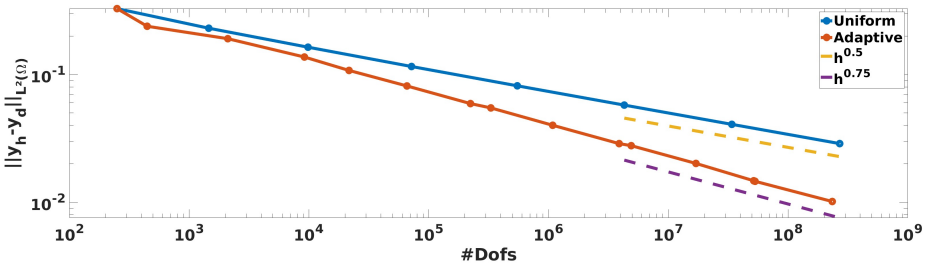

Fig. 1. Comparison of convergence using uniform and adaptive refinements.

Level	Adaptive			Uniform		
	#Dofs	$\|y_h - y_d\|_{L^2(\Omega)}$	Its	#Dofs	$\|y_h - y_d\|_{L^2(\Omega)}$	Its
L_1	250	3.28255e−1	12	250	3.28255e−1	12
L_2	446	2.38883e−1	118	1,458	2.30561e−1	99
L_3	2,102	1.90941e−1	171	9,826	1.63827e−1	137
L_4	9,170	1.37227e−1	204	71,874	1.15682e−1	141
L_5	21,620	1.07761e−1	202	549,250	8.16986e−2	138
L_6	65,828	8.14778e−2	201	4,293,378	5.77278e−2	132
L_7	223,162	5.93121e−2	198	33,949,186	4.08032e−2	123
L_8	330,422	5.48633e−2	201	270,011,394	2.88463e−2	114
L_9	1,084,164	4.01323e−2	199			
L_{10}	3,891,800	2.88341e−2	191			
L_{11}	4,907,338	2.77892e−2	208			
L_{12}	17,034,046	2.01781e−2	191			
L_{13}	51,731,508	1.47666e−2	186			
L_{14}	53,049,534	1.46249e−2	188			
L_{15}	234,045,680	1.01634e−2	181			

Table 1. Comparison of the PB-PCG solver on both adaptive and uniform refinements.

4 Conclusions

We have shown that the mass matrix M_h and, therefore, suitable diagonal representations D_h such as $\mathrm{diag}(M_h)$ or $\mathrm{lump}(M_h)$ are robust preconditioners for the Schur complement $S_{\varrho h}$ in the case of the variable choice of the regularization parameter $\varrho(x) = h_\tau^4$ for $x \in \tau$ and for all $\tau \in \mathcal{T}_h$. Together with a similar, but appropriately scaled diagonal preconditioner $D_{\varrho h}$ for $M_{\varrho h}$, the corresponding BP-PCG is a robust and efficient solver for the reduced discrete optimality system (4) in the case of both constant and variable L_2 regularizations which correspond to uniform and adaptive mesh refinements, respectively. These theoretical results are well supported by the numerical results. The numerical results given in [4] show a similar behavior for the variable energy regularization, but a rigorous analysis is still missing. Furthermore, in applications, these solvers should be embedded in a nested iteration strategy.

References

1. Axelsson, O., Neytcheva, M., Ström, A.: An efficient preconditioning method for state box-constrained optimal control problems. J. Numer. Math. **26**(4), 185–207 (2018)
2. Benzi, M., Golub, G., Liesen, J.: Numerical solution of saddle point problems. Acta Numer. **14**, 1–137 (2005)
3. Bramble, J.H., Pasciak, J.E.: A preconditioning technique for indefinite systems resulting from mixed approximations of elliptic problems. Math. Comp. **50**(181), 1–17 (1988)
4. Langer, U., Löscher, R., Steinbach, O., Yang, H.: An adaptive finite element method for distributed elliptic optimal control problems with variable energy regularization. Tech. Rep. arXiv:2209.08811, arXiv.org (2022)
5. Langer, U., Löscher, R., Steinbach, O., Yang, H.: Robust finite element discretization and solvers for distributed elliptic optimal control problems. Comput. Methods Appl. Math. (2023). https://doi.org/10.1515/cmam-2022-0138
6. Langer, U., Steinbach, O., Yang, H.: Robust discretization and solvers for elliptic optimal control problems with energy regularization. Comput. Meth. Appl. Math. **22**(1), 97–111 (2022)
7. Lions, J.L.: Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles. Dunod Gauthier-Villars, Paris (1968)
8. Mardal, K.A., Winther, R.: Preconditioning discretizations of systems of partial differential equations. Numer. Linear Algebra Appl. **18**(1), 1–40 (2011)
9. Neumüller, M., Steinbach, O.: Regularization error estimates for distributed control problems in energy spaces. Math. Methods Appl. Sci. **44**, 4176–4191 (2021)
10. Pearson, J., Wathen, A.: A new approximation of the Schur complement in preconditioners for PDE-constrained optimization. Numer. Linear Algebra Appl. **12**(5), 816–829 (2012)
11. Schiela, A., Ulbrich, S.: Operator preconditioning for a class of inequality constrained optimal control problems. SIAM J. Optim. **24**(1), 435–466 (2014)
12. Zulehner, W.: Analysis of iterative methods for saddle point problems: a unified approach. Math. Comp. **71**(238), 479–505 (2002)
13. Zulehner, W.: Nonstandard norms and robust estimates for saddle point problems. SIAM J. Matrix Anal. Appl. **32**(2), 536–560 (2011)