Technische Universität Graz

Numerical Tests for the Recovery of the Gravity Field by Fast Boundary Element Methods

G. Of, B. Schneider

Berichte aus dem Institut für Numerische Mathematik

Bericht 2006/4

Technische Universität Graz

Numerical Tests for the Recovery of the Gravity Field by Fast Boundary Element Methods

G. Of, B. Schneider

Berichte aus dem Institut für Numerische Mathematik

Bericht 2006/4

Technische Universität Graz
Institut für Numerische Mathematik
Steyrergasse 30
A 8010 Graz

WWW: `http://www.numerik.math.tu-graz.at`

Numerical Tests for the Recovery of the Gravity Field by Fast Boundary Element Methods

G. Of and B. Schneider

Institute for Computational Mathematics, Graz University of Technology,
Steyrergasse 30, 8010 Graz, Austria,
of@tugraz.at

Abstract The purpose of this paper is to test the applicability of a fast boundary element method in the context of geoid computations of the gravity. The fast multipole method is the method of choice due to the its advantageous property of a fast evaluation in the post-processing. Several sets of trianglar meshes for the approximation of the unit sphere and several modifications of the prediscribed data have been tested. Also, adapted settings of the fast multipole method have been applied. Finally, the potential of fast boundary element method for this kind of applcations is shown.

1 The single layer potential ansatz and the fast multipole method

First numerical tests have been executed to evaluate the potential use of fast approximation techniques like the fast multipole method [3], adaptive cross approximation [1] and hierarchical matrix arithmetics [4]. As test problem for geoid computations of the gravity, a single layer potential ansatz

$$\frac{1}{4\pi}\int_\Gamma \frac{1}{|x_\ell - y|} t(y) ds_y = f(x_\ell)$$

was chosen to recover the predescribed data $f(x_\ell)$ for $\ell = 1, \ldots, M$ by an unknown density function t. The surface Γ of the unit sphere is approximated by several sets of plane triangles. The unknown density function $t(x)$ is approximated by a linear combination of piecewise constant trial functions,

$$t_h(y) = \sum_{k=1}^{N} t_k \varphi_k(y).$$

The resulting system of linear equations is given by

$$V_h \underline{t} = \underline{f} \qquad \text{with} \qquad V_h[\ell, k] = \frac{1}{4\pi}\int_{\tau_k} \frac{1}{|x_\ell - y|} ds_y \qquad \text{for } k = 1, \ldots, N; \ell = 1, \ldots, M.$$

This overdetermined system of linear equations is solved in the shape of the normal equations

$$V_h^\top V_h \underline{u} = V_h^\top \underline{f}$$

by the conjugate gradient method.

The two main ideas of the fast multipole method [3] are the separation of variables by an expansion of the kernel and the use of a cluster hierarchy to compute these expansions efficiently. The splitting of the kernel is realized by the expansion

$$\frac{1}{4\pi}\frac{1}{|x-y|} \approx \frac{1}{4\pi}\sum_{n=0}^{p}\sum_{m=-n}^{n} |x|^n Y_n^{-m}(x/|x|) \frac{Y_n^m(y/|y|)}{|y|^{n+1}}$$

of the kernel in spherical harmonics

$$Y_n^{\pm m}(x/|x|) = \sqrt{\frac{(n-m)!}{(n+m)!}}(-1)^m \frac{d^m}{dx_3^m} P_n(x_3/|x|) \left(\frac{x_1}{|x|} \pm i \frac{x_2}{|x|}\right)^m$$

for $m \geq 0$. This expansion converges only for $d|x| < |y|$ with some nearfield parameter $d > 1$. Therefore, the nearfield part has to be realized by the standard approach while the multipole approximation is applied for the farfield. The matrix times vector multiplication is realized by

$$\underline{\widetilde{w}}_\ell = \sum_{k \in \mathrm{NF}(\ell)} V_h[\ell, k]\, t_k + \sum_{n=0}^{p} \sum_{m=-n}^{n} |x_\ell|^n Y_n^{-m}(x_\ell/|x_\ell|) \underbrace{\sum_{k \in \mathrm{FF}(\ell)} \frac{t_k}{4\pi} \int_{\tau_k} \frac{Y_n^m(y/|y|)}{|y|^{n+1}} ds_y}_{=L_n^m(\ell)} .$$

Unfortunately, the coefficients $L_n^m(\ell)$ depend on the evaluation point x_ℓ. Therefore, an efficient algorithm [3] has to be used for the computation of these coefficients. This algorithm is based on cluster hierarchies built upon the set of triangles τ_k and the set of evaluation points x_ℓ. For a detailed description of the algorithm see [6], for example. An analysis of the computational effort is given in [5].

2 Numerical tests without the fast multipole method

All computations in this section have been executed on a personal computer with an Intel Pentium M 1.3 GHz processor and 512 MB SDRAM.

2.1 Reference computation

As a model problem, the unit sphere is approximated by several sets of plane triangles. In the reference configuration, the approximation of the sphere starts from a discretization by two square pyramides fit together at their base with eight triangles. Each triangle is divided into four triangles and the new nodes are projected onto the sphere to create a finer discretization. Figure 1 shows the third refinement level of the sphere with 512 triangles.

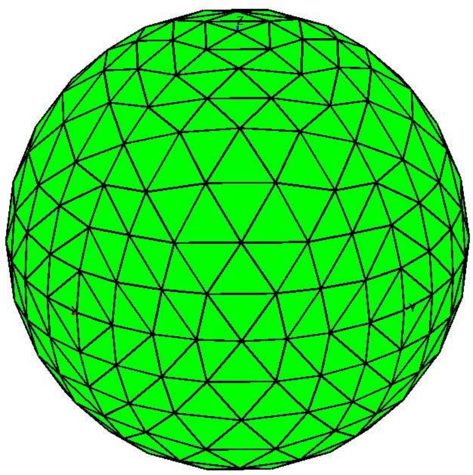


Figure 1: Third refinement level of the sphere with 512 triangles

The satellites positions are arranged along orbits on a sphere with radius 1.05 and the same center as the unit sphere. Only one fictitious measurement datum is used at each pole. The fundamental solution

$$\frac{1}{|x - x_0|} \qquad \text{with } x_0 = (0,0,0)^\top \tag{1}$$

is chosen as measurement data. The relative accuracy is set to 10^{-8} in the conjugate gradient method to solve the normal equations. For all calculations in this section, the fast multipole method is not used. Further, no preconditioning is applied. In the subsequent tests of this section, only the variations of these settings are mentioned. All other settings are chosen as described here. The results of these subsequent computations are compared with the results of the reference configuration in Table 1.

N	M	geo	setup	it	solve	error
	18	2, 10, 0.2	< 1	1	< 1	1.97e-02
8	44	3, 16, 0.188	< 1	1	< 1	1.83e-02
	82	4, 22, 0.182	< 1	1	< 1	2.17e-02
	62	3, 22, 0.15	< 1	4	< 1	4.89e-03
32	162	5, 34, 0.156	1	4	1	4.74e-03
	322	8, 42, 0.19	< 1	3	< 1	5.48e-03
	268	7, 40, 0.175	< 1	35	< 1	1.01e-03
128	640	11, 60, 0.183	1	27	1	1.19e-03
	1262	15, 86, 0.174	1	27	1	1.22e-03
	1038	14, 76, 0.184	2	146	4	1.31e-04
512	2598	22, 120, 0.183	5	119	9	1.61e-04
	5042	30, 170, 0.176	10	109	17	1.81e-04
2048	4202	28, 152, 0.184	31	348	99	1.11e-05
	10236	43, 240, 0.179	75	370	251	1.16e-05

Table 1: Results for the reference configuration.

In all tables, N denotes the number of triangles of the discretization of the sphere and M is the number of measurement points in the orbits of the satellite. In the column "geo", the number of orbits, the number of measurement points per orbit and the ratio of orbits to satellites are listed. "setup" and "solve" are the computational times for setting up the system of linear equations and for solving the normal equations. "it" is the number of iterations needed for solving the normal equations by the conjugate gradient method with a relative accuracy of 10^{-8}. The "error" of the approximation is measured by the average absolute value of the difference of the chosen reference solution and the computed approximation of all measurement points.

The time "setup" for setting up the system of linear equations depends linearly on the number N of satellites and the number M of measurement points. The number of iterations is growing fast with the increase of these numbers. The time "solve" for solving the normal equations increases correspondingly. The average error of the approximation is reduced in each refinement step significantly. This effect is related to the excellent convergence of the approximations computed by boundary element methods inside the computational domain.

2.2 Variations of the radius of the satellite orbits

Now, the radius of the satellite orbits is reduced to 1.01. The corresponding results are given in Table 2. The reduced distance of the surface and the measurement points causes an increase of the average error. The number of iterations goes down or stays constant, except the setting of 2048 triangles and 4202 measurement positions.

N	M	geo	setup	it	solve	error
8	18	2, 10, 0.2	< 1	1	< 1	2.497643e-02
	44	3, 16, 0.188	< 1	1	< 1	2.338679e-02
	82	4, 22, 0.182	< 1	1	< 1	2.665912e-02
32	62	3, 22, 0.15	< 1	4	< 1	4.89e-03
	162	5, 34, 0.156	1	4	1	6.41e-03
	322	8, 42, 0.19	< 1	3	< 1	7.70e-03
128	268	7, 40, 0.175	< 1	30	< 1	1.90e-03
	640	11, 60, 0.183	< 1	27	< 1	2.12e-03
	1262	15, 86, 0.174	< 1	25	1	2.18e-03
512	1038	14, 76, 0.184	2	123	4	4.16e-04
	2598	22, 120, 0.183	5	69	7	5.05e-04
	5042	30, 170, 0.176	10	69	14	5.64e-04
2048	4202	28, 152, 0.184	31	373	104	9.57e-05
	10236	43, 240, 0.179	76	211	178	1.04e-04

Table 2: Results for the reduced radius of the orbits.

An enlargement of the radius of the orbits to 1.1 leads to a decrease of the error, see Table 3. The number of iterations increases or is constant for all discretizations except for the one of 2048 elements. In this case the number of iterations decreases.

N	M	geo	setup	it	solve	error
8	18	2, 10, 0.2	< 1	1	< 1	1.53e-02
	44	3, 16, 0.188	< 1	1	< 1	1.42e-02
	82	4, 22, 0.182	< 1	1	< 1	1.72e-02
32	62	3, 22, 0.15	< 1	4	< 1	3.35e-03
	162	5, 34, 0.156	< 1	4	< 1	3.32e-03
	322	8, 42, 0.19	< 1	3	< 1	3.71e-03
128	268	7, 40, 0.175	< 1	35	< 1	5.03e-04
	640	11, 60, 0.183	1	28	1	6.07e-04
	1262	15, 86, 0.174	< 1	29	< 1	6.09e-04
512	1038	14, 76, 0.184	2	193	5	3.30e-05
	2598	22, 120, 0.183	5	175	11	4.30e-05
	5042	30, 170, 0.176	9	180	19	4.87e-05
2048	4202	28, 152, 0.184	31	277	83	3.71e-06
	10236	43, 240, 0.179	73	314	217	3.86e-06

Table 3: Results for the increased radius of the orbits.

2.3 Shifted singularity of the fundamental solution

Table 4 shows the results when the measurement data is given by the fundamental solution (1) with a shifted singularity, where $x_0 = (0.001, 0, 0)^\top$. This shift shall simulate a slight distribution of the measurement data. The errors are almost the same but the number of iterations increases.

N	M	geo	setup	it	solve	error
8	18	2, 10, 0.2	< 1	2	< 1	1.97e-02
	44	3, 16, 0.188	< 1	2	< 1	1.83e-02
	82	4, 22, 0.182	< 1	2	< 1	2.17e-02
32	62	3, 22, 0.15	1	9	1	4.89e-03
	162	5, 34, 0.156	< 1	10	1	4.74e-03
	322	8, 42, 0.19	< 1	8	< 1	5.48e-03
128	268	7, 40, 0.175	< 1	44	< 1	1.01e-03
	640	11, 60, 0.183	1	34	1	1.19e-03
	1262	15, 86, 0.174	1	34	1	1.22e-03
512	1038	14, 76, 0.184	2	184	4	1.31e-04
	2598	22, 120, 0.183	4	145	9	1.61e-04
	5042	30, 170, 0.176	10	130	17	1.81e-04
2048	4202	28, 152, 0.184	30	332	94	1.12e-05
	10236	43, 240, 0.179	73	434	274	1.15e-05

Table 4: Results for a shift of the singularity to $x_0 = (0.001, 0, 0)^\top$.

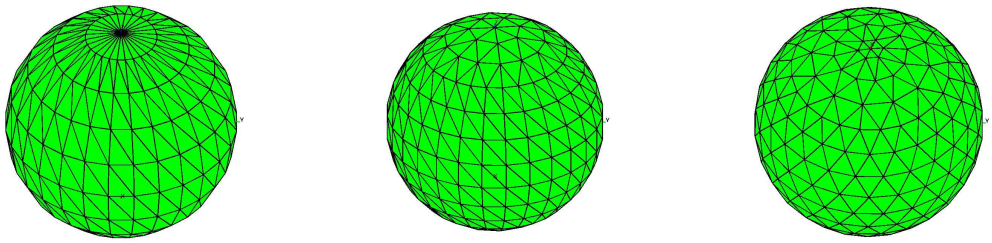
Almost the same result are obtained for a shift of the singularity to $x_0 = (0.01, 0, 0)^\top$, see Table 5. The only noticeable difference appears in the number of iterations for the finest discretization.

N	M	geo	setup	it	solve	error
8	18	2, 10, 0.2	< 1	2	< 1	1.97e-02
	44	3, 16, 0.188	< 1	2	< 1	1.83e-02
	82	4, 22, 0.182	< 1	2	< 1	2.18e-02
32	62	3, 22, 0.15	< 1	9	< 1	4.88e-03
	162	5, 34, 0.156	1	10	1	4.74e-03
	322	8, 42, 0.19	< 1	10	< 1	5.48e-03
128	268	7, 40, 0.175	1	44	1	1.01e-03
	640	11, 60, 0.183	1	34	1	1.19e-03
	1262	15, 86, 0.174	1	34	1	1.22e-03
512	1038	14, 76, 0.184	2	196	5	1.31e-04
	2598	22, 120, 0.183	5	140	9	1.61e-04
	5042	30, 170, 0.176	10	135	18	1.82e-04
2048	4202	28, 152, 0.184	30	434	113	1.09e-05
	10236	43, 240, 0.179	74	357	239	1.17e-05

Table 5: Results for a shift of the singularity to $x_0 = (0.001, 0, 0)^\top$.

2.4 Other discretizations of the sphere

In this subsection, several discretizations of the sphere are tested. For the first discretization, the sphere is split up by angles of longitude and latitude. Each of the corresponding segments is subdivided into two triangles, see Figure 2a. The angles of longitude are arranged in a way that each orbit of the satellites is in the middle of two neighboring angles of longitude. The results for these meshes are given in Table 6. For comparable numbers of triangles and measurement points, the number of iterations is reduced significantly compared to the reference values of Table 1. On the other hand, the computed errors are larger.



(a) equiangular (b) modified equiangular (c) uniform, rotational

Figure 2: Several discretizations of the sphere.

N	M	geo	setup	it	solve	error
504	1038	14, 76, 0.184	2	26	3	5.21e-04
528	2598	22, 120, 0.183	5	10	5	1.44e-03
480	5042	30, 170, 0.176	8	4	8	3.42e-03
600	5042	30, 170, 0.176	10	6	11	2.18e-03
2016	4202	28, 152, 0.184	29	133	54	5.78e-05
2064	10236	43, 240, 0.179	72	55	97	2.44e-04

Table 6: Results for adapted meshes.

The second alternative discretization is very similar to the first one. The only difference is that at the poles the number of triangles is reduced, see Figure 2b. The corresponding results are given in Table 7. Again, the number of iterations decreases and the errors increase compared to the reference values of Table 1. In comparison to the results of Table 6, the more regular elements at the poles help to decrease the average error. But the modified meshes fit worse to the measurement points. So the number of iterations is larger again.

N	M	geo	setup	it	solve	error
532	1038	14, 76, 0.184	2	97	4	2.03e-04
484	2598	22, 120, 0.183	5	39	6	7.03e-04
572	2598	22, 120, 0.183	6	48	7	5.29e-04
540	5042	30, 170, 0.176	9	15	10	9.97e-04
2072	4202	28, 152, 0.184	3	225	74	3.49e-05
2100	10236	43, 240, 0.179	75	139	140	1.20e-04

Table 7: Results for adapted meshes with less triangles at the poles.

The setting of Figure 2a seems to be closest to some kind of interpolation scheme of the boundary element method. But the bad regularity of the corresponding meshes reduces the accuracy of the solution.

The third alternative set of meshes, see Figure 2c, is constructed by cutting the sphere in slices and discretizing the part of the sphere surface of each each slice separately. The algorithm uses triangles of almost the same size. This discretization has the opposite effect as the discretizations before, see Table 8. The number of iterations increases and the error goes down slightly in comparison to Table 1.

N	M	geo	setup	it	solve	error
504	1038	14, 76, 0.184	2	489	8	1.18e-04
	2598	22, 120, 0.183	5	428	18	1.50e-04
	5042	30, 170, 0.176	9	202	21	1.67e-04
2016	4202	28, 152, 0.184	29	633	148	8.98e-06
	10236	43, 240, 0.179	72	509	301	1.00e-05

Table 8: Results for a mesh constructed by rotation.

2.5 Several measurements per pole

Since in reality, the number of measurement points at the poles is rather large, the next test is executed for multiple measurement points at the poles. This causes only a slight increase of the error and of the number of iterations as shown in Table 9.

N	M	geo	setup	it	solve	error
8	20	2, 10, 0.2	< 1	1	< 1	2.21e-02
	48	3, 16, 0.188	< 1	1	< 1	2.23e-02
	88	4, 22, 0.182	< 1	1	< 1	2.56e-02
32	66	3, 22, 0.15	< 1	4	< 1	5.24e-03
	170	5, 34, 0.156	< 1	4	< 1	5.01e-03
	336	8, 42, 0.19	< 1	3	< 1	5.65e-03
128	280	7, 40, 0.175	< 1	34	< 1	1.13e-03
	660	11, 60, 0.183	1	30	1	1.21e-03
	1290	15, 86, 0.174	1	27	1	1.25e-03
512	1064	14, 76, 0.184	3	155	5	1.37e-04
	2640	22, 120, 0.183	5	119	9	1.61e-04
	5100	30, 170, 0.176	10	109	16	1.83e-04
2048	4256	28, 152, 0.184	31	351	100	1.12e-05
	10320	43, 240, 0.179	75	391	257	1.16e-05

Table 9: Results for multiple measurement points at the poles.

3 Multipole method

In this section, the fast multipole method is used to speed up the calculations. Since a computer with more RAM had to be used for finer meshes, the reference calculations of Table 1 had to be redone in order to be able to compare the figures of the standard method and of the fast multipole method. The values of the redone computations are summarized in Table 10. All calculations in this section have been executed on a personal computer with an 3.2 GHz INTEL Pentium 4 processor and 2 GB of RAM.

N	M	geo	setup	it	solve	error
	18	2, 10, 0.2	1	1	1	1.97e-02
8	44	3, 16, 0.188	< 1	1	< 1	1.83e-02
	82	4, 22, 0.182	< 1	1	< 1	2.17e-02
	62	3, 22, 0.15	1	4	1	4.89e-03
32	162	5, 34, 0.156	< 1	4	< 1	4.74e-03
	322	8, 42, 0.19	< 1	3	< 1	5.48e-03
	268	7, 40, 0.175	< 1	35	< 1	1.01e-03
128	640	11, 60, 0.183	1	27	1	1.19e-03
	1262	15, 86, 0.174	1	27	1	1.22e-03
	1038	14, 76, 0.184	2	149	3	1.31e-04
512	2598	22, 120, 0.183	3	109	5	1.61e-04
	5042	30, 170, 0.176	5	110	9	1.81e-04
	4202	28, 152, 0.184	18	322	60	1.11e-05
2048	10236	43, 240, 0.179	43	395	161	1.16e-05
	20282	60, 340, 0.176	88	364	310	1.20e-05

Table 10: Results for the reference configuration on a faster computer.

3.1 Fast multipole method without preconditioning

The same computations and further refinement steps have been executed by using the fast multipole method. The used settings for the parameters of the fast multipole method are given in Table 11.

N	MPLEVEL	DEGREESL	CMP
512	2	4	1.5
2048	3	4	2.0
8192	4	4	2.6
32768	5	5	2.8

Table 11: Settings of the fast multipole method.

"MPLEVEL" is the maximum level used in the cluster tree of the fast multipole method. "DEGRREESL" denotes the expansion degree used for the kernel approximation, while "CMP" is the parameter which controls the size of the nearfield. The results of the fast multipole method are given in Table 12.

The fast multipole method has not been applied to the coarse discretizations since the standard method is faster for this small discretization sizes due to some overhead of the fast multipole method. The numbers of iterations rise compared to the standard boundary element method. This is a somehow surprising because in boundary element methods the accuracy of the fast multipole method used here gives the same number of iterations as a standard method. But this

N	M	geo	setup	it	solve	error
	1038	14, 76, 0.184	< 1	230	7	1.30e-04
512	2598	22, 120, 0.183	1	162	6	1.60e-04
	5042	30, 170, 0.176	1	143	7	1.79e-04
	4202	28, 152, 0.184	3	454	63	1.09e-05
2048	10236	43, 240, 0.179	4	414	74	1.21e-05
	20282	60, 340, 0.176	9	436	104	1.23e-05
	16310	54, 304, 0.177	12	560	346	3.06e-06
8192	41110	86, 480, 0.179	27	503	411	3.90e-06
	81362	120, 680, 0.177	52	501	561	3.95e-06
	65450	108, 608, 0.177	48	525	2755	2.94e-06
32768	163820	171, 960, 0.178	116	557	3419	2.91e-06
	328154	242, 1358, 0.178	231	523	4004	2.93e-06

Table 12: Results for the reference configuration using the fast multipole method.

may be due to the worse condition number of the system of normal equations. For the meshes of 512 and 2048 triangles, the fast multipole method is faster than the standard method for the larger numbers M of measurement points. This is due to the fact, that the number M mainly effects the number of evaluations and not the number of translation and conversion operations of the fast multipole method. For the further refined meshes, their is also a speedup to the expected times for solving the normal equations. One would expect more than 930 seconds for solving the system of 8194 triangles and 16310 measurement points by a standard boundary element method. At this stage, the number M of measurement data effects the time for solving the system of linear equations only moderately, since the computational time is dominated by the part of the algorithm which is independent of the number of evaluation points. The numbers of iterations still increase but seem to be bounded somehow. The numbers for the fast multipole method are by far not optimal, since a rather simple implementation has been used. More involved implementations may speedup the solving times by a factor of about four. In this case, the advantage of the fast multipole method over the standard approach would be even larger.

Since the error reduction is not optimal anymore for the finer meshes, the accuracy of the fast multipole method was increase, see Table 13. The following table shows that an increase of DEGREESL and CMP leads to a reduction of the error.

N	MPLEVEL	DEGREESL	CMP
8192	4	5	3

Table 13: Settings for more accurate computations.

This leads to a reduction of the error but also to an increase of the computational times, see Table 14. Since the measured average errors are close to the relative error, the accuracy is very high. The average error is reduced to an accuracy of about $3 \cdot 10^{-6}$. This maximal accuracy is not related to the discretization error and the error induced by the fast multipole approximation but to the relative accuracy used in the conjugate gradient method. Therefore, a better fast multipole approximation does not pay off and its accuracy can be reduced for finer discretizations to speed up the computational times.

3.2 Fast multipole method with fixed nearfield parameter

Now, the nearfield parameter “CMP” is fixed to make it possible to calculate finer discretizations of the sphere. The approximation error of the fast multipole method is controlled by the parameter

N	M	geo	setup	it	solve	error
	16310	54, 304, 0.177	15	505	665	2.85e-06
8192	41110	86, 480, 0.179	34	475	786	3.19e-06
	81362	120, 680, 0.177	68	497	1016	3.01e-06

Table 14: results of the fast multipole multipole method with higher accuracy.

"DEGREESL" for the expansion, see Table 15.

N	MPLEVEL	DEGREESL	CMP
512	2	2	2.0
2048	3	4	2.0
8192	4	6	2.0
32768	5	8	2.0
131072	6	10	2.0

Table 15: Settings of the fast multipole method with fixed CMP.

Table 16 gives the results of the fast multipole method for the fixed nearfield parameter. In comparison with Table 12, the computational times for solving the normal equations are larger. This shows that this choice of parameters is not optimal. The same effect is observed in boundary element methods. On the other hand, this choice of parameters enables the computation of problems of larger sizes. Surprisingly the numbers of iterations increase only slightly or are even reduced for the largest problems sizes. Again, there is no error reduction on the finer meshes anymore, as the relative accuracy of the conjugate gradient method limits the overall accuracy of the approximation.

N	M	geo	setup	it	solve	error
	1038	14, 76, 0.184	< 1	348	3	1.36e-04
512	2598	22, 120, 0.183	1	246	5	1.64e-04
	5042	30, 170, 0.176	3	157	7	1.89e-04
	4202	28, 152, 0.184	2	454	61	1.09e-05
2048	10236	43, 240, 0.179	4	414	71	1.21e-05
	20282	60, 340, 0.176	8	436	99	1.23e-05
	16310	54, 304, 0.177	9	519	758	2.70e-06
8192	41110	86, 480, 0.179	19	499	852	3.17e-06
	81362	120, 680, 0.177	40	499	991	3.00e-06
	65450	108, 608, 0.177	30	509	7860	2.87e-06
32768	163820	171, 960, 0.178	71	503	8200	3.12e-06
	328154	242, 1358, 0.178	141	429	7721	3.55e-06
131072	262226	216, 1216, 0.178	99	472	57966	2.96e-06

Table 16: Results for a fixed nearfield parameter.

The parameter "DEGREESL" of the expansion length of the fast multipole approximation is now fixed at eight to do computation for even finer meshes and more measurement points. These parameters of the fast multipole method are given in Table 17.

The results for the reduced accuracy of the approximation and finer meshes are given in Table 18. Even though the accuracy of the fast multipole method has been reduced, the error for

N	MPLEVEL	DEGREESL	CMP
131072	6	8	2.0
524288	7	8	2.0

Table 17: Settings for larger problem sizes.

131072 triangles and 262226 measurement points is not increased but even slightly reduced. This is a further indication that the remaining errors are not due to the fast multipole approximation but caused by other approximation errors. The increase of the computational times for the last refinement step is lower as expected from theory since the accuracy has not been adopted to the problem size.

N	M	geo	setup	it	solve	error
131072	262226	216, 1216, 0.178	96	498	30301	2.82e-06
	655958	342, 1920, 0.178	222	477	31789	2.95e-06
524288	1048898	432, 2430, 0.178	162	458	74266	3.10e-06

Table 18: Results with reduced accuracy of the approximation.

A smaller nearfield in the fast multipole method than usual does not pay off due to larger computational times for small problem sizes. But together with a finer clustering technique resulting in a deeper cluster tree, the smaller nearfield gives the possibility to compute larger problems and to reduce the computational times for large problems. Corresponding results are given in Table 19.

N	M	settings	setup	it	solve	error
131072	262226	2.0, 8, 6, 10	96	498	30301	2.82e-06
		2.0, 8, 7, 1	22	516	77634	2.72e-06
524288	1048898	2.0, 8, 7, 10	(162)	458	(74266)	3.10e-06
		2.0, 4, 7, 1	35	556	17532	4.33e-06
524288	2622722	2.0, 7, 7, 10	212	510	71120	2.91e-06
		2.0, 4, 7, 10	199	523	17615	4.47e-06
		2.0, 4, 7, 1	55	523	18030	4.55e-06
524288	5247312	2.0, 4, 7, 7	250	485	18475	4.53e-06
		2.0, 4, 7, 1	89	493	18759	4.49e-06

Table 19: Results of the fast multipole boundary element method with finer clustering.

The parameters of the fast multipole method have been adopted to create a finer cluster structure. These parameters are given in the column “settings”. The first value is the nearfield parameter CMP and has been chosen as 2.0. The second parameter is “DEGREESL”. A reduced expansion degree has only a small influence on the accuracy of the approximation but speeds up the computation. The third parameter is the maximal cluster depth “MPLEVEL”. The forth and last parameter is the maximal number of elements in a single cluster. In the used adaptive scheme, each cluster is refined into smaller clusters until the number of elements in the cluster is smaller than the given maximal number or the maximal cluster depth is reached. Due to these settings the error is increased slightly. But on the other hand, the computational times are reduced by a factor of two even though the problem size is increased. This effect is due to the fact that the size of the 524288 triangles and the size of the clusters on the finest cluster level of the fast multipole method are small enough to resolve the distance between the sphere and the measurement points. Therefore the size of the nearfield matrix is reduced and the computation is speeded up. For example, only about 14 million nearfield entries have to be saved of about 2751 billion matrix entries. This corresponds to about 0.0005 percent. This effect is not obtained for 131072 triangles as the triangles and so the corresponding clusters are not small enough to cause this effect.

4 Conclusions and Final Remarks

The numerical tests have shown that the boundary element method gives a good and fast converging approximation of the predescribed data. The use of the fast multipole method enables the computation of large problem sizes. These first test have shown the potential of fast boundary element methods to approximate the gravity field of the earth.

The effect of the vanishing nearfield can be enhanced by adopted clustering techniques which take better care of the fact that the source and target points are separated. This will further speed up the computational times and will further reduce the memory requirement such that even larger problems can be considered. A trickier implementation of the fast multipole method will give some extra speedup of the computation. First tests for the preconditioning of the system show promising results and give strong hints that the numbers of iterations can be reduced by preconditioning techniques based on hierarchical matrix arithmetics [2]. Even a direct hierarchical matrix solver may be attainable. An analysis of the properties of the involved operator should give helpful hints for the development of these fast solution techniques. Even though the method has been tested only for uniform distributions so far, it can be applied to locally adapted distributions and locally refined discretizations for a better local resolution.

Acknowledgement *This work has been supported by the Austrian Science Fund 'Österreichischen Forschungsförderungsgesellschaft m.b.H. (FFG)' under the grant 'The Austrian Geoid 2007 (GEOnAUT)'.*

References

[1] M. Bebendorf, S. Rjasanow (2003): Adaptive low–rank approximation of collocation matrices. Computing, 70, 1–24.

[2] M. Bebendorf (2005): Hierarchical LU decomposition-based preconditioners for BEM. Computing, 74(3), 225-247.

[3] L. Greengard, V. Rokhlin (1987): A fast algorithm for particle simulations. J. Comput. Phys., 73, 325–348.

[4] W. Hackbusch (1999): A sparse matrix arithmetic based on $\mathcal{H}$–matrices. Part I: Introduction to $\mathcal{H}$–matrices. Computing, 62, 89–108.

[5] B. Muth, G. Of, P. Eberhard, O. Steinbach (2006): Collision detection for complicated polyhedra using the fast multipole method or ray crossing. Arch. Appl. Mech., accepted.

[6] G. Of, O. Steinbach, W.L. Wendland (2006): The fast multipole method for the symmetric boundary integral formulation. IMA J. Numer. Anal., 26, 272–296.

Erschienene Preprints ab Nummer 2005/1

2005/1	O. Steinbach	Numerische Mathematik 1. Vorlesungsskript.
2005/2	O. Steinbach	Technische Numerik. Vorlesungsskript.
2005/3	U. Langer G. Of O. Steinbach W. Zulehner	Inexact Fast Multipole Boundary Element Tearing and Interconnecting Methods
2005/4	U. Langer G. Of O. Steinbach W. Zulehner	Inexact Data–Sparse Boundary Element Tearing and Interconnecting Methods
2005/5	U. Langer O. Steinbach W. L. Wendland	Fast Boundary Element Methods in Industrial Applications Söllerhaus Workshop, 25.–28.9.2005, Book of Abstracts.
2005/6	U. Langer A. Pohoata O. Steinbach	Dual–Primal Boundary Element Tearing and Interconnecting Methods
2005/7	O. Steinbach (ed.)	Jahresbericht 2004/2005
2006/1	S. Engleder O. Steinbach	Modified Boundary Integral Formulations for the Helmholtz Equation
2006/2	O. Steinbach (ed.)	2nd Austrian Numerical Analysis Day, Book of Abstracts
2006/3	B. Muth G. Of P. Eberhard O. Steinbach	Collision Detection for Complicated Polyhedra Using the Fast Multipole Method or Ray Crossing