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Modified combined field integral equations for electromagnetic scattering

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Abstract

The boundary integral formulation of exterior boundary value problems for the Maxwell system may not be equivalent to the original uniquely solvable problem if the wave number corresponds to an eigenvalue of an associated interior eigenvalue problem. To avoid these spurious modes one may use a combined boundary integral approach. To analyze the resulting boundary integral equations in the energy function spaces suitable regularizations have to be introduced. Here we formulate and analyze an alternative modified boundary integral equation which is based on the use of standard boundary integral operators only. In particular we can avoid the use of the Hodge decomposition in both the analysis and the implementation. A first numerical example shows the applicability of the proposed approach.

1 Introduction

The modeling of electromagnetic scattering at a perfect conductor in the exterior of a bounded domain $\Omega \subset \mathbb{R}^3$ leads to the Dirichlet boundary value problem [11, 18, 19]

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x) + \kappa^2 \mathbf{U}(x) = 0 \qquad \text{for } x \in \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}, \tag{1.1}$$

$$\mathbf{n}_x \times (\mathbf{U}(x) \times \mathbf{n}_x) = \mathbf{g}(x) \qquad \text{for } x \in \Gamma = \partial\Omega \tag{1.2}$$

where κ is purely imaginary, i.e. $\kappa = ik$ with $k > 0$. This more general notation is used since we will define some boundary integral operators for real κ too. In addition to the exterior boundary value problem (1.1) we need to formulate the radiation condition of electromagnetic scattering, i.e. the Silver–Müller radiation condition

$$\lim_{r=|x|\to\infty} \int_{\partial B_r} |\mathbf{curl}\,\mathbf{U}(x) \times \mathbf{n}_x - ik\mathbf{U}(x)|^2 ds_x = 0. \tag{1.3}$$

Note that the exterior Dirichlet boundary value problem (1.1)–(1.3) admits a unique solution. According to the partial differential operator in (1.1) we can formulate Green's first formula which is valid for sufficiently smooth functions as

$$\begin{aligned}\int\limits_{\Omega} \mathbf{curl}\,\mathbf{curl}\,\mathbf{U}(x)\cdot\overline{\mathbf{V}}(x)\,dx &= \int\limits_{\Omega} \mathbf{curl}\,\mathbf{U}(x)\cdot\mathbf{curl}\,\overline{\mathbf{V}}(x)\,dx \\ &\quad - \int\limits_{\Gamma} (\mathbf{curl}\,\mathbf{U}(x)_{|\Gamma}\times\mathbf{n}_x)\cdot(\mathbf{n}_x\times(\overline{\mathbf{V}(x)}_{|\Gamma}\times\mathbf{n}_x))\,ds_x.\end{aligned} \tag{1.4}$$

Based on (1.4) related Sobolev spaces and corresponding trace operators can be introduced [3, 4, 5, 6, 7], these results will be summarized in Section 2. Then, the well known Stratton–Chu representation formula will be discussed which implies the definition of appropriate potential and boundary integral operators [5, 7, 10, 12, 15, 16, 19]. The corresponding boundary integral equations can be used for a numerical treatment of the problem by means of boundary element methods [5, 7, 10, 11, 12, 19]. But although the exterior boundary value problem (1.1)–(1.3) is uniquely solvable, the standard boundary integral equations are not uniquely solvable if the wave number κ corresponds to an eigenvalue of an associated interior eigenvalue problem. To avoid these spurious modes Brakhage and Werner [1] introduced a combined boundary integral approach for the acoustic problem in 1965. In the same year Panich discussed this approach for the electromagnetic case [20]. But the analysis of the approach of Brakhage and Werner is applicable for smooth boundaries only. Hence modified boundary integral equations were discussed in [9] for the acoustic case and in [8] for the electromagnetic case. Note that all of these approaches are based on using some compact regularisation operators. In [14] an alternative approach was introduced for the acoustic case which does not rely on compact perturbations. Here we want to generalize this idea to obtain modified combined boundary integral equations for the electromagnetic case. In addition we will not made use on a Hodge decomposition in both the analysis and the implementation.

The paper is structured as follows: In Section 2 we first summarize the definitions of Sobolev spaces to handle the variational formulation of the Maxwell system, and introduce potential operators and related boundary integral operators as needed later. We also discuss standard boundary integral approaches to solve the exterior Dirichlet boundary value problem, and comment on combined and already existent stabilized boundary integral formulations. An alternative modified boundary integral equation is formulated and analyzed in Section 3. In particular, we present a new boundary integral formulation which is based on the use of standard, and therefore already available boundary integral operators, and which is stable for all wave numbers. In Section 4 we describe a first numerical example to show the applicability of the proposed approach. We finally end up with some conclusions and an outlook on ongoing work.

2 Function spaces and boundary integral equations

The formulation of boundary integral equations for the Maxwell system requires the use of the correct function spaces. Here we will only recall the definitions and the properties of Sobolev spaces for the Maxwell system, for a more detailed description see, e.g. [3, 4].

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedron [3] with a Lipschitz boundary $\Gamma = \partial\Omega$ which is the union of plane faces Γ_i, i.e. $\Gamma = \bigcup_i \Gamma_i$ where $\mathbf{n}_i$ is the exterior normal vector on Γ_i.

The partial differential equation in (1.1) and Green's first formula (1.4) motivate the definition of the energy space

$$\mathbf{H}(\mathbf{curl}\,,\Omega) := \{\mathbf{V} \in \mathbf{L}_2(\Omega) : \mathbf{curl}\,\mathbf{V} \in \mathbf{L}_2(\Omega)\}$$

as well as the space of the natural solutions

$$\mathbf{H}(\mathbf{curl}^{\,2},\Omega) := \{\mathbf{V} \in \mathbf{H}(\mathbf{curl}\,,\Omega) : \mathbf{curl}\,\mathbf{curl}\,\mathbf{V} \in \mathbf{L}_2(\Omega)\}\,.$$

In addition we need to introduce appropriate Sobolev spaces on the boundary. For $|s| \leq 1$ and for scalar functions on the boundary the usual Sobolev spaces are denoted by $H^s(\Gamma)$. Let us define the Dirichlet traces

$$\gamma_D\mathbf{U} := \mathbf{n} \times (\mathbf{U}_{|\Gamma} \times \mathbf{n}) = \mathbf{n} \times \gamma_\times\mathbf{U}, \quad \gamma_\times\mathbf{U} := \mathbf{U}_{|\Gamma} \times \mathbf{n}$$

and the Neumann trace

$$\gamma_N\mathbf{U} := \mathbf{curl}\,\mathbf{U}_{|\Gamma} \times \mathbf{n}$$

which all are mappings into tangential spaces. Hence we introduce the space

$$\mathbf{L}_{2,t}(\Gamma) := \{\mathbf{u} \in \mathbf{L}_2(\Gamma) : \mathbf{u} \cdot \mathbf{n} = 0\}$$

of tangential $\mathbf{L}_2(\Gamma)$ integrable functions. For higher order Sobolev spaces we use the piecewise definition

$$\mathbf{H}^s_{pw,t}(\Gamma) := \{\mathbf{u} \in \mathbf{L}_{2,t}(\Gamma) : \mathbf{u} \in \mathbf{H}^s(\Gamma_k)\,, k = 1, \ldots, N_\Gamma\}.$$

The trace spaces $\gamma_D\mathbf{H}^1(\Omega)$ and $\gamma_\times\mathbf{H}^1(\Omega)$ are denoted by $\mathbf{H}^{1/2}_{\parallel}(\Gamma)$ and $\mathbf{H}^{1/2}_{\perp}(\Gamma)$ respectively, for an alternative definition see [3]. The dual spaces with respect to $\mathbf{L}_{2,t}(\Gamma)$ are denoted by $\mathbf{H}^{-1/2}_{\parallel}(\Gamma)$ and $\mathbf{H}^{-1/2}_{\perp}(\Gamma)$.

Before introducing the trace spaces of $\mathbf{H}(\mathbf{curl}\,\Omega)$ we need to define some boundary differential operators. Here we just give definitions for smooth boundaries, for Lipschitz polyhedrons see [3, 4]. For scalar functions u and vector fields $\mathbf{U}$ which are given on Γ we denote by $\widetilde{u}$ and $\widetilde{\mathbf{U}}$ arbitrary bounded extensions into a three–dimensional neighborhood of Γ. Then we can define the boundary differential operators

$$\begin{aligned}
\nabla_\Gamma\, u &:= [\mathbf{n} \times (\nabla\widetilde{u} \times \mathbf{n})]_{|\Gamma}\,,\\
\mathbf{curl}_\Gamma\, u &:= [\mathbf{curl}\,(\widetilde{u}\,\mathbf{n})]_{|\Gamma}\,,\\
\mathrm{curl}_\Gamma\,\mathbf{U} &:= \left[\mathbf{n} \cdot \mathbf{curl}\,\widetilde{\mathbf{U}}\right]_{|\Gamma},\\
\mathrm{div}_\Gamma\,\mathbf{U} &:= \left[\mathrm{div}\,\widetilde{\mathbf{U}}\right]_{|\Gamma}.
\end{aligned}$$

Note that the operator $\mathbf{curl}_\Gamma$ is the adjoint of curl_Γ, and ∇_Γ is the adjoint of $-\mathrm{div}_\Gamma$. With the help of these operators we can finally define the Hilbert spaces

$$\mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) := \left\{ \mathbf{u} \in \mathbf{H}_\perp^{-1/2}(\Gamma) : \mathrm{curl}_\Gamma \, \mathbf{u} \in \mathbf{H}^{-1/2}(\Gamma) \right\},$$
$$\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \left\{ \mathbf{u} \in \mathbf{H}_\|^{-1/2}(\Gamma) : \mathrm{div}_\Gamma \, \mathbf{u} \in \mathbf{H}^{-1/2}(\Gamma) \right\}.$$

These spaces are dual to each other with respect to $\mathbf{L}_{2,t}(\Gamma)$, and represent the trace spaces $\gamma_D \mathbf{H}(\mathbf{curl}, \Omega)$ and $\gamma_\times \mathbf{H}(\mathbf{curl}, \Omega)$, respectively. Furthermore, there holds the following theorem [3].

Theorem 2.1 *The operators*

$$\gamma_D : \mathbf{H}(\mathbf{curl}, \Omega) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma),$$
$$\gamma_N : \mathbf{H}(\mathbf{curl}\,\mathbf{curl}, \Omega) \to \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

are linear, continuous and surjective.

Now we are able to introduce some potential and boundary integral operators which are relevant for electromagnetic scattering [10]. The solution of the exterior Dirichlet boundary value problem (1.1)–(1.3) can be described by using the Stratton–Chu representation formula [16]

$$\mathbf{U}(x) \;=\; -\boldsymbol{\Psi}_M^\kappa(\gamma_D^c \mathbf{U})(x) - \boldsymbol{\Psi}_S^\kappa(\gamma_N^c \mathbf{U})(x) \quad \text{for } x \in \Omega^c \tag{2.1}$$

where the Maxwell single layer potential is given by

$$\boldsymbol{\Psi}_S^\kappa(\boldsymbol{\mu}) := \boldsymbol{\Psi}_A^\kappa(\boldsymbol{\mu}) - \frac{1}{\kappa^2}\mathbf{grad}\, \Psi_V^\kappa(\mathrm{div}_\Gamma(\boldsymbol{\mu})),$$

and the Maxwell double layer potential is defined by

$$\boldsymbol{\Psi}_M^\kappa(\boldsymbol{\lambda})(x) := \mathbf{curl}\, \boldsymbol{\Psi}_A^\kappa(\boldsymbol{\lambda} \times \mathbf{n})(x).$$

The operators $\boldsymbol{\Psi}_A^\kappa$ and $\boldsymbol{\Psi}_S^\kappa$ are the vectorial and the scalar single layer potentials which are given by

$$\boldsymbol{\Psi}_A^\kappa(\boldsymbol{\lambda})(x) := \int\limits_\Gamma g_\kappa(x,y)\boldsymbol{\lambda}(y) ds_y, \quad \Psi_V^\kappa(\lambda)(x) := \int\limits_\Gamma g_\kappa(x,y)\lambda(y) ds_y$$

whereas $g_\kappa(x,y)$ is the fundamental solution of the Helmholtz equation,

$$g_\kappa(x,y) = \frac{e^{\kappa|x-y|}}{|x-y|}.$$

To use an indirect approach to represent the solution of (1.1)–(1.3) the following result is essential [10, 12].

Theorem 2.2 *The Maxwell single and double layer potentials are solutions of the partial differential equation in* (1.1) *and fulfill the Silver–Müller radiation condition* (1.3)*. Moreover, the following mapping properties are valid:*

$$\mathbf{\Psi}_S^\kappa : \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^c),$$
$$\mathbf{\Psi}_M^\kappa : \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \Omega \cup \Omega^c).$$

Hence we can represent the solution of the exterior Dirichlet boundary value problem (1.1)–(1.3) either by the single layer potential

$$\mathbf{U}(x) = \mathbf{\Psi}_S^\kappa(\boldsymbol{\mu})(x) \quad \text{for } x \in \Omega^c \tag{2.2}$$

or by using the double layer potential

$$\mathbf{U}(x) = \mathbf{\Psi}_M^\kappa(\boldsymbol{\lambda})(x) \quad \text{for } x \in \Omega^c. \tag{2.3}$$

To find the yet unknown density functions $\boldsymbol{\mu} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ and $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ we have to formulate appropriate boundary integral equations which can be derived from the Dirichlet boundary condition (1.2). For this we first use the trace operators γ_D and γ_N as given in Theorem 2.1 to define related boundary integral operators, in particular for the interior trace we obtain

$$\begin{aligned}
\gamma_D \mathbf{\Psi}_S^\kappa \boldsymbol{\mu}(x) &=: \mathsf{S}_\kappa \boldsymbol{\mu}(x), \\
\gamma_D \mathbf{\Psi}_M^\kappa \boldsymbol{\lambda}(x) &=: \left(\frac{1}{2} I + \mathsf{C}_\kappa\right) \boldsymbol{\lambda}(x), \\
\gamma_N \mathbf{\Psi}_S^\kappa \boldsymbol{\mu}(x) &=: \left(\frac{1}{2} I + \mathsf{B}_\kappa\right) \boldsymbol{\mu}(x), \\
\gamma_N \mathbf{\Psi}_M^\kappa \boldsymbol{\lambda}(x) &=: \mathsf{N}_\kappa \boldsymbol{\lambda}(x),
\end{aligned}$$

while for the exterior trace we get

$$\begin{aligned}
\gamma_D^c \mathbf{\Psi}_S^\kappa \boldsymbol{\mu}(x) &=: \mathsf{S}_\kappa \boldsymbol{\mu}(x), \\
\gamma_D^c \mathbf{\Psi}_M^\kappa \boldsymbol{\lambda}(x) &=: \left(-\frac{1}{2} I + \mathsf{C}_\kappa\right) \boldsymbol{\lambda}(x), \\
\gamma_N^c \mathbf{\Psi}_S^\kappa \boldsymbol{\mu}(x) &=: \left(-\frac{1}{2} I + \mathsf{B}_\kappa\right) \boldsymbol{\mu}(x), \\
\gamma_N^c \mathbf{\Psi}_M^\kappa \boldsymbol{\lambda}(x) &=: \mathsf{N}_\kappa \boldsymbol{\lambda}(x).
\end{aligned}$$

While the single layer boundary integral operator

$$\mathsf{S}_\kappa : \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

and the hypersingular integral operator

$$\mathsf{N}_\kappa : \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$$

are self–adjoint with respect to the complex inner product, the double layer potentials C_κ and B_κ are related to each other as follows.

Lemma 2.3 [10] *For all* $\boldsymbol{\mu} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ *and* $\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *there holds*

$$\langle \mathsf{B}_\kappa \boldsymbol{\mu}, \boldsymbol{\lambda} \rangle \;=\; -\langle \boldsymbol{\mu}, \mathsf{C}_\kappa \boldsymbol{\lambda} \rangle.$$

When using the single layer potential (2.2) we have to find $\boldsymbol{\mu} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ by solving the boundary integral equation

$$\mathsf{S}_\kappa \boldsymbol{\mu}(x) \;=\; \mathbf{g}(x) \quad \text{for } x \in \Gamma \tag{2.4}$$

while for the double layer potential (2.3) $\boldsymbol{\lambda} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ is the solution of the boundary integral equation

$$-\frac{1}{2}\boldsymbol{\lambda}(x) + \mathsf{C}_\kappa \boldsymbol{\lambda}(x) \;=\; \mathbf{g}(x) \quad \text{for } x \in \Gamma. \tag{2.5}$$

When applying the exterior Dirichlet and the exterior Neumann trace to the Stratton–Chu representation formula (2.1) we obtain a system of boundary integral equations,

$$\begin{aligned} \gamma_D^c \mathbf{U} &= \quad -\mathsf{S}_\kappa \gamma_N^c \mathbf{U} \quad &+& \quad (\tfrac{1}{2}I - \mathsf{C}_\kappa)\gamma_D^c \mathbf{U}, \\ \gamma_N^c \mathbf{U} &= (\tfrac{1}{2}I - \mathsf{B}_\kappa)\gamma_N^c \mathbf{U} \quad &+& \quad -\mathsf{N}_\kappa \gamma_D^c \mathbf{U}. \end{aligned} \tag{2.6}$$

In particular, to describe the solution of the exterior Dirichlet boundary value problem (1.1)–(1.3) we may use the first equation in (2.6) to find $\gamma_N^c \mathbf{U} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$\mathsf{S}_\kappa \gamma_N^c \mathbf{U}(x) \;=\; -\frac{1}{2}\mathbf{g}(x) - \mathsf{C}_\kappa \mathbf{g}(x) \quad \text{for } x \in \Gamma. \tag{2.7}$$

Proposition 2.4 [11] *If* $k^2 = \lambda$ *is an eigenvalue of the interior Dirichlet eigenvalue problem*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_\lambda(x) = \lambda \mathbf{U}_\lambda(x) \quad \textit{for } x \in \Omega, \quad \gamma_D \mathbf{U}_\lambda(x) = 0 \quad \textit{for } x \in \Gamma, \tag{2.8}$$

then $\gamma_N \mathbf{U}_\lambda(x)$ *is in the kernel of* S_κ *and* $(-\frac{1}{2}I + \mathsf{B}_\kappa)$*, i.e.*

$$\mathsf{S}_\kappa \gamma_N \mathbf{U}_\lambda \;=\; 0, \quad (\frac{1}{2}I - \mathsf{B}_\kappa)\gamma_N \mathbf{U}_\lambda \;=\; 0$$

where $\kappa = ik$*.*

On the other hand, if k^2 *is not an eigenvalue of the interior Dirichlet eigenvalue problem* (2.8)*, then* $\mathsf{S}_\kappa \mathbf{w} = 0$ *implies* $\mathbf{w} = 0$*.*

Hence, if $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem (2.8), we conclude that the single layer potential operator S_κ is not invertible, and therefore, the boundary integral equations (2.4) and (2.7) are in general not solvable. However, due to

$$\langle -\frac{1}{2}\mathbf{g} - \mathsf{C}_\kappa \mathbf{g}, \gamma_N \mathbf{U}_\lambda \rangle \;=\; \langle \mathbf{g}, (-\frac{1}{2}I + \mathsf{B}_\kappa)\gamma_N \mathbf{U}_\lambda \rangle \;=\; 0$$

we conclude that the right hand side of the boundary integral equation (2.7) is in the image of the single layer potential S_κ, i.e. the boundary integral equation (2.7) of the direct approach is solvable, but the solution is not unique.

Proposition 2.5 [11] *If $k^2 = \mu$ is an eigenvalue of the interior Neumann eigenvalue problem*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{U}_\mu(x) = \mu \mathbf{U}_\mu(x) \quad \textit{for } x \in \Omega, \quad \gamma_N \mathbf{U}_\mu(x) = 0 \quad \textit{for } x \in \Gamma, \tag{2.9}$$

then $\gamma_D \mathbf{U}_\mu(x)$ is in the kernel of N_κ and $(\frac{1}{2}I - \mathsf{C}_\kappa)$, i.e.

$$\mathsf{N}_\kappa \gamma_D \mathbf{U}_\mu \;=\; 0, \quad (\frac{1}{2}I - \mathsf{C}_\kappa)\gamma_D \mathbf{U}_\mu \;=\; 0$$

where $\kappa = ik$.

Hence, if $k^2 = \mu$ is an eigenvalue of the interior Neumann eigenvalue problem (2.9), we conclude that the boundary integral operator $\frac{1}{2}I - \mathsf{C}_\kappa$ is not invertible, and therefore the boundary integral equation (2.5) of the indirect approach is in general not solvable.

To overcome the problem of non–solvability of boundary integral equations due to interior eigenfrequencies one may use a combined approach such as the formulation of Brakhage and Werner who introduced a combined field integral equation for the acoustic scattering problem [1]. The same idea was used by Panich in [20] for the electromagnetic case. In general, the idea is to consider complex linear combinations of the single and double layer potential, i.e.

$$\mathbf{U}(x) \;=\; -i\eta \boldsymbol{\Psi}_S^\kappa \mathbf{w}(x) - \boldsymbol{\Psi}_M^\kappa \mathbf{w}(x) \quad \text{for } x \in \Omega^c$$

where $\eta \in \mathbb{R}_+$ is some parameter to be chosen. The unknown density $\mathbf{w} \in L_2(\Gamma)$ can then be determined from the resulting boundary integral equation

$$\gamma_D^c \mathbf{U}(x) = -i\eta \mathsf{S}_\kappa \mathbf{w}(x) + (\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{w}(x) \;=\; \mathbf{g}(x) \quad \text{for } x \in \Gamma \tag{2.10}$$

which can be proved to be uniquely solvable if the boundary $\Gamma = \partial\Omega$ is sufficiently smooth. But this proof is essentially based on the compactness of the double layer potential operator C_κ which is not satisfied if Ω is a Lipschitz polyhedron. Hence one may introduce a regularisation operator $B : \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that the stabilized boundary integral equation

$$\gamma_D^c \mathbf{U}(x) = -i\eta \mathsf{S}_\kappa \mathbf{w}(x) + (\frac{1}{2}I - \mathsf{C}_\kappa)B\mathbf{w}(x) \;=\; \mathbf{g}(x) \quad \text{for } x \in \Gamma \tag{2.11}$$

admits a unique solution $w \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. A suitable compact operator B was introduced by Buffa and Hiptmair in [8]. The unique solvability of the stabilized boundary integral equation (2.11) is then based on a generalized Gårdings inequality for the single layer potential S_κ, and on the injectivity of the composed boundary integral operator in (2.11). But this approach requires an appropriate splitting, i.e. a Hodge decomposition, of the space $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ to achieve such a generalized Gårdings inequality, which does not carry over to a Galerkin discretization of the stabilized boundary integral equation.

Therefore further considerations are needed to establish a stability and error analysis of the stabilized formulation [8, 17]. Moreover, a numerical implementation of this approach seems to be a difficult task.

Hence, in the next section we will describe an alternative approach which is based on standard boundary integral operators only. To analyze the proposed modified boundary integral formulation we will need some auxiliary results as given in the following.

Due to the boundary integral equations (2.6) we define the Calderon projector

$$\mathcal{C} = \begin{pmatrix} \frac{1}{2}I - \mathsf{C}_\kappa & -\mathsf{S}_\kappa \\ -\mathsf{N}_\kappa & \frac{1}{2}I - \mathsf{B}_\kappa \end{pmatrix}$$

which satisfies the projection property

$$\mathcal{C}^2 \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} = \mathcal{C} \begin{pmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{pmatrix} \tag{2.12}$$

for all $\boldsymbol{\lambda} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\boldsymbol{\mu} \in \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. As a corollary of the projection property (2.12) we then conclude the relations

$$\mathsf{S}_\kappa \mathsf{N}_\kappa = \frac{1}{4}I - \mathsf{C}_\kappa^2, \tag{2.13}$$

$$\mathsf{N}_\kappa \mathsf{S}_\kappa = \frac{1}{4}I - \mathsf{B}_\kappa^2, \tag{2.14}$$

$$-\mathsf{N}_\kappa \mathsf{C}_\kappa = \mathsf{B}_\kappa \mathsf{N}_\kappa, \tag{2.15}$$

$$-\mathsf{C}_\kappa \mathsf{S}_\kappa = \mathsf{S}_\kappa \mathsf{B}_\kappa. \tag{2.16}$$

If the single layer potential operator S_κ is invertible we can define the Steklov–Poincaré operator

$$\mathsf{T}_\kappa := \mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa) \quad : \quad \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \tag{2.17}$$

which allows an alternative symmetric representation

$$\mathsf{T}_\kappa := \mathsf{N}_\kappa + (\frac{1}{2}I + \mathsf{B}_\kappa)\mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa). \tag{2.18}$$

Theorem 2.6 [5] *The operators*

$$\mathsf{A}_0 \;=\; \gamma_D^c \boldsymbol{\Psi}_A^0 \;:\; \mathbf{H}_\parallel^{-1/2}(\Gamma) \to \mathbf{H}_\parallel^{1/2}(\Gamma)$$

and

$$V_0 \;=\; \gamma_D^c \Psi_V^0 \;:\; H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

are self–adjoint as well as $\mathbf{H}_\parallel^{-1/2}(\Gamma)$– and $H^{-1/2}(\Gamma)$–elliptic, respectively. Moreover, for real $\kappa < 0$ the single layer potential

$$\mathsf{S}_\kappa : \mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

is $\mathbf{H}_\parallel^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$–elliptic and selfadjoint.

3 Modified boundary integral equations

In this section we propose an alternative approach of a modified boundary integral equation without using a compact operator B. Because of symmetry reasons we choose

$$B = \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa) : \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma)$$

whereas $\mathsf{S}_0^* : \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma) \to \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma)$ is given by

$$\mathsf{S}_0^*\mathbf{u} \;:=\; \mathbf{n} \times \mathsf{A}_0(\mathbf{u} \times \mathbf{n}) + \mathbf{curl}_\Gamma\, V_0\, \mathrm{curl}_\Gamma\, \mathbf{u}.$$

By using Theorem 2.6 one can prove that S_0^* is $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma)$–elliptic and selfadjoint.

Now we can describe the solution of the exterior Dirichlet boundary value problem (1.1)–(1.3) by

$$\mathbf{U}(x) \;=\; \mathbf{\Psi}_S^\kappa \mathbf{w}(x) - i\eta \mathbf{\Psi}_M^\kappa B\mathbf{w}(x) \quad \text{for } x \in \Omega^c.$$

When applying the exterior Dirichlet trace we find the unknown density $\mathbf{w} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma)$ from the modified boundary integral equation

$$\mathsf{Z}_\kappa \mathbf{w}(x) \;=\; \mathsf{S}_\kappa \mathbf{w}(x) + i\eta(\frac{1}{2}I - \mathsf{C}_\kappa)\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}(x) = \mathbf{g}(x) \quad \text{for } x \in \Gamma. \tag{3.1}$$

To establish the unique solvability of the modified boundary integral equation (3.1) we first prove that Z_κ is coercive. In difference to the usual approach we show the coercivity in the second part, because the single layer potential S_κ does not fulfill a Gårdings inequality. Note that Buffa and Hiptmair used a Hodge decomposition which is not possible in our case.

To prove the coercivity of the operator Z_κ we first define an appropriate equivalent norm in $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma)$ by, see Theorem 2.6,

$$\|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} := \sqrt{\langle \mathsf{S}_\kappa^{-1}\mathbf{u}, \mathbf{u}\rangle}, \quad \mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma).$$

As in the case of a formally elliptic partial differential operator [22] we can prove a contraction property of the double layer potential $\frac{1}{2}I - \mathsf{C}_\kappa$.

Theorem 3.1 *For all* $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma)$ *and for all real* $\kappa < 0$ *there holds*

$$(1 - c_K)\|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} \leq \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} \leq c_K \|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}}$$

where

$$c_K = \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^S c_1^N} < 1,$$

and c_1^S, c_1^N *are the ellipticity constants of the single layer potential* S_κ *and of the hypersingular operator* N_κ.

Proof: For $\mathbf{u} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ with $\|\mathbf{u}\|_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} > 0$ we first have

$$\|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|^2_{\mathsf{S}_\kappa^{-1}} \;=\; \langle \mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}, (\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\rangle \;=\; \langle \mathsf{T}_\kappa \mathbf{u}, \mathbf{u}\rangle - \langle \mathsf{N}_\kappa \mathbf{u}, \mathbf{u}\rangle$$

where the Steklov–Poincaré operator T_κ is defined as in (2.18). Let

$$J : \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

be the Riesz operator, then

$$A := J\,\mathsf{S}_\kappa^{-1} \;:\; \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

is self–adjoint and $\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$–elliptic.

Hence we can consider the splitting $A = A^{1/2}A^{1/2}$ to obtain

$$\begin{aligned}
\langle \mathsf{T}_\kappa \mathbf{u}, \mathbf{u}\rangle &= \langle \mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}, \mathbf{u}\rangle \\
&= \langle J\mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}, \mathbf{u}\rangle_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \\
&= \langle A^{1/2}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}, A^{1/2}\mathbf{u}\rangle_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \\
&\le \|A^{1/2}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \|A^{1/2}\mathbf{u}\|_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} .
\end{aligned}$$

With

$$\begin{aligned}
\|A^{1/2}\mathbf{v}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} &= \langle A^{1/2}\mathbf{v}, A^{1/2}\mathbf{v}\rangle_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \\
&= \langle J\mathsf{S}_\kappa^{-1}\mathbf{v}, \mathbf{v}\rangle_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \;=\; \langle \mathsf{S}_\kappa^{-1}\mathbf{v}, \mathbf{v}\rangle \;=\; \|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}}
\end{aligned}$$

we then obtain

$$\langle \mathsf{T}_\kappa \mathbf{u}, \mathbf{u}\rangle \;\le\; \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} \|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}}.$$

On the other hand, for the hypersingular boundary integral operator we have

$$\langle \mathsf{N}_\kappa \mathbf{u}, \mathbf{u}\rangle \;\ge\; c_1^N \|\mathbf{u}\|^2_{\mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)} \;\ge\; c_1^N c_1^S \langle \mathsf{S}_\kappa^{-1}\mathbf{u}, \mathbf{u}\rangle \;=\; c_1^N c_1^S \|\mathbf{u}\|^2_{\mathsf{S}_\kappa^{-1}}.$$

Altogether, this gives

$$\|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|^2_{\mathsf{S}_\kappa^{-1}} \;=\; \langle \mathsf{T}_\kappa \mathbf{u}, \mathbf{u}\rangle - \langle \mathsf{N}_\kappa \mathbf{u}, \mathbf{u}\rangle \;\le\; \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} \|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} - c_1^N c_1^S \|\mathbf{u}\|^2_{\mathsf{S}_\kappa^{-1}}$$

which is equivalent to

$$\left(\frac{a}{b}\right)^2 - \frac{a}{b} + c_1^N c_1^S \;\le\; 0$$

where

$$a := \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} \ge 0, \quad b := \|\mathbf{u}\|_{\mathsf{S}_\kappa^{-1}} > 0.$$

Hence we finally conclude

$$\frac{1}{2} - \sqrt{\frac{1}{4} - c_1^N c_1^S} \;\leq\; \frac{a}{b} \;\leq\; \frac{1}{2} + \sqrt{\frac{1}{4} - c_1^N c_1^S}$$

which gives the assertion. ■

A similar estimate can also be shown for the operator $\frac{1}{2}I + \mathsf{C}_\kappa$.

Theorem 3.2 *For* $\mathbf{v} \in \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *there holds*

$$(1 - c_K)\|\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} \;\leq\; \|(\frac{1}{2}I + \mathsf{C}_\kappa)\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} \;\leq\; c_K\|\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}}.$$

Proof: With the contraction property of $\frac{1}{2}I - \mathsf{C}_\kappa$ we obtain

$$\begin{aligned}
\|\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} &= \|(\frac{1}{2}I + \mathsf{C}_\kappa)\mathbf{v} + (\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} \\
&\leq \|(\frac{1}{2}I + \mathsf{C}_\kappa)\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} + \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} \\
&\leq \|(\frac{1}{2}I + \mathsf{C}_\kappa)\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}} + c_K\|\mathbf{v}\|_{\mathsf{S}_\kappa^{-1}}
\end{aligned}$$

and therefore the first inequality. On the other hand, by using the representations (2.17) and (2.18) we get

$$\begin{aligned}
\|(\frac{1}{2}I + \mathsf{C}_\kappa)\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} &= \|(I - (\frac{1}{2}I - \mathsf{C}_\kappa))\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} \\
&= \|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} + \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} - 2\langle \mathsf{S}_\kappa^{-1}(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}, \mathbf{v}\rangle \\
&= \|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} + \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} - 2\langle \mathsf{T}_\kappa\mathbf{v}, \mathbf{v}\rangle \\
&= \|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} - \|(\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} - 2\langle \mathsf{N}_\kappa\mathbf{v}, \mathbf{v}\rangle \\
&\leq [1 - (1 - c_K)^2 - 2c_1^S c_1^N]\, \|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}} \;=\; c_K^2\|\mathbf{v}\|^2_{\mathsf{S}_\kappa^{-1}}
\end{aligned}$$

and therefore the upper estimate. ■

As for the operators $\frac{1}{2} \pm \mathsf{C}_\kappa$ we can prove related estimates for the operators $\frac{1}{2} \pm \mathsf{B}_\kappa$ when considering an equivalent norm in $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ which is induced by the single layer potential S_κ, i.e. for $\kappa < 0$ there holds

$$(1 - c_K)\,\|\mathbf{w}\|_{\mathsf{S}_\kappa} \;\leq\; \|(\frac{1}{2}I \pm \mathsf{B}_\kappa)\mathbf{w}\|_{\mathsf{S}_\kappa} \;\leq\; c_K\,\|\mathbf{w}\|_{\mathsf{S}_\kappa}. \tag{3.2}$$

for all $\mathbf{w} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$.

For $\mathbf{u} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ we finally define the operator

$$\mathsf{S}_{\kappa,0}\mathbf{u} \;:=\; \mathsf{A}_0\mathbf{u} - \frac{1}{\kappa^2}\nabla_\Gamma V_0\, \mathrm{div}_\Gamma\,\mathbf{u}.$$

Now we are able to prove the coercivity of the operator Z_κ.

Theorem 3.3 *The operator*

$$\mathsf{Z}_\kappa \;=\; \mathsf{S}_\kappa + i\eta(\frac{1}{2}I - \mathsf{C}_\kappa)\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_k) \;:\; \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma) \to \mathbf{H}_\perp^{-1/2}(\mathrm{curl}_\Gamma\,,\Gamma)$$

satisfies a Gårdings inequality, i.e. there holds

$$\mathrm{Im}[\langle \mathsf{Z}_\kappa \boldsymbol{\mu}, \boldsymbol{\mu}\rangle + c_1(\boldsymbol{\mu}, \boldsymbol{\mu})] \;\geq\; c_Z\,\|\boldsymbol{\mu}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)}$$

for all $\boldsymbol{\mu} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)$ *with a positive constant* c_Z *where* $c_1(\boldsymbol{\mu}, \boldsymbol{\mu})$ *is a compact bilinear form.*

Proof: Since $\langle \mathsf{S}_{\kappa,0}\mathbf{w}, \mathbf{w}\rangle$ is real, the same holds true for the duality product

$$\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}\rangle \;\in\; \mathbb{R}.$$

Because of the contraction property (3.2) we get for some $\kappa' < 0$

$$\|(\frac{1}{2}I + \mathsf{B}_{\kappa'})\mathbf{w}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)} \;\geq\; c\,\|\mathbf{w}\|_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)}$$

for all $\mathbf{w} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)$. Since the operator S_0^{*-1} is $\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)$–elliptic, we have

$$\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_{\kappa'})\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_{\kappa'})\mathbf{w}\rangle \;\geq\; c\,\|\mathbf{w}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)}$$

for all $\mathbf{w} \in \mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)$. The operator Z_κ can now be written in the following form

$$\begin{aligned}\mathsf{Z}_\kappa \;&=\; \mathsf{S}_{\kappa,0} + \underbrace{(\mathsf{S}_\kappa - \mathsf{S}_{\kappa,0})}_{\text{compact}} + i\eta\Big((\frac{1}{2}I - \mathsf{C}_{\kappa'})\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_{\kappa'})\\ &\quad + \underbrace{(\mathsf{C}_{\kappa'} - \mathsf{C}_\kappa)\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa) + (\frac{1}{2}I - \mathsf{C}_{\kappa'})\mathsf{S}_0^{*-1}(\mathsf{B}_\kappa - \mathsf{B}_{\kappa'})}_{\text{compact}}\Big)\end{aligned}$$

which implies

$$\begin{aligned}\mathrm{Im}\left[\langle \mathsf{Z}_\kappa\mathbf{w}, \mathbf{w}\rangle + c_1(\mathbf{w}, \mathbf{w})\right] \;&=\; \mathrm{Im}\left[\langle \mathsf{S}_{\kappa,0}\mathbf{w}, \mathbf{w}\rangle + i\eta\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_k)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_k)\mathbf{w}\rangle\right]\\ &=\; \mathrm{Im}\left[\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_k)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_k)\mathbf{w}\rangle\right]\\ &\geq\; c\,\|\mathbf{w}\|^2_{\mathbf{H}_\|^{-1/2}(\mathrm{div}_\Gamma\,,\Gamma)}.\end{aligned}$$

■

Hence, to use Fredholms alternative to establish the unique solvability of the modified boundary integral equation (3.1) it remains to prove the injectivity of the operator Z_κ.

Theorem 3.4 *If the wave number $\kappa = ik$ is imaginary then there holds*

$$\mathrm{Im}[\langle \mathsf{S}_\kappa \mathbf{u}, \mathbf{u}\rangle] \,\geq\, 0$$

for all $\mathbf{u} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma)$.

Proof: From Green's first formula we have

$$\int_\Omega \mathbf{curl}\,\mathbf{U}(x) \cdot \mathbf{curl}\,\mathbf{V}(x) dx - \int_\Gamma \gamma_N \mathbf{U}(x) \cdot \gamma_D \mathbf{V}(x) ds_x \,=\, k^2 \int_\Omega \mathbf{U}(x) \cdot \mathbf{V}(x) dx.$$

For $\mathbf{V} = \overline{\mathbf{U}}$ it follows that

$$\int_\Omega \left(|\mathbf{curl}\,\mathbf{U}(x)|^2 - k^2 |\mathbf{U}(x)|^2 \right) dx \,=\, \int_\Gamma \gamma_N \mathbf{U}(x) \cdot \gamma_D \overline{\mathbf{U}}(x) ds_x.$$

Let $\mathbf{U}(x) = \mathbf{\Psi}_S^\kappa \mathbf{w}(x)$, $x \in \Omega$, be a solution of the partial differential equation (1.1), i.e. we have for $x \in \Gamma$

$$\begin{aligned} \gamma_N \mathbf{\Psi}_S^\kappa \mathbf{w}(x) &= \frac{1}{2}\mathbf{w}(x) + \mathsf{B}_\kappa \mathbf{w}(x), \\ \gamma_D \mathbf{\Psi}_S^\kappa \mathbf{w}(x) &= \mathsf{S}_\kappa \mathbf{w}(x). \end{aligned}$$

To handle the exterior domain Ω^c we consider a sphere B_R such that $\Omega \subset B_R$ is satisfied. Moreover, let $\Omega_R = B_R \setminus \overline{\Omega}$. As above we then have

$$\begin{aligned} \int_{\Omega_R} \left(|\mathbf{curl}\,\mathbf{U}(x)|^2 - k^2 |\mathbf{U}(x)|^2 \right) dx &= \int_{\partial\Omega_R} \gamma_N \mathbf{U}(x) \cdot \gamma_D \overline{\mathbf{U}}(x) ds_x \\ &= \int_{\partial B_R} \gamma_N \mathbf{U}(x) \cdot \gamma_D \overline{\mathbf{U}}(x) ds_x - \int_\Gamma \gamma_N^c \mathbf{U}(x) \cdot \gamma_D^c \overline{\mathbf{U}}(x) ds_x. \end{aligned}$$

For the exterior traces of $\mathbf{U}(x) = \mathbf{\Psi}_S^\kappa \mathbf{w}(x)$, $x \in \Omega^c$, we have for $x \in \Gamma$

$$\begin{aligned} \gamma_N^c \mathbf{\Psi}_S^\kappa \mathbf{w}(x) &= -\frac{1}{2}\mathbf{w}(x) + \mathsf{B}_\kappa \mathbf{w}(x), \\ \gamma_D^c \mathbf{\Psi}_S^\kappa \mathbf{w}(x) &= \mathsf{S}_\kappa \mathbf{w}(x). \end{aligned}$$

Hence we find by summing up the above expressions and when inserting the jump conditions

$$\int_{B_R} \left(|\mathbf{curl}\,\mathbf{U}(x)|^2 - k^2 |\mathbf{U}(x)|^2 \right) dx \,=\, \langle \mathbf{w}, \mathsf{S}_\kappa \mathbf{w}\rangle + \int_{\partial B_R} \gamma_N \mathbf{U}(x) \cdot \gamma_D \overline{\mathbf{U}}(x) ds_x$$

and therefore

$$\mathrm{Im}[\langle \mathbf{w}, \mathsf{S}_\kappa \mathbf{w}\rangle] \,=\, -\mathrm{Im}\left[\int_{\partial B_R} \gamma_N \mathbf{U}(x) \cdot \gamma_D \overline{\mathbf{U}}(x) ds_x \right].$$

From the Silver–Müller radiation condition, i.e.

$$\lim_{r=|x|\to 0} \int\limits_{\partial B_R} |\mathbf{curl}\,\mathbf{U}(x) \times \mathbf{n} - ik(\mathbf{n} \times \mathbf{U}(x)) \times \mathbf{n}|^2 ds_x \;=\; 0,$$

we further conclude

$$\begin{aligned}
\int\limits_{\partial B_R} |\gamma_N \mathbf{U} - ik\gamma_D \mathbf{U}|^2 ds_x \;&=\; \int\limits_{\partial B_R} \left(|\gamma_N \mathbf{U}|^2 + |ik\gamma_D \mathbf{U}|^2 - 2\mathrm{Re}[\gamma_N \mathbf{U} \cdot \overline{ik\gamma_D \mathbf{U}}]\right) ds_x \\
&=\; \int\limits_{\partial B_R} \left(|\gamma_N \mathbf{U}|^2 + |k\gamma_D \mathbf{U}|^2 - 2k\,\mathrm{Im}\left[\gamma_N \mathbf{U} \cdot \gamma_D \overline{\mathbf{U}}\right]\right) ds_x \\
&=\; \int\limits_{\partial B_R} \left(\|\gamma_N \mathbf{U}\|^2 + \|k\gamma_D \mathbf{U}\|^2\right) ds_x + 2k\,\mathrm{Im}[\langle \mathbf{w}, \mathsf{S}_\kappa \mathbf{w}\rangle]
\end{aligned}$$

and therefore

$$\lim_{r=|x|\to\infty} \int\limits_{\partial B_R} |\gamma_N \mathbf{U} - ik\gamma_D \mathbf{U}|^2 ds_x \;=\; 0$$

which implies

$$2k\,\mathrm{Im}[\langle \mathbf{w}, \mathsf{S}_\kappa \mathbf{w}\rangle] \;\leq\; 0$$

and thus

$$2k\,\mathrm{Im}[\langle \mathsf{S}_\kappa \mathbf{w}, \mathbf{w}\rangle] \;\geq\; 0.$$

■

Now we are in a position to prove the injectivity of Z_κ.

Theorem 3.5 *For an imaginary wave number $\kappa = ik$ the modified boundary integral operator*

$$\mathsf{Z}_\kappa \;=\; \mathsf{S}_\kappa + i\eta(\frac{1}{2}I - \mathsf{C}_\kappa)\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa) \;:\; \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma) \to \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma\,, \Gamma)$$

is injective.

Proof: Let $w \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma\,, \Gamma)$ be a solution of the homogeneous equation

$$\mathsf{Z}_\kappa \mathbf{w}(x) \;=\; 0 \quad \text{for } x \in \Gamma.$$

Then it follows that

$$0 \;=\; \langle \mathsf{Z}_\kappa \mathbf{w}, \mathbf{w}\rangle \;=\; \langle \mathsf{S}_\kappa \mathbf{w}, \mathbf{w}\rangle + i\eta\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}\rangle$$

and therefore

$$\mathrm{Im}\left[\langle \mathsf{S}_\kappa \mathbf{w}, \mathbf{w}\rangle + i\eta \langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}\rangle\right] \;=\; 0.$$

By using Theorem 3.4 we then get

$$\eta\langle \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}\rangle \;=\; -\mathrm{Im}[\langle \mathsf{S}_\kappa \mathbf{w}, \mathbf{w}\rangle] \;\leq\; 0$$

and hence we conclude

$$(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w} \;=\; 0.$$

But then we also have

$$\mathsf{S}_\kappa \mathbf{w}(x) \;=\; 0 \quad \text{for } x \in \Gamma$$

which only admits a non–trivial solution if $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem (2.8) implying

$$(\frac{1}{2}I - \mathsf{B}_{\pm\sqrt{\lambda}})\gamma_N \mathbf{U}_\lambda(x) \;=\; 0,$$

or

$$(\frac{1}{2}I + \mathsf{B}_{\pm\sqrt{\lambda}})\mathbf{w}(x) \;=\; 0, \quad (\frac{1}{2}I - \mathsf{B}_{\pm\sqrt{\lambda}})\mathbf{w}(x) \;=\; 0.$$

Hence we conclude $\mathbf{w} = 0$ for all frequencies k. ■

When combining the coercivity (Theorem 3.3) and the injectivity (Theorem 3.4) of the operator Z_κ we therefore conclude the unique solvability of the modified boundary integral equation (3.1). The related variational formulation is to find $\mathbf{w} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$\langle \mathsf{S}_\kappa \mathbf{w}, \boldsymbol{\tau}\rangle + i\eta\langle (\frac{1}{2}I - \mathsf{C}_\kappa)\mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, \boldsymbol{\tau}\rangle \;=\; \langle \mathbf{g}, \boldsymbol{\tau}\rangle. \tag{3.3}$$

is satisfied for all test functions $\boldsymbol{\tau} \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$. Note that the variational problem (3.3) has a similar structure as the symmetric boundary integral representation of the Steklov–Poincaré operator. Due to the composite structure a direct Galerkin discretization of (3.3) will not be possible. Hence we introduce

$$\mathbf{z} \;=\; \mathsf{S}_0^{*-1}(\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$$

which is the unique solution of the variational problem such that

$$\langle \mathsf{S}_0^* \mathbf{z}, \mathbf{v}\rangle \;=\; \langle (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, \mathbf{v}\rangle$$

is satisfied for all $\mathbf{v} \in \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. Finally we obtain a saddle point formulation to find $(\mathbf{w}, \mathbf{z}) \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that

$$\begin{array}{rcccl} \langle \mathsf{S}_\kappa \mathbf{w}, \boldsymbol{\tau}\rangle & + & i\eta\langle (\frac{1}{2}I - \mathsf{C}_\kappa)\mathbf{z}, \boldsymbol{\tau}\rangle & = & \langle \mathbf{g}, \boldsymbol{\tau}\rangle \\ -\langle (\frac{1}{2}I + \mathsf{B}_\kappa)\mathbf{w}, \mathbf{v}\rangle & + & \langle \mathsf{S}_0^* \mathbf{z}, \mathbf{v}\rangle & = & 0 \end{array} \tag{3.4}$$

is satisfied for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathbf{H}_{\|}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \times \mathbf{H}_{\perp}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. Since the modified boundary integral equation (3.1) is the Schur complement system of the mixed formulation (3.4) the unique solvability of (3.4) follows immediately.

Remark 3.6 *In this paper we just presented a modified boundary integral formulation for the exterior Dirichlet boundary value problem* (1.1)–(1.3)*. For an exterior Neumann boundary value problem a similar modified formulation can be derived and analyzed as well* [23].

4 Numerical example

As a numerical example to show the applicability of the proposed approach we consider the exterior Dirichlet boundary value problem (1.1)–(1.3) where $\Omega = (0,1)^3$ is the unit cube whose boundary $\Gamma = \partial\Omega$ is decomposed into N triangular plane elements. For this domain we can easily deduce the eigenvalues and eigenfrequencies of the interior Dirichlet eigenvalue problem. In particular we will consider the smallest eigenvalue which corresponds to the wave number $k = \sqrt{2}\pi \approx 4.44288$. As exact solution of the exterior Dirichlet boundary value problem (1.1)–(1.3) we consider [2]

$$\mathbf{U}(x) = \left[\frac{\kappa^2 r^2 + \kappa r + 1}{r^3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\kappa^2 r^2 + 3\kappa r + 3}{r^5} (x_1 - \hat{x}_1) \begin{pmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \\ x_3 - \hat{x}_3 \end{pmatrix} \right] e^{\kappa r}$$

for $x \in \Omega^c$, where the source point is $\hat{x} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^\top \in \Omega$, and $r = |x - \hat{x}|$. For a comparison of different approaches we consider the indirect single layer potential ansatz leading to the boundary integral equation (2.4), the proposed modified formulation ($\eta = 1$) where we have to solve (3.1), and a direct approach which results in the boundary integral equation (2.7). In all cases the Galerkin discretization is done by using linear Raviart–Thomas elements, see e.g. [2, 21] for details. The resulting linear systems are solved by a GMRES method with a relative error reduction of $\varepsilon = 10^{-8}$. Then we compute approximate solutions $\mathbf{U}_h$ and the related pointwise error in the evaluation point $\bar{x} = (1.4, 1.8, 2.0)^\top \in \Omega^c$. All results are documented in Table 1.

	indirect, (2.4)		modified, (3.1)		direct, (2.7)	
N	Iter	$\lvert\mathbf{U}(\bar{x}) - \mathbf{U}_h(\bar{x})\rvert$	Iter	$\lvert\mathbf{U}(\bar{x}) - \mathbf{U}_h(\bar{x})\rvert$	Iter	$\lvert\mathbf{U}(\bar{x}) - \mathbf{U}_h(\bar{x})\rvert$
72	53	7.64	110	1.27632	53	0.64908
288	107	10.85	197	0.19541	107	0.19153
1152	238	15.52	280	0.04874	209	0.04677
4608	554	43.20	403	0.01308	469	0.01222
18432			665	0.00730	834	0.00529

Table 1: Number of GMRES iterations and pointwise error.

It is obvious that the indirect single layer potential approach fails since the wave number k corresponds to an eigenvalue of the interior Dirichlet eigenvalue problem. The results of the modified formulation (3.1) and of the direct approach (2.7) are comparable in this example. However, for the latter one has to ensure a solvability condition also in the discrete case which requires in general the knowledge of the related eigenfrequency. Here we only considered a direct Galerkin discretization of (2.7) which may fail in more general situations.

Related to the numerical results there are several points to be discussed, first of all the numerical analysis to establish the quadratic order of pointwise convergence. Moreover, we have to investigate a suitable choice of the scaling parameter $\eta \in \mathbb{R}_+$, and the construction of efficient preconditioned iterative solution methods. It is obvious that these questions are strongly related to the case of exterior boundary value problems for the Helmholtz equation [13].

5 Conclusions

In this paper we have described and analyzed a modified boundary integral equation to solve an exterior Dirichlet boundary value problem for the Maxwell system which is stable for all wave numbers. Note that a similar formulation can be given in the case of an exterior Neumann boundary value problem as well. The proposed regularization operator relies on boundary integral operators which are already available when considering standard boundary integral equations for the Maxwell system. In particular we avoid to use a Hodge decomposition in both the analysis and in the implementation. The modified boundary integral equation is finally reformulated as a saddle point formulation which allows a direct Galerkin discretization. A first numerical example shows the applicability of the proposed approach.

In a forthcoming paper we will present the numerical analysis of the related boundary element method to solve the saddle point formulation (3.4). This may also include the use of fast boundary element methods, and the design of preconditioned iterative solution strategies to solve the resulting linear systems of algebraic equations.

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