Technische Universität Graz

A fully consistent equal–order finite element method for incompressible flow problems

D. R. Q. Pacheco, R. Schussnig, O. Steinbach, T.–P. Fries

Berichte aus dem
Institut für Angewandte Mathematik

Bericht 2020/3

Technische Universität Graz

A fully consistent equal–order finite element method for incompressible flow problems

D. R. Q. Pacheco, R. Schussnig, O. Steinbach, T.–P. Fries

Berichte aus dem Institut für Angewandte Mathematik

Bericht 2020/3

Technische Universität Graz
Institut für Angewandte Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.applied.math.tugraz.at`

© Alle Rechte vorbehalten. Nachdruck nur mit Genehmigung des Autors.

A fully consistent equal-order finite element method for incompressible flow problems

Douglas R. Q. Pacheco [1], **Richard Schussnig** [2], **Olaf Steinbach** [3] and **Thomas-Peter Fries** [4]

[1,3] Institute of Applied Mathematics, Graz University of Technology, Graz, Austria

[2,4] Institute of Structural Analysis, Graz University of Technology, Graz, Austria

[1,2,3,4] Graz Center of Computational Engineering, Graz University of Technology, Graz, Austria

Abstract

Due to simplicity in implementation, data structure and meshing, elements with first-order interpolation of velocity and pressure are by far the most common choice in finite-element-based flow simulations. Although such equal-order pairs are known to be unstable in the Ladyzhenskaya-Babuška-Brezzi (LBB) sense, a variety of stabilisation techniques exist to circumvent that and yield accurate solutions. One of the most popular methods is the pressure stabilisation Petrov-Galerkin (PSPG) formulation, which consists of relaxing the incompressibility constraint with a weighted residual of the momentum equation. Yet, the PSPG method can perform poorly for low-order elements due to loss of consistency. This happens because first-order polynomial spaces are unable to approximate the second-order derivatives required to completely evaluate the stabilisation term. Alternative techniques exist, but they normally either require expensive projections and/or unconventional data structures, or lead to suboptimal convergence. In this context, we present a new technique that rewrites the second-order viscous term in the residual as a first-order boundary term, thereby preserving full consistency even for linear-linear elements. Our method has a similar structure to standard residual-based formulations, but the stabilisation term is computed globally instead of only in element interiors. This results in a scheme that does not require relaxing incompressibility, thereby leading to improved results. The new method is simple to implement and more robust than PSPG. Various numerical examples are provided to showcase the performance of this novel approach in comparison to existing ones.

1. Introduction

In the early years of Computational Mechanics, there was considerable scepticism about whether the Finite Element Method was a suitable technique for flow simulations. However, what was regarded as poor performance was rather the consequence of a numerical phenomenon commonly known as instability [1]. A classical example of an unstable formulation results from the use of linear interpolation for velocity and pressure in incompressible flows. The reason is that equal-order pairs violate the Ladyzhenskaya-Babuška-Brezzi (LBB) condition, which dictates a compatibility requirement between mixed finite element spaces [2].

A major breakthrough came from Hood and Taylor [3], who discovered that going one order higher in velocity allows stable, optimally convergent approximations. There is, however, great practical appeal in the use of first-order interpolation for both flow quantities. The first so-called stabilised formulations allowing equal-order pairs were developed by Brezzi and Pitkäranta [4] and Hughes et al. [5] Both consist of perturbing the continuity equation in order to introduce a non-zero pressure-pressure block in the system, thereby breaking its saddle-point structure. The latter formulation[5] – often called pressure stabilisation Petrov-Galerkin (PSPG) [6] – has over the former[4] the advantage of being residual-based, i.e., the added perturbation is proportional to the residual of the momentum equation. This means that the stabilisation term is smaller in regions where the solution is accurate enough, quickly vanishing as the numerical solution converges to the exact one. The formulation is thus said to be fully consistent, as it is satisfied by the solution of the continuous problem. Other residual-based methods similar to PSPG are also available [7, 8, 9]. Those are all efficient methods offering simple implementation, good accuracy and low computational cost, but they have something else in common: for linear elements, the velocity Laplacian in the residual cannot be approximated, which spoils consistency (in fact, nearly all of these techniques reduce to the same penalty method in that case). This can have serious impact on the quality of the approximations and spoil robustness [10, 11].

The consistent reformulation for PSPG proposed by Droux and Hughes [10] require quasi-uniform meshes, which is of course too restrictive. Other fully consistent stabilisation methods have been proposed by Codina and Blasco [12], Jansen et al. [11] and Bochev and Gunzburger [13]; yet, they demand vector projections that largely increase the size of the problem to be solved. The pressure Poisson formulation of Johnston and Liu [14] is also consistent, but suffers from suboptimal convergence. In this context, we seek a robust stabilisation method which preserves full consistency for first-order elements but still has low computational cost and high accuracy, while also keeping a simple implementation (i.e., without the need for internal face loops, macroelements or additional projections).

The basic idea is to rewrite the Laplacian form present in the weighted residual as a first-order term. We achieve this by replacing the Navier-Stokes system by an equivalent boundary value problem (BVP), extending the idea presented by Liu [15]. In our novel approach, the stabilisation term is computed globally (instead of element-wise), whereas the continuity equation is handled in an element-weighted manner. With this, it is possible to construct a stabilised formulation that has a similar structure to other residual-based ones, but without relaxing incompressibility. We further show that our method is closely related to the pressure Poisson formulation[14], with the strong residual of the divergence-free constraint as an added penalty term. Numerical examples are provided in two and three dimensions, using three types of low-order elements. The results show a clear gain in robustness and accuracy with respect to the standard PSPG method.

It is important to note that the type of stabilisation to which we refer herein is not to be confused with other residual-based techniques such as SUPG [16], grad–div [17] or artificial diffusion [18]. Those methods aim to remedy other sources of instability/inaccuracy, and can be appropriately combined with the present one for specific flow problems and regimes.

The rest of the paper is organised as follows. In Section 2 we state the problem and briefly comment on usual issues. We start Section 3 by presenting the classical PSPG method and illustrating the matter of inconsistency for linear finite element spaces. Following that, our new stabilised formulation is presented in strong and weak forms.

Section 4 deals with discretisation and solution aspects, and Section 5 presents several numerical examples systematically comparing the performances of our new method and existing ones. We finally summarise our findings and draw important remarks.

2. Problem statement

As a model problem, we consider the homogeneous Dirichlet setting for the stationary incompressible Navier-Stokes system, in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3:

$$\begin{aligned} (\rho\nabla\mathbf{u})\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p &= \rho\mathbf{g} \;\text{ in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \;\text{ in } \Omega, \\ \mathbf{u} &= \mathbf{0} \;\text{ on } \Gamma := \partial\Omega, \end{aligned} \tag{1}$$

where $\mathbf{g}$ is a given volumetric force, $\mathbf{u}$ is the flow velocity, p is the pressure, and ρ and μ are the fluid's density and dynamic viscosity, respectively. In the pure Dirichlet case we apply the usual pressure scaling $\int_\Omega p \, \mathrm{d}\Omega = 0$. The standard variational formulation for this problem is: Given $\mathbf{g} \in X'$, find $(\mathbf{u}, p) \in X \times Y$ such that for all $(\mathbf{w}, q) \in X \times Y$

$$(\mathbf{w}, (\rho\nabla\mathbf{u})\mathbf{u}) + (\nabla\mathbf{w}, \mu\nabla\mathbf{u}) - (\nabla\cdot\mathbf{w}, p) = (\mathbf{w}, \rho\mathbf{g}), \tag{2}$$

$$(q, \nabla\cdot\mathbf{u}) = 0, \tag{3}$$

with $X = [H_0^1(\Omega)]^d$, X' is the dual space of X, and $Y = L_0^2(\Omega) := \left\{q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}\Omega = 0\right\}$. Let $X_h \subset X$ and $Y_h \subset Y$ be discrete velocity and pressure spaces. The Bubnov-Galerkin finite element formulation reads: Given $\mathbf{g} \in X'$, find $(\mathbf{u}_h, p_h) \in X_h \times Y_h$ such that for all $(\mathbf{w}_h, q_h) \in X_h \times Y_h$

$$(\mathbf{w}_h, (\rho\nabla\mathbf{u}_h)\mathbf{u}_h) + (\nabla\mathbf{w}_h, \mu\nabla\mathbf{u}_h) - (\nabla\cdot\mathbf{w}_h, p_h) = (\mathbf{w}_h, \rho\mathbf{g}), \tag{4}$$

$$(q_h, \nabla\cdot\mathbf{u}_h) = 0. \tag{5}$$

The unique solvability in the infinite-dimensional case is not sufficient to guarantee that the discrete problem is also uniquely solvable, as X_h and Y_h must also be chosen carefully. In fact, for the simplest case where first-order elements are used for velocity and pressure (or equal-order interpolation in general), the resulting system is generally not uniquely solvable (in other words, not invertible) [1]. One way out is to use quadratic interpolation for velocity while keeping the pressure linear, which is the case of the well-known Taylor-Hood elements. Alternatively, it is possible to use equal-order pairs if the variational formulation is appropriately modified so as to break its saddle-point structure. Those are the so-called stabilised formulations, which are the focus here. We remark that, although the original problem requires only $p \in L^2(\Omega)$, the use of continuous basis functions for pressure in the discrete case assumes $p \in H^1(\Omega)$. This is so, regardless of whether stable or stabilised methods are employed, and in particular in the case of the new stabilisation method that will be presented herein using piecewise linear basis functions.

3. Stabilised finite element formulations

3.1. The pressure stabilisation Petrov-Galerkin method (PSPG)

The pressure stabilisation Petrov-Galerkin (PSPG) method devised by Hughes et al. [5] is probably the most popular stabilisation approach for incompressible flow simulations. It consists of keeping the momentum equation (4) as it is, while relaxing the incompressibility constraint (5) with an element-weighted residual of the momentum equation:

$$(q_h, \nabla \cdot \mathbf{u}_h) + \sum_{e=1}^{N_e} (\delta_e \nabla q_h, \nabla p_h - \mu \Delta \mathbf{u}_h + (\rho \nabla \mathbf{u}_h)\mathbf{u}_h - \rho \mathbf{g})_{\Omega_e} = 0, \tag{6}$$

where N_e is the number of elements Ω_e and δ_e is a positive parameter dependent on the element size h_e; let us also define $h := \max\{h_e\}$. For $\delta_e = \alpha h_e^2/\mu$ and sufficiently large α, the system formed by Eqs. (4) and (6) is stable for equal-order pairs [1, 6].

The PSPG method is an efficient tool for incompressible flow simulations, offering computational simplicity and low cost. Nevertheless, it presents a major drawback when low-order interpolation is used. For first-order triangular and rectangular elements (and their three-dimensional equivalents), the velocity Laplacian in the residual vanishes and Eq. (6) reduces to

$$(q_h, \nabla \cdot \mathbf{u}_h) + \sum_{e=1}^{N_e} (\delta_e \nabla q_h, \nabla p_h + (\rho \nabla \mathbf{u}_h)\mathbf{u}_h - \rho \mathbf{g})_{\Omega_e} = 0. \tag{7}$$

Since the computation of the residual is incomplete, this stabilisation is often said to be inconsistent [10] or weakly consistent [11]. The global method itself is still consistent, since $(\mathbf{u}_h, p_h) \to (\mathbf{u}, p)$ as $h \to 0$ (of course, provided that α is large enough for stability). What is thus meant here by "inconsistency" is the fact that the added residual does not converge to zero, or, in other words, the solution $(\mathbf{u}, p)$ of ((2),(3)) in general does not satisfy ((2),(7)). Although this loss of consistency does not damage the stability of the method or its asymptotic convergence, it induces unphysical pressure boundary conditions [6] and can lead to considerable loss of accuracy, often resulting in poor approximations [10, 11]. In this context, devising a fully consistent reformulation of the PSPG method is a relevant task.

3.2. A new pressure-Poisson-based stabilisation

The first idea is to try relaxing the divergence-free constraint with a different form of the momentum equation, namely,

$$(\rho \nabla \mathbf{u})\mathbf{u} + \mu \nabla \times (\nabla \times \mathbf{u}) + \nabla p = \rho \mathbf{g}, \tag{8}$$

which is equivalent to the standard Laplacian form [19, 20]. This comes from a simple manipulation of the second-order term:

$$\Delta \mathbf{u} \equiv \nabla (\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) = -\nabla \times (\nabla \times \mathbf{u}).$$

Furthermore,

$$\nabla q \cdot [-\nabla \times (\nabla \times \mathbf{u})] \equiv \nabla \cdot [\nabla q \times (\nabla \times \mathbf{u})] - (\nabla \times \nabla q) \cdot (\nabla \times \mathbf{u}) = \nabla \cdot [\nabla q \times (\nabla \times \mathbf{u})],$$

since $\nabla \times \nabla q \equiv \mathbf{0}$. This allows us to write

$$\begin{aligned}(\nabla q, \mu \nabla \times \nabla \times \mathbf{u})_{\Omega_e} &= -\int_{\Omega_e} \nabla \cdot [\nabla q \times (\mu \nabla \times \mathbf{u})] \, \mathrm{d}\Omega = -\int_{\Gamma_e} \mathbf{n} \cdot [\nabla q \times (\mu \nabla \times \mathbf{u})] \, \mathrm{d}\Gamma \\ &= \langle \nabla q \times \mathbf{n}, \mu \nabla \times \mathbf{u} \rangle_{\Gamma_e},\end{aligned}$$

where $\Gamma_e := \partial\Omega_e$ and $\mathbf{n}$ is the outward unit normal vector on Γ_e. The corresponding stabilised formulation would then be: Given $\mathbf{g} \in X'$, find $(\mathbf{u}_h, p_h) \in X_h \times Y_h$ such that for all $(\mathbf{w}_h, q_h) \in X_h \times Y_h$

$$
\begin{aligned}
(\mathbf{w}_h, (\rho\nabla\mathbf{u}_h)\mathbf{u}_h) + (\nabla\mathbf{w}_h, \mu\nabla\mathbf{u}_h) - (\nabla\cdot\mathbf{w}_h, p_h) - (\mathbf{w}_h, \rho\mathbf{g}) = 0,\\
(q_h, \nabla\cdot\mathbf{u}_h) + \sum_{e=1}^{N_e} \delta_e \left[(\nabla q_h, \nabla p_h + (\rho\nabla\mathbf{u}_h)\mathbf{u}_h - \rho\mathbf{g})_{\Omega_e} + \langle \nabla q_h \times \mathbf{n}, \mu\nabla\times\mathbf{u}_h \rangle_{\Gamma_e} \right] = 0.
\end{aligned}
\tag{9}
$$

We thus seem to have managed to rewrite the incomputable (for linear elements) second-order term as a computable first-order boundary term. Yet, as we will show next, this reformulation, as it is, does not offer an improvement with respect to PSPG. Let us assume simplicial elements with linear shape functions. If that is the case, all derivatives appearing in the formulation are piecewise constant, so that

$$
\langle \nabla q_h \times \mathbf{n}, \mu\nabla\times\mathbf{u}_h \rangle_{\Gamma_e} = [\nabla q_h \times (\mu\nabla\times\mathbf{u}_h))] \cdot \int_{\Gamma_e} \mathbf{n}\ \mathrm{d}\Gamma = 0,
$$

since $\int_S \mathbf{n}\ \mathrm{d}S = \mathbf{0}$ for any closed region S. Hence, the modified viscous term vanishes again for linear elements, recovering the (inconsistent) PSPG formulation. Nonetheless, we will show how a similar rewriting of the viscous term can be used globally (rather than element-wise) in order to render the method consistent.

As in most stabilisation techniques [4, 8, 15], a modified system of equations will be our starting point. We propose the following BVP to replace the Navier-Stokes system (2):

$$
(\rho\nabla\mathbf{u})\mathbf{u} - \mu\Delta\mathbf{u} + \nabla p = \rho\mathbf{g} \quad \text{in } \Omega, \tag{10}
$$

$$
\Delta p = \nabla\cdot[\rho\mathbf{g} - (\rho\nabla\mathbf{u})\mathbf{u}] + \gamma\mu\nabla\cdot\mathbf{u} \quad \text{in } \Omega, \tag{11}
$$

$$
\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma, \tag{12}
$$

$$
\frac{\partial p}{\partial n} = \mathbf{n}\cdot[\rho\mathbf{g} - (\rho\nabla\mathbf{u})\mathbf{u} - \mu\nabla\times(\nabla\times\mathbf{u})] \quad \text{on } \Gamma, \tag{13}
$$

where γ is some given positive function in $L^2(\Omega)$ and $\frac{\partial p}{\partial n} := \mathbf{n}\cdot\nabla p$. We wish to prove that, for sufficiently regular p, $\mathbf{u}$ and $\mathbf{g}$, this system is equivalent to the original Navier-Stokes problem (2). Since the momentum equation is kept unchanged, all we need to prove is that this new system implies the incompressibility constraint $\nabla\cdot\mathbf{u} = 0$ (and also the way back, which is easier to prove). The first step is to apply the divergence operator to both sides of Eq. (10), yielding

$$
\Delta p = \nabla\cdot[\rho\mathbf{g} - (\rho\nabla\mathbf{u})\mathbf{u} + \mu\Delta\mathbf{u}] = \nabla\cdot[\rho\mathbf{g} - (\rho\nabla\mathbf{u})\mathbf{u}] + \mu\Delta\left(\nabla\cdot\mathbf{u}\right), \tag{14}
$$

which is the so-called pressure Poisson equation (PPE). Subtracting Eq. (14) from Eq. (11) leads to a diffusion-reaction equation in the variable $\phi := \nabla\cdot\mathbf{u}$:

$$
\Delta\phi - \gamma\phi = 0 \ \text{ in } \Omega. \tag{15}
$$

The boundary condition (BC) for this equation can be obtained by dotting both sides of

Eq. (10) with $\mathbf{n}$ and subtracting the result from Eq. (13), which gives

$$0 = \mathbf{n} \cdot [\Delta \mathbf{u} + (\nabla \times \nabla \times \mathbf{u})] = \mathbf{n} \cdot [\nabla (\nabla \cdot \mathbf{u})] = \frac{\partial \phi}{\partial n}. \tag{16}$$

We have thus a homogeneous diffusion-reaction equation with zero Neumann boundary conditions. The result is the trivial solution $\phi \equiv 0$, that is, $\nabla \cdot \mathbf{u} = 0$ in Ω, as wanted. We omit the proof of the opposite direction, as it is similar and simpler.

Now, we can derive a variational formulation for the equivalent system. We begin by multiplying Eq. (11) by a test function $q \in H^1(\Omega)$ and integrating over Ω:

$$(\mu\gamma q, \nabla \cdot \mathbf{u}) - (q, \Delta p - \nabla \cdot [\rho \mathbf{g} - (\rho \nabla \mathbf{u})\mathbf{u}]) = 0. \tag{17}$$

Applying integration by parts to the second term leads to

$$(\mu\gamma q, \nabla \cdot \mathbf{u}) + (\nabla q, \nabla p + (\rho \nabla \mathbf{u})\mathbf{u} - \rho \mathbf{g}) - \left\langle q, \frac{\partial p}{\partial n} + \mathbf{n} \cdot [(\rho \nabla \mathbf{u})\mathbf{u} - \rho \mathbf{g}] \right\rangle_\Gamma = 0. \tag{18}$$

Now, we substitute the Neumann boundary condition (13) to get

$$(\mu\gamma q, \nabla \cdot \mathbf{u}) + (\nabla q, \nabla p + (\rho \nabla \mathbf{u})\mathbf{u} - \rho \mathbf{g}) + \mu \langle q, \mathbf{n} \cdot [\nabla \times (\nabla \times \mathbf{u})] \rangle_\Gamma = 0. \tag{19}$$

The divergence theorem can be used to write

$$\langle q, \mathbf{n} \cdot [\nabla \times (\nabla \times \mathbf{u})] \rangle_\Gamma = \int_\Omega \nabla \cdot [q \nabla \times (\nabla \times \mathbf{u})] \, \mathrm{d}\Omega,$$

but

$$\nabla \cdot [q \nabla \times (\nabla \times \mathbf{u})] = \nabla q \cdot [\nabla \times (\nabla \times \mathbf{u})] + q \nabla \cdot [\nabla \times (\nabla \times \mathbf{u})] = \nabla q \cdot [\nabla \times (\nabla \times \mathbf{u})].$$

Moreover,

$$\begin{aligned} \int_\Omega \nabla q \cdot [\nabla \times (\nabla \times \mathbf{u})] \, \mathrm{d}\Omega &= \int_\Omega \nabla \cdot [(\nabla \times \mathbf{u}) \times \nabla q] + (\nabla \times \nabla q) \cdot (\nabla \times \mathbf{u}) \, \mathrm{d}\Omega \\ &= \int_\Omega \nabla \cdot [(\nabla \times \mathbf{u}) \times \nabla q] \, \mathrm{d}\Omega = \langle \nabla q \times \mathbf{n}, \nabla \times \mathbf{u} \rangle_\Gamma . \end{aligned} \tag{20}$$

In deriving Eq. (20), we require $\nabla \times \nabla q \in [L^2(\Omega)]^d$, that is, $\nabla q \in H(\mathrm{curl}, \Omega)$. Fortunately, this condition is fulfilled by $q \in H^1(\Omega)$ [15], which can be shown through a Helmholtz decomposition of $H(\mathrm{curl}, \Omega)$ (*c.f.* Ref.[21], Theorem 29). Hence, our variational formulation finally reads: Given $\mathbf{g} \in X'$, find $(\mathbf{u}, p) \in \tilde{X} \times Y$ such that for all $(\mathbf{w}, q) \in X \times \tilde{Y}$

$$(\mathbf{w}, (\rho \nabla \mathbf{u})\mathbf{u}) + (\nabla \mathbf{w}, \mu \nabla \mathbf{u}) - (\nabla \cdot \mathbf{w}, p) = (\mathbf{w}, \rho \mathbf{g}), \tag{21}$$

$$(\mu\gamma q, \nabla \cdot \mathbf{u}) + (\nabla q, \nabla p) + (\nabla q, (\rho \nabla \mathbf{u})\, \mathbf{u}) + \langle \nabla q \times \mathbf{n}, \mu \nabla \times \mathbf{u} \rangle_\Gamma = (\nabla q, \rho \mathbf{g}), \tag{22}$$

with $X = [H_0^1(\Omega)]^d$, $Y = H^1(\Omega) \cap L_0^2(\Omega)$ and

$$\tilde{X} = \left\{ \mathbf{w} \in X : \; (\nabla \times \mathbf{w})\,|_\Gamma \in L^2(\Gamma) \right\}, \tag{23}$$

$$\tilde{Y} = \left\{ q \in Y : \; \mathbf{n} \times \nabla q|_\Gamma \in L^2(\Gamma) \right\}. \tag{24}$$

Although these spaces may seem somewhat unusual in the continuous level, their regularity requirements are fulfilled by standard C^0 finite element spaces [14].

This formulation is quite similar to PSPG, and even more similar to the boundary integral modification proposed by Brezzi and Douglas Jr. [8]. Apart from the treatment of the viscous term, the only difference is that here the stabilisation term is computed globally, instead of only in element interiors. The price to pay is the H^1–regularity requirement for the pressure spaces (trial and test). This prohibits the use of discontinuous pressure spaces, but that is in general not a problem since continuous elements are used in most practical applications.

A last question to answer is how to define the function γ that acts as a weight for the divergence bilinear form. Notice that γ appears in the reaction term of Eq. (15), so it must be positive in order to guarantee $\nabla \cdot \mathbf{u} \equiv 0$. For conformity, we need simply $\gamma \in L^2(\Omega)$. However, γ must be chosen appropriately if we desire optimal velocity convergence. For that, we compare our formulation to the modified PSPG form of Brezzi and Douglas Jr. [8], which is defined for quasi-uniform meshes. In that case, their relaxed continuity equation is written as

$$(q, \nabla \cdot \mathbf{u}) + \frac{\alpha h^2}{\mu} \left[(\nabla q, \nabla p) + (\nabla q, (\rho \nabla \mathbf{u})\, \mathbf{u}) + \langle q\mathbf{n}, \mu \Delta \mathbf{u} \rangle_\Gamma - (\nabla q, \rho \mathbf{g}) \right] = 0, \tag{25}$$

where α is the standard PSPG stabilisation parameter. As in our method, they treat the viscous contribution as a boundary term – but with a second-order operator, which again reduces the method to standard PSPG for linear elements. Comparing our form (22) to theirs (25) (or any classical stabilised formulation fitting the present framework [4, 5, 7, 8, 9, 12]) leads to a natural choice for γ, namely, $\gamma = (\alpha h^2)^{-1}$, or $\gamma_e = (\alpha h_e^2)^{-1}$ for non-uniform meshes. This specific power on the mesh size parameter is chosen for optimal convergence [6].

We are finally in position to state our finite element formulation. Considering a more general BVP with Dirichlet BC $\mathbf{u}|_{\Gamma_D} = \mathbf{u}_D$ and natural BC $[-p\mathbf{n} + (\mu \nabla \mathbf{u})\, \mathbf{n}]\,|_{\Gamma_N} = \mathbf{t}_N$, the discrete problem reads: Given $\mathbf{g} \in X'$, find $(\mathbf{u}_h, p_h) \in X_h \times Y_h$, with $\mathbf{u}_h|_{\Gamma_D} = \mathbf{u}_D$, such that for all $(\mathbf{w}_h, q_h) \in X_h \times Y_h$, with $\mathbf{w}_h|_{\Gamma_D} = \mathbf{0}$,

$$\begin{gathered}
(\mathbf{w}_h, (\rho \nabla \mathbf{u}_h)\mathbf{u}_h) + (\nabla \mathbf{w}_h, \mu \nabla \mathbf{u}_h) - (\nabla \cdot \mathbf{w}_h, p_h) = (\mathbf{w}_h, \rho \mathbf{g}) + \int_{\Gamma_N} \mathbf{w} \cdot \mathbf{t}_N \, \mathrm{d}\Gamma, \\
(\nabla q_h, \nabla p_h + (\rho \nabla \mathbf{u}_h)\, \mathbf{u}_h) + \langle \nabla q_h \times \mathbf{n}, \mu \nabla \times \mathbf{u}_h \rangle_\Gamma + \sum_{e=1}^{N_e} \frac{\mu}{\alpha h_e^2} (q_h, \nabla \cdot \mathbf{u}_h)_{\Omega_e} = (\nabla q_h, \rho \mathbf{g}),
\end{gathered} \tag{26}$$

with X_h and Y_h taken as C^0 finite element spaces for conformity.

It is worth remarking that, if $\gamma \to 0$, that is, $\alpha \to \infty$, the pressure Poisson formulation by Johnston and Liu [14] is recovered. Their method replaces the continuity equation by the PPE completely, which leads to a stable but suboptimally convergent scheme when equal-order pairs are used [15, 22]. Hence, our formulation can be given two quite distinct interpretations. On the one hand, it can be viewed as a consistent modification of PSPG dealing with mesh size effects in the divergence term rather than in the stabilisation term; on the other hand, one can also see our method as a PPE-based formulation[15, 23] with an added term penalising violations of the divergence-free constraint. An important consequence is that, differently from the existing residual-based methods, ours does not relax incompressibility. In the standard formulations, $\alpha \to \infty$ leads to the stabilising term replacing completely the continuity equation, i.e., the system no longer conserves mass (not even approximately). In our formulation, $\alpha \to \infty$ means completely replacing the continuity equation by the PPE, which also enforces incompressibility, but in a lower-order way (*c.f.* Refs. [15, 24, 23] for excellent discussions on how different forms of the PPE can be used to enforce conservation of mass). We also remark that, differently from other residual-based formulations, our method does not require second deratives of shape functions, which represents an advantage from the standpoint of implementation and data structure.

4. Discretisation and solution

Standard continuous finite element spaces of equal order for velocity and pressure are considered herein. The corresponding shape functions will be denoted by ψ_i, $i = 1, ..., N$, with N being the number of nodes in the mesh. Moreover, only first-order elements are considered in the numerical examples, both simplicial and non-simplicial.

4.1. The Stokes system

The Stokes solution is obtained by dropping the convective term $(\rho\nabla\mathbf{u})\,\mathbf{u}$. After discretisation, this leads to the linear algebraic system

$$\begin{bmatrix} \mathbf{K} & -\mathbf{B}^\top \\ \frac{1}{\alpha}\tilde{\mathbf{B}} + \mathbf{L} & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{Bmatrix} = \begin{Bmatrix} \underline{\mathbf{b}} \\ \underline{f} \end{Bmatrix}, \tag{27}$$

where $\mathbf{K}$, $\mathbf{B}$ and $\underline{\mathbf{b}}$ are the usual stiffness matrix, divergence matrix and forcing vector coming from the standard Galerkin formulation of the Stokes system[6]. Matrix $\mathbf{A}$ is a standard Laplacian stiffness matrix (without boundary conditions) and matrices $\mathbf{L}$ and $\tilde{\mathbf{B}}$ have a block structure:

$$\mathbf{L} = \left[\mathbf{L}^1 \; ... \; \mathbf{L}^d\right], \quad \tilde{\mathbf{B}} = \left[\tilde{\mathbf{B}}^1 \; ... \; \tilde{\mathbf{B}}^d\right], \tag{28}$$

with

$$L_{ij}^k = \mu \sum_{m=1}^{d} \int_\Gamma (\delta_{mk} - 1) \left(n_m \frac{\partial \psi_i}{\partial x_k} - n_k \frac{\partial \psi_i}{\partial x_m} \right) \frac{\partial \psi_j}{\partial x_m} \, \mathrm{d}\Gamma, \tag{29}$$

$$\tilde{B}_{ij}^k = \mu \sum_{e=1}^{N_e} \frac{1}{h_e^2} \int_{\Omega_e} \psi_i \frac{\partial \psi_j}{\partial x_k} \, \mathrm{d}\Omega, \tag{30}$$

where n_k is the k-th spatial component of the normal vector $\mathbf{n}$, and δ is the Kronecker delta. The entries of the forcing term $\underline{f}$ are given by

$$f_i = \int_\Omega \rho \nabla\psi_i \cdot \mathbf{g} \, \mathrm{d}\Omega. \tag{31}$$

4.2. The Navier-Stokes system

When the convection term is kept, an iterative scheme is necessary to solve the resulting non-linear problem. To obtain such a scheme, standard methods can be readily applied, since the term we add in comparison to classical PSPG is linear. We use the following Picard method: after an initial guess $\left(\underline{\mathbf{u}}^0, \underline{p}^0\right)$, the iterations follow as

$$\begin{bmatrix} \mathbf{K} + \mathbf{C}\left(\underline{\mathbf{u}}^n\right) & -\mathbf{B}^\top \\ \frac{1}{\alpha}\tilde{\mathbf{B}} + \mathbf{L} + \mathbf{H}\left(\underline{\mathbf{u}}^n\right) & \mathbf{A} \end{bmatrix} \begin{Bmatrix} \underline{\mathbf{u}}^{n+1} \\ \underline{p}^{n+1} \end{Bmatrix} = \begin{Bmatrix} \underline{\mathbf{b}} \\ \underline{f} \end{Bmatrix}, \tag{32}$$

where $\mathbf{C}$ is a block-diagonal matrix with d identical blocks $\mathbf{c}$ given by

$$c_{ij} = \int_\Omega \psi_i \nabla\psi_j \cdot \mathbf{u}_h \, \mathrm{d}\Omega, \tag{33}$$

and $\mathbf{H} = \left[\mathbf{H}^1 \, ... \, \mathbf{H}^d\right]$, with

$$H_{ij}^k = \int_\Omega \frac{\partial\psi_i}{\partial x_k} \nabla\psi_j \cdot \mathbf{u}_h \, \mathrm{d}\Omega. \tag{34}$$

After each iteration, Aitken relaxation is applied in order to provide the iterative solver with quadratic convergence [25] (*c.f.* Ref.[26] for details).

4.3. Solving the linear system

The linear systems (27) and (32) can be easily solved using direct methods when considering two-dimensional problems, but in three dimensions the resulting memory requirements and computational complexity can quickly become prohibitive. Thus, we present here an iterative technique suitable to handle the problem at hand. It is based on a flexible GMRES method with right preconditioner $\mathcal{P}^{-1}$ for the Navier-Stokes system[27]:

$$\mathcal{P}^{-1} := \begin{bmatrix} (\mathbf{K}+\mathbf{C})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{B}^\top \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix}, \tag{35}$$

using the Schur complement defined as $\mathbf{S} := \mathbf{A} + (\frac{1}{\alpha}\tilde{\mathbf{B}} + \mathbf{L} + \mathbf{H})\,(\mathbf{K}+\mathbf{C})^{-1}\,\mathbf{B}^\top$. For the Stokes system, the convective terms in $\mathcal{P}^{-1}$ and $\mathbf{S}$ are simply zero. Explicitly computing the Schur complement $\mathbf{S}$ is considered too costly, so the key factor for achieving robust and fast convergence lies in its efficient approximation. We choose $\mathbf{S}^{-1} \approx \mu\mathbf{M}_p^{-1}$, where $\mathbf{M}_p$ is the mass matrix in the pressure space[27, 28]. This choice is suitable for the diffusion-dominated case, which is the focus of this contribution. The actions of the inverses in (35) applied to some iterate can be approximated by single V-cycles of an algebraic multigrid method (e.g., utilizing the BoomerAMG package [29] via deal.II [30]). The rows corresponding to the pressure test functions are multiplied by element-averaged factors τ_i defined per node as

$$\tau_i := \left(\sum_{e=1}^{N_i} |\Omega_e|\right)^{-1} \sum_{e=1}^{N_i} \frac{\alpha h_e^2}{\mu} |\Omega_e|, \tag{36}$$

where $|\Omega_e| := \int_{\Omega_e} \mathrm{d}\Omega$ and N_i is the number of elements sharing vertex i.

5. Numerical examples

In this section, we use different benchmark examples to assess the performance of our new method in comparison to classical ones. Three types of first-order elements are considered: triangular, quadrilateral and hexahedral. In order to measure approximation erros, we define a normalised L^2 norm

$$\|p - p_h\|_0 := \frac{\|p - p_h\|_{L^2(\Omega)}}{\|p\|_{L^2(\Omega)}},$$

and analogously for $\mathbf{u}_h$. The spatial coordinates (x_1, x_2, x_3) will be denoted by (x, y, z).

5.1. Stokes flow in L-shaped domain

We first consider the Stokes problem in the L-shaped illustrated in Figure 1, with $\mu = \rho = 1$, homogeneous Dirichlet boundary conditions, zero mean pressure and body force $\mathbf{g}$ given by

$$\mathbf{g} = \begin{Bmatrix} 8\pi^2 \sin 4\pi y \\ 8\pi^2 \left(4\cos 4\pi y - 1\right) \sin 4\pi x \end{Bmatrix}.$$

The analytical solution to this problem is

$$\mathbf{u} = \begin{Bmatrix} \sin 4\pi y \sin^2 2\pi x \\ -\sin 4\pi x \sin^2 2\pi y \end{Bmatrix}, \quad p = 4\pi \sin 4\pi x \sin 4\pi y.$$

The first goal is to compare our new approach and PSPG, regarding robustness with respect to the stabilisation parameter α. For that, we consider a uniform mesh consisting of 6912 square elements, as depicted in Figure 1. The pressure error for a wide range of stabilisation parameters is shown in Figure 2, and important considerations can be drawn. The expected behaviour is observed: the error is large for small values of α, decreases as α is increased, reaches a minimum, then starts growing again and eventually settles at a finite value. Both formulations yield virtually identical results for small α, since the divergence-free constraint dominates over the stabilisation term. However, the reasons why each method reaches a limiting performance for $\alpha \to \infty$ are distinct. In the PSPG formulation, the error becomes very high because a large α leads to over-relaxation of incompressibility; in our formulation, $\alpha \to \infty$ leads to the pressure Poisson equation completely replacing the divergence-free constraint, which does not violate (discrete) incompressibility but results in suboptimal convergence [22]. Therefore, the error for large α is much higher for PSPG than for the present formulation. Moreover, the former's lack of consistency has an important impact on robustness. The numerical solution is very sensitive to the stabilisation parameter, with the error growing very fast when α moves away from its optimal value in either direction. Conversely, in our formulation the error varies very little for $\alpha \in [10^{-1}, 10^2]$, i.e., there is much more freedom in parameter selection. This is a crucial feature for a stabilisation technique, since the optimal parameters are often problem-dependent. The velocity errors are shown in Figure 3. Here, too, the numerical approximation proves considerably less sensitive to the stabilisation parameter in our formulation than in the classical one.

5.2. Kovasznay flow problem

We now consider a non-linear problem: the Kovasznay flow benchmark [31]. It is one of the only known analytical solutions to the Navier-Stokes problem with $\mathbf{g} = \mathbf{0}$, and models the behaviour of laminar flow past cylinders. The solution in $\Omega = \left(-\frac{1}{2}, \frac{1}{2}\right)^2$ is

$$\mathbf{u} = \begin{Bmatrix} 1 - e^{kx} \cos 2\pi y \\ \frac{k}{2\pi} e^{kx} \sin 2\pi y \end{Bmatrix}, \quad p = \frac{e^k - e^{2kx}}{2}, \tag{37}$$

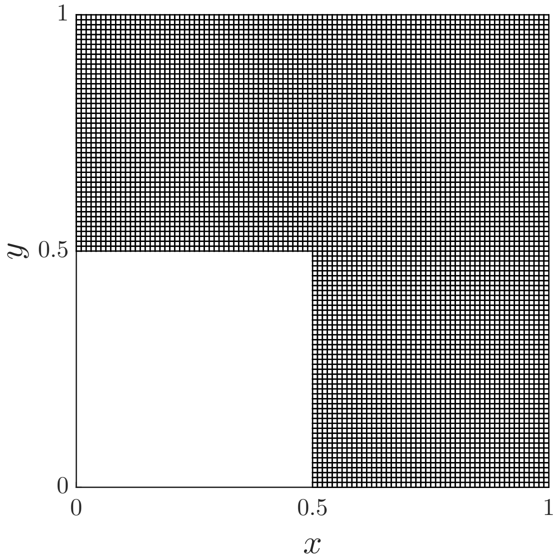

Figure 1: Stokes problem: uniform quadrilateral mesh.

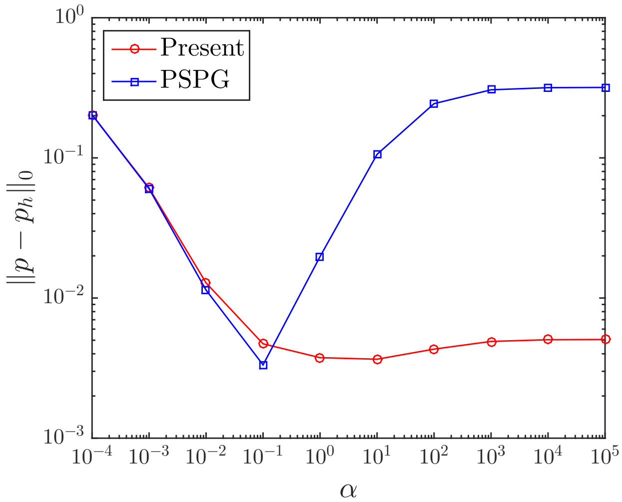

Figure 2: Stokes problem: pressure error with respect to stabilisation parameter.

where Re is the Reynolds number and $k = \frac{\mathrm{Re}}{2} - \sqrt{\left(\frac{\mathrm{Re}}{2}\right)^2 + (2\pi)^2}$. We solve the corresponding Dirichlet problem, this time using triangular elements. An h–refinement study is performed; the coarsest mesh is shown in Figure 4, and then five levels of uniform refinement are considered. The pressure and velocity errors for $\mathrm{Re} = 100$ and $\alpha = 1$ are shown in Figure 5. It can be seen that, although both PSPG and our formulation converge with similar rates, the former takes longer to reach the asymptotic behaviour, which leads to larger errors. The reason for this "delayed" convergence is the fact that the inconsistent pressure boundary conditions induced by the PSPG formulation only become negligible as the mesh size goes to zero [6]. Another important conclusion from the convergence plots is the remarkable performance improvement of our approach with respect to the pure PPE formulation[14], attained with the addition of a simple term to penalise large velocity divergences. It is also relevant to assess how well the conservation of mass is fulfilled. In order to quantify that, we plot in Figure 6 the norm of $\nabla \cdot \mathbf{u}_h$, which should

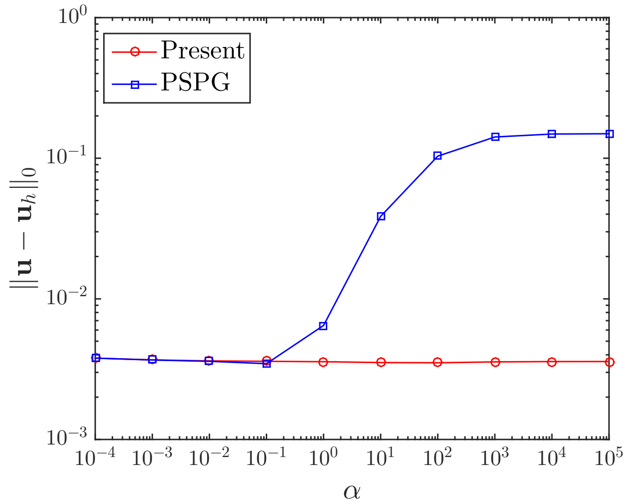

Figure 3: Stokes problem: velocity error with respect to stabilisation parameter.

be zero for an exactly incompressible flow. Here, too, our method performs better than the others.

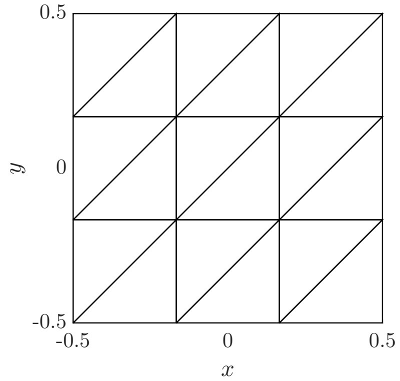

Figure 4: Kovasznay benchmark: coarsest mesh used in the refinement study.

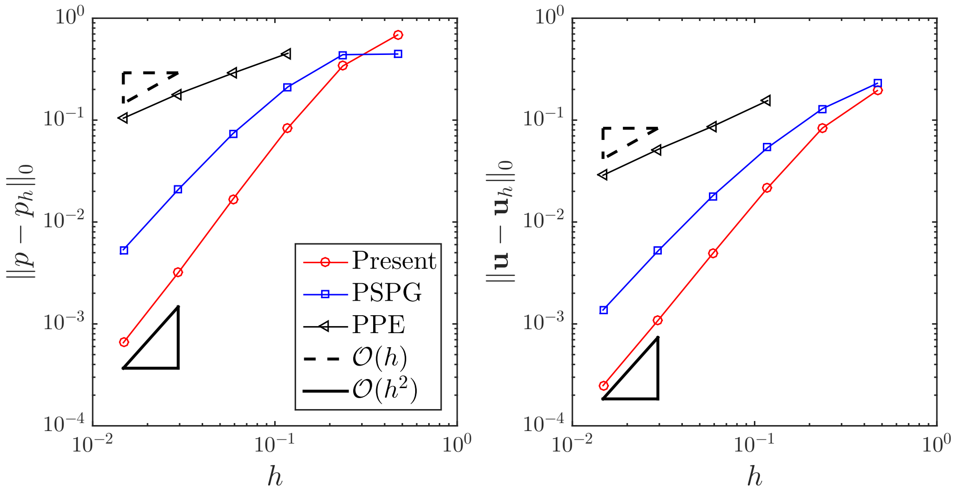

Figure 5: Kovasznay benchmark: uniform refinement study.

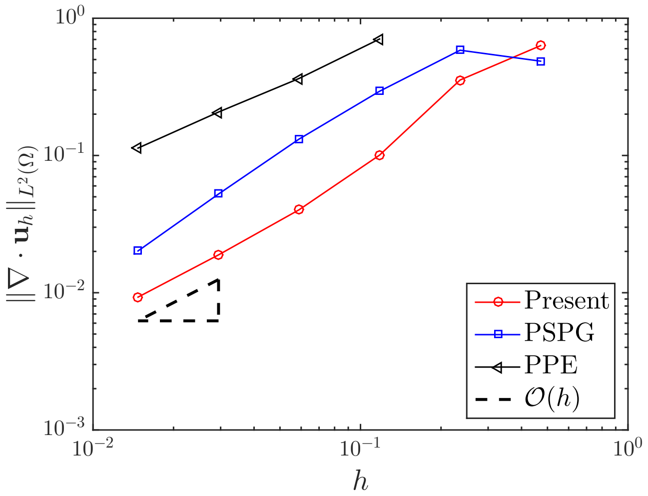

Figure 6: Kovasznay benchmark: conservation of mass for uniform refinement.

The influence of the stabilisation parameter has also been investigated for this example, and the results are shown in Figure 7 (using the second finest mesh). Once again, our consistent method proves considerably more robust than standard PSPG. While the latter requires a stabilisation parameter between 10^{-3} and 10^{-2} to yield smalls errors, the former maintains practically the minimum errors for any $\alpha \in [10^{-3}, 1]$. It is also important to remark that here the optimal α for the PSPG formulation differs by two orders of magnitude with respect to the previous example. At $\alpha = 10^{-1}$, which is close to the value often recommended for PSPG [1], the pressure and velocity errors are already 40% and 30% larger, respectively, than those attained by our method.

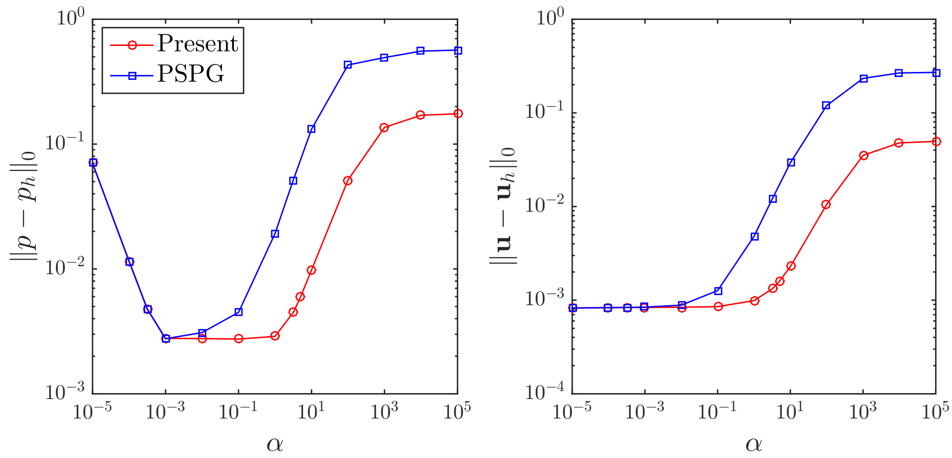

Figure 7: Kovasznay benchmark: error with respect to stabilisation parameter.

To further illustrate how the lack of consistency can impact the quality of the approximation, we show in Figure 8 the pressure isolines for the Kovasznay problem with $\mathrm{Re} = 40$, $\alpha = 100$ and the finest mesh considered in the convergence study. This is a particularly good example for illustrating such effects because the exact solution has prefectly vertical isolines. Note that the stabilisation parameter is deliberately chosen outside of the optimal interval for both formulations, so as to critically test their robustness. We see that PSPG yields completely distorted lines all over the domain, whereas in our consistent approach there is only a mild distortion close to the edges.

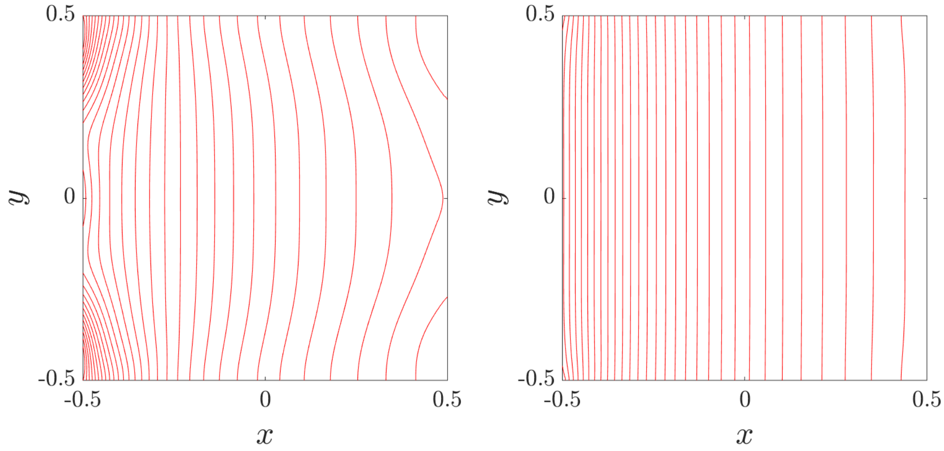

Figure 8: Kovasznay benchmark: pressure isolines for PSPG (left) and present formulation (right).

5.3. Poiseuille flow in three dimensions

We now consider a three-dimensional example on graded hexahedral meshes. The domain is a cylinder defined as

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3 : \; x^2 + y^2 < R^2, \; 0 < z < L\right\}.$$

The corresponding Poiseuille flow solution is

$$\mathbf{u} = \left\{0, 0, \frac{2Q}{\pi R^2}\left(1 - \frac{x^2 + y^2}{R^2}\right)\right\}^\top, \quad p = \frac{8\mu Q L}{\pi R^4}\left(1 - \frac{z}{L}\right),$$

in which Q is a given volumetric flow rate and R and L are the pipe's radius and length, respectively. There are no body forces. As boundary conditions for the numerical solution, we use the analytical velocity profile at the inlet $z = 0$, zero velocity on the wall ($x^2 + y^2 = R^2$), and zero natural BC ($\mathbf{t} = \mathbf{0}$) at the outlet $z = L$. We begin once again by comparing the methods regarding the effect of the stabilisation parameter α. For this, we consider a test case with $\frac{L}{R} = 3$, Re $= 150$ and a graded mesh with 92160 elements and 96657 nodes (see Figure 9). For this example, too, our method clearly outperforms PSPG in accuracy and robustness, as revealed by the error plot in Figure 10.

Next, a convergence study is performed for the present method. The four graded meshes considered are depicted in Figure 11 (frontal view). The refinement in the z direction is uniform, starting with an element length of $L/3$. We consider a normalised problem with $\frac{2R}{L} = \frac{\rho Q}{\mu R} = 1$. As for the two-dimensional examples, we use a direct solver here. Table 1 shows the errors for the four meshes. Once again, the estimated orders of convergence (eoc) are quadratic. It is important to draw some remarks regarding the convergence rates. From the approximation standpoint, the polynomial degree for p_h should be one less than for $\mathbf{u}_h$ if optimal pressure convergence is desired, since we are looking for $p \in L^2(\Omega)$ but $\mathbf{u} \in \left[H^1(\Omega)\right]^d$. When using equal orders, all that can be guaranteed for the pressure is linear convergence in $L^2(\Omega)$ [5, 8]. In fact, the order is known in practice to range between 1 and 2, depending on the flow regime [6]. Therefore, the quaratic convergence experienced here for the pressure should be seen as an "initial"

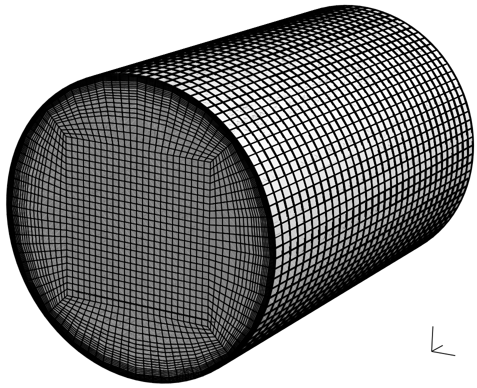

Figure 9: Poiseuille flow: graded mesh used for the first test case.

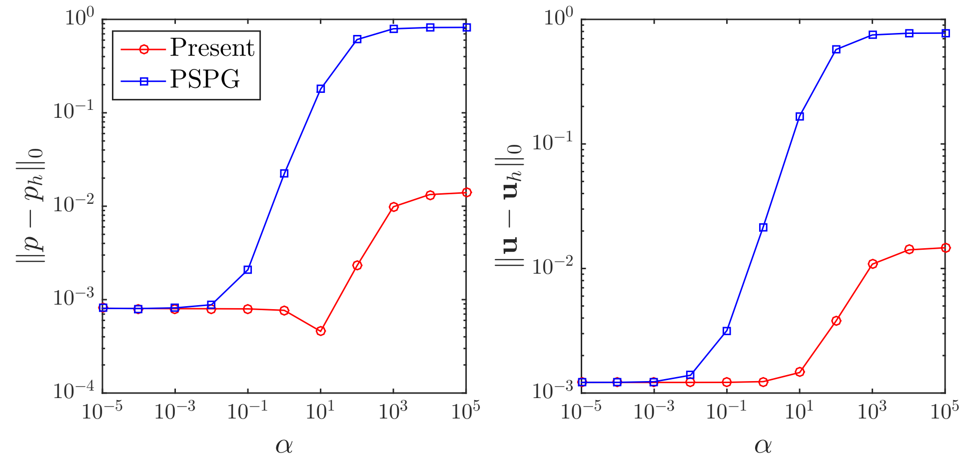

Figure 10: Poiseuille flow: error with respect to stabilisation parameter.

higher-order convergence[32] and cannot be expected to hold indefinitely. This applies of course not only to our method, but also to all popular equal-order stabilised methods[4, 5, 13, 12, 8, 33].

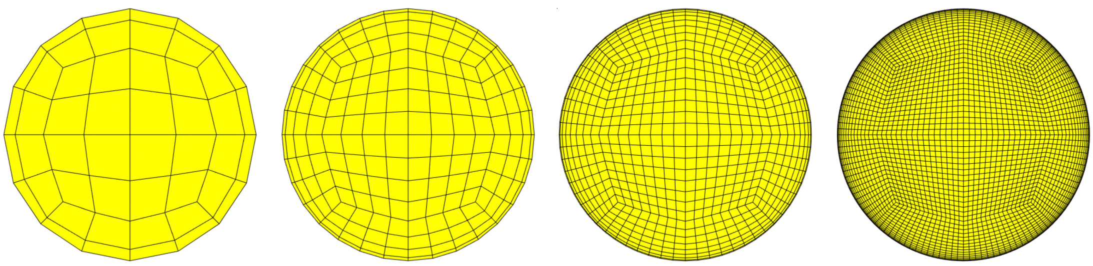

Figure 11: Frontal view of the graded meshes used for the Poiseuille flow problem.

6. Concluding remarks

This work has presented a consistent pressure-stabilised formulation for incompressible flows allowing first-order velocity-pressure pairs. The method has been derived by replacing the standard Navier-Stokes equations by an equivalent system containing a weighted average of the continuity equation and the pressure Poisson equation. Using this equivalent boundary value problem and appropriate vector calculus identities, we are able to

Table 1: Poiseuille flow: convergence study for graded meshes.

Mesh	$\|\mathbf{u}-\mathbf{u}_h\|_0$	eoc	$\|p-p_h\|_0$	eoc
1	7.2×10^{-2}	–	3.5×10^{-2}	–
2	1.3×10^{-2}	2.4	6.0×10^{-3}	2.5
3	3.3×10^{-3}	2.0	1.5×10^{-3}	2.0
4	8.3×10^{-4}	2.0	3.6×10^{-4}	2.0

rewrite the second-order viscous term as a first-order boundary term, thereby preserving consistency even for linear elements. To the best of our knowledge, this is the first stabilisation technique which is fully consistent for first-order elements, without requiring the definition of auxiliary variables and projections, or unconventional data structures such as macroelements, patches and internal face loops. Various numerical examples have been provided to allow a comparison between our formulation and existing ones, revealing a clear improvement in accuracy and robustness. We hope that this new technique can offer the CFD community a practical, efficient alternative to some existing methods such as PSPG. Future and ongoing developments include a generalisation to fluids with variable viscosity, as well as a systematic numerical analysis to provide theoretical stability estimates, which could provide useful bounds for the stabilisation parameter.

Acknowledgements

The authors acknowledge Graz University of Technology for the financial support of the Lead-project: Mechanics, Modeling and Simulation of Aortic Dissection.

References

[1] J. Donea and A. Huerta. *Finite Element Methods for Flow Problems.* John Wiley & Sons, New York, 2003.

[2] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44. Springer, 2013.

[3] P. Hood and C. Taylor. Navier-Stokes equations using mixed interpolation. In J.T. Oden, O.C. Zienkiewicz, R.H. Gallagher, and C. Taylor, editors, *Finite element methods in flow problems*, pages 57–66. . UAH Press, 1974.

[4] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximations of the Stokes equations. In W. Hackbusch, editor, *Efficient solutions of elliptic systems*, pages 11–19. . Notes on Numerical Fluid Mechanics, 1984.

[5] T. J. R. Hughes, L. P. Franca, and M. Balestra. A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: a stable Petrov-Galerkin formulation of the stokes problem accommodating equal-order interpolations. *Computer Methods in Applied Mechanics and Engineering*, 59 (1):85–99, 1986.

[6] V. John. *Finite element methods for incompressible flow problems.* Springer, 2016.

[7] T. J. R. Hughes and L. P. Franca. A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces. *Computer Methods in Applied Mechanics and Engineering*, 65(1):85–96, nov 1987.

[8] F. Brezzi and J. Douglas Jr. Stabilized mixed methods for the Stokes problem. *Numerische Mathematik*, 53(1-2):225–235, jan 1988.

[9] J. Douglas Jr. and J. Wang. An Absolutely Stabilized Finite Element Method for the Stokes Problem. *Mathematics of Computation*, 52(186):495, apr 1989.

[10] J.-J. Droux and T. J. R. Hughes. A boundary integral modification of the Galerkin least squares formulation for the Stokes problem. *Computer Methods in Applied Mechanics and Engineering*, 113(1-2):173–182, mar 1994.

[11] K. E. Jansen, S. S Collis, C. Whiting, and F. Shaki. A better consistency for low-order stabilized finite element methods. *Computer Methods in Applied Mechanics and Engineering*, 174(1-2):153–170, may 1999.

[12] R. Codina and J. Blasco. A finite element formulation for the Stokes problem allowing equal velocity-pressure interpolation. *Computer Methods in Applied Mechanics and Engineering*, 143(3-4):373–391, apr 1997.

[13] P. Bochev and M. Gunzburger. An Absolutely Stable Pressure-Poisson Stabilized Finite Element Method for the Stokes Equations. *SIAM Journal on Numerical Analysis*, 42(3):1189–1207, 2004.

[14] H. Johnston and J.-G. Liu. Accurate, stable and efficient NavierStokes solvers based on explicit treatment of the pressure term. *Journal of Computational Physics*, 199 (1):221–259, sep 2004.

[15] J. Liu. Open and traction boundary conditions for the incompressible NavierStokes equations. *Journal of Computational Physics*, 228(19):7250–7267, oct 2009.

[16] L. P. Franca and S. L. Frey. Stabilized finite element methods: II. The incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 99(2-3):209–233, sep 1992.

[17] L. P. Franca and T. J. R. Hughes. Two classes of mixed finite element methods. *Computer Methods in Applied Mechanics and Engineering*, 69(1):89–129, jul 1988.

[18] R. Hartmann. Higher-order and adaptive discontinuous Galerkin methods with shock-capturing applied to transonic turbulent delta wing flow. *International Journal for Numerical Methods in Fluids*, 72(8):883–894, jul 2013.

[19] P. M. Gresho. Some current CFD issues relevant to the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 87(2-3):201–252, 1991.

[20] J. G. Heywood, R. Rannacher, and S. Turek. Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 22(5):325–352, 1996.

[21] J. Schöberl. Numerical methods for Maxwell equations. *Lecture note*, 2009. URL `https://www.asc.tuwien.ac.at/ schoeberl/wiki/lva/notes/maxwell.pdf`.

[22] J.-G. Liu, J. Liu, and R. L. Pego. Stable and accurate pressure approximation for unsteady incompressible viscous flow. *Journal of Computational Physics*, 229(9): 3428–3453, may 2010.

[23] D. Shirokoff and R. R. Rosales. An efficient method for the incompressible Navier-Stokes equations on irregular domains with no-slip boundary conditions, high order up to the boundary. *Journal of Computational Physics*, 230(23):8619–8646, sep 2011.

[24] P. M. Gresho and R. L. Sani. On pressure boundary conditions for the incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 7 (10):1111–1145, 1987.

[25] V.M. Verzhbitski and I.F. Yumanova. On the quadratic convergence of the Aitken Δ^2 process. *Computational Mathematics and Mathematical Physics*, 51(10):1659–1663, 2011.

[26] U. Küttler and W. A. Wall. Fixed-point fluid-structure interaction solvers with dynamic relaxation. *Computational Mechanics*, 43(1):61–72, 2008.

[27] S. Turek. *Efficient Solvers for Incompressible Flow Problems: An Algorithmic and Computational Approach*. Springer Berlin, 1999.

[28] J. Cahouet and J.P. Chabard. Some fast 3d finite element solvers for the generalized stokes problem. *International Journal for Numerical Methods in Fluids*, 8(8):869–895, 1988.

[29] V. E. Henson and U. M. Yang. Boomeramg: a parallel algebraic multigrid solver and preconditioner. *Applied Numerical Mathematics*, 41:155–177, 2000.

[30] G. Alzetta, D. Arndt, W. Bangerth, V. Boddu, B. Brands, D. Davydov, R. Gassmoeller, T. Heister, L. Heltai, K. Kormann, M. Kronbichler, M. Maier, J.-P. Pelteret, B. Turcksin, and D. Wells. The `deal.II` library, version 9.0. *Journal of Numerical Mathematics*, 26(4):173–183, 2018.

[31] L. I. G. Kovasznay. Laminar flow behind a two-dimensional grid. *Mathematical Proceedings of the Cambridge Philosophical Society*, 44(1):58–62, jan 1948.

[32] Olaf Steinbach. A note on initial higher-order convergence results for boundary element methods with approximated boundary conditions. *Numerical Methods for Partial Differential Equations*, 16(6):581–588, 2000.

[33] C. R. Dohrmann and P. B. Bochev. A stabilized finite element method for the Stokes problem based on polynomial pressure projections. *International Journal for Numerical Methods in Fluids*, 46(2):183–201, sep 2004.