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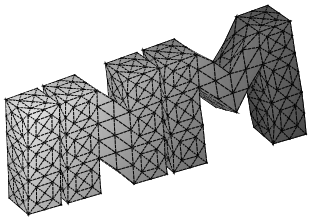

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Circulant matrices and FFT methods for the fast evaluation of Newton potentials in BEM

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Abstract

In this paper we describe and analyze a fast approach for the evaluation of the Newton potential for inhomogeneous partial differential equations in the particular case of two–dimensional circular domains. The method is based on a suitable mesh dicretization of the domain which enables to write the Newton potential in terms of matrix–vector multiplication. Moreover, this multiplication can be speed up by utilizing the fast Fourier transform (FFT) due to the circulant structure of the matrices. We present an error analysis for the fully discretized equations and some numerical results for the scalar Yukawa equation, and for the system of linear elasticity of Yukawa type.

1 Introduction

In this paper we develop and analyze a boundary element method for the solution of the scalar Yukawa problem in the particular case of a two–dimensional circular domain. But, these results can also be applied in various type of problems having a logarithmic kernels as the principal part [9, 10] such as the time dependent heat equation, time dependent diffusion equation or time dependent elasticity equations after an implicit time discretization. Therefore, reliable and efficient numerical algorithms for the solution of the scalar Yukawa equation can be of great use in many different areas of solid mechanics.

In particular we include in the second part of this paper a plane linear elasticity model problem of Yukawa type. The analysis of this problem as well as the error estimate are similar to the above mentioned problem except the case of the double layer and the hypersingular integral operators which present some higher singularities and need to be treated in a special way [12, 13, 17].

In the first part we consider the inhomogeneous mixed boundary value problem in a two–dimensional simply connected domain Ω (in particular a two–dimensional disc). Applying the boundary integral equation method, the partial differential equations are reduced equivalently to boundary integral equations on the boundary curve [23]. Due to the shape of the domain, the boundary (circle) can easily be represented by a one–periodical parametrization [20, 22]. Making use of this parametrization the eigensystems of the boundary integral operators can be derived [2, 3, 14], which are crucial results for the computation of eigenvalues of the discrete operators. The boundary integral equations are approximated by using a Galerkin method with the help of B–splines as basis functions. This yields an equivalent algebraic linear system involving dense matrices, but having circulant structure [4, 5, 6]. The entries of those matrices are computed explicitly and efficiently. Furthermore, the circulant property enables us to use the discrete Fourier matrix as preconditioner within an iterative solver or the fast Fourier transform (FFT) as direct solver [8, 16, 20, 21, 22]. However, the boundary element formulation (BEM) loses at a first glance its attractiveness due to the fact that the equation is inhomogeneous which requires an integration over the whole domain.

During the past two decades, much effort has been devoted to dealing with this issue in the BEM community. One of the most widely used methods in engineering is the dual reciprocity method (DRM) introduced by Nardini and Brebbia in 1982 [19]. This method transfers the domain integrals to boundary integrals. The main idea is to approximate the right hand side, for example by radial basis functions (RBF) [7], which help to determine a particular solution of the partial differential equation. Furthermore, particular solutions can be computed by finite difference methods or by finite element methods [11] by embedding the domain into an auxiliary domain and solving the inhomogeneous equation with homogeneous Dirichlet boundary conditions. The recent development for the evaluation of the Newton potential is the fast multipole method [15, 18, 23]. The main idea of this technique is based on the multipole expansion of the fundamental solution in the far field of the evaluation point. In [18], the computational domain is divided into the far and near field so that in the near field the evaluation is done by using a standard collocation approach, while in the far field the multipole expansion is used. Additionally, an error analysis is given. In this work we present and analyze a new approach for an efficient evaluation of the Newton potential in the boundary element method of the mixed boundary value problem by using the Steklov-Poincaré operator. The technique we present here is based on a special mesh discretization of the circular domain. First, at each level of refinement the disc is split into M rings in an adaptive way. Second, a uniform mesh is defined on each ring in such a way that the mesh on the external ring has the same size as the boundary mesh. Further, on each ring the right hand side is approximated by piecewise constant functions. This enables us to write the Newton potential vector on each ring in terms of a matrix-vector multiplication. Moreover, the FFT can be used to speed up this process due to the circulant structure of the matrices on each ring [8, 16, 20, 21, 22]. Here we provide a complete error analysis containing not only the theoretical Galerkin error but also the errors due to the numerical approximation of the Newton potential. In the second part the above approach is extended for the solution of a Dirichlet problem in linear elasticity

of Yukawa type.

The paper is organized as follows: In Sect. 2 we describe the considered scalar mixed boundary value problem of Yukawa type, establish the boundary integral equations and derive the eigensystems of the operators involved. In Sect. 3 we are interested in the standard Galerkin procedure for the boundary integral equations formulated in Sect. 2 with the help of one–periodical B–splines. The focus of Sect. 4 is the evaluation of the Newton potential for the scalar Yukawa problem by using the method we described above and the presentation of the numerical error analysis. In Sect. 5, we present some numerical results for the scalar Yukawa problems. The main focus of Sect. 6 is the extension of the procedure we elaborated in Sect. 4 for the linear elasticity problem of Yukawa type.

2 Scalar Yukawa type boundary value problems

As a model problem we consider the mixed boundary value problem for the Yukawa partial differential equation

$$\alpha^2 u - \Delta u = f \text{ in } \Omega, \quad u = g_D \text{ on } \Gamma_D, \quad \partial_n u := \frac{\partial}{\partial n} u = g_N \text{ on } \Gamma_N, \tag{2.1}$$

where $\alpha \in \mathbb{R}_+$ is the so called generalized wave number and n represents the outer unit normal to the boundary $\Gamma = \partial\Omega$ which is divided into two mutually disjoint parts Γ_D and Γ_N. Here we are interested in the particular case when $\Omega \subset \mathbb{R}^2$ is a circular domain. The solution u of the partial differential equation in (2.1) is given by the representation formula for $x \in \Omega$, see, e.g., [23],

$$u(x) = \int_\Gamma U^*(x,y)\partial_n u(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) u(y) ds_y + \int_\Omega U^*(x,y) f(y) dy, \tag{2.2}$$

where $U^*(x,y)$ denotes the fundamental solution of the Yukawa partial differential operator in two dimensions given by

$$U^*(x,y) = \frac{1}{2\pi} K_0(\alpha|x-y|). \tag{2.3}$$

Note that

$$\begin{aligned} K_0(r) &= (\ln 2 - \mathbf{E} - \ln r) I_0(r) + \sum_{k=1}^{\infty} \left[\left(\sum_{j=1}^{k} \frac{1}{j} \right) \frac{1}{(k!)^2} \left(\frac{r}{2} \right)^{2k} \right], \\ I_0(r) &= 1 + \sum_{k=1}^{\infty} \frac{1}{(k!)^2} \left(\frac{r}{2} \right)^{2k}, \end{aligned}$$

are the second and first kind modified Bessel functions respectively [1], and

$$\mathbf{E} = \lim_{n\to\infty} \left[\sum_{j=1}^{n} \frac{1}{j} - \ln n \right] \approx 0.57721566490\ldots$$

represents the so–called Euler–Mascheroni constant.

2.1 Boundary integral equations

The representation formula (2.2) states that u is determined uniquely for a point $x \in \Omega$ if the complete Cauchy data $[u, \partial_n u]_{|\Gamma}$ and the source f are known. But, it turns out that $[u, \partial_n u]_{|\Gamma}$ are given only partially on the boundary Γ, that are $u|_{|\Gamma_D} = g_D$ and $\partial_n u|_{|\Gamma_N} = g_N$. Therefore, we have to determine $u|_{|\Gamma_N}$ and $\partial_n u|_{|\Gamma_D}$. By proceeding as in [23], we first apply the Dirichlet trace operator to (2.2) to obtain, on a smooth boundary Γ,

$$u(x) = (V\partial_n u)(x) + \frac{1}{2}u(x) - (Ku)(x) + (N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{2.4}$$

Further, by applying the normal derivative again to (2.2) this gives

$$\partial_n u(x) = \frac{1}{2}\partial_n u(x) + (K'\partial_n u)(x) + (Du)(x) + (N_1 f)(x) \quad \text{for } x \in \Gamma. \tag{2.5}$$

In (2.4) and (2.5) we have used the standard notations for boundary integral operators, i.e., $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer integral operator, $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the double layer integral operator with its adjoint $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, and $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the hypersingular boundary integral operator which are defined for $x \in \Gamma$ as

$$\begin{aligned} (Vt)(x) &= \int_\Gamma U^*(x,y)t(y)ds_y, & (Ku)(x) &= \int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y, \\ (K't)(x) &= \int_\Gamma \frac{\partial}{\partial n_x}U^*(x,y)t(y)ds_y, & (Du)(x) &= -\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}U^*(x,y)u(y)ds_y. \end{aligned}$$

Moreover, $N_0 : \widetilde{H}^{-1}(\Omega) \to H^{1/2}(\Gamma)$ and $N_1 : \widetilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma)$ are the Newton or volume potentials which are given for $x \in \Gamma$ as

$$(N_0 f)(x) = \int_\Omega U^*(x,y)f(y)dy, \quad (N_1 f)(x) = \frac{\partial}{\partial n_x}\int_\Omega U^*(x,y)f(y)dy.$$

Recall that the single layer integral operator V and the hypersingular integral operator D satisfy the following properties.

Lemma 2.1 *The single layer integral operator $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and the hypersingular boundary integral operator $D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are self–adjoint and elliptic on $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively, i.e.*

$$\langle V\tau, \tau\rangle_\Gamma \geq c_1^V \|\tau\|^2_{H^{-1/2}(\Gamma)} \quad \textit{for all } \tau \in H^{-1/2}(\Gamma) \textit{ with } c_1^V > 0$$

and

$$\langle Dv, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H^{1/2}(\Gamma) \textit{ with } c_1^D > 0.$$

Proof. See, e.g., [23], in the case of the Laplace equation. For the more general case of the Yukawa equation with $\alpha \in \mathbb{R}_+$ we need to use in addition the properties of the modified Bessel functions [1]. ∎

Since the single layer integral operator V is $H^{-1/2}(\Gamma)$–elliptic, the unique solvability of the boundary integral equation (2.4) with respect to the normal derivative $\partial_n u$ follows immediately, i.e.

$$\partial_n u = V^{-1}\Big(\frac{1}{2}I + K\Big)u - V^{-1}N_0 f \quad \text{on } \Gamma. \tag{2.6}$$

But, by using the representation (2.6) in a Galerkin discretization may lead to a non–symmetric discrete matrix for the symmetric Dirichlet to Neumann map. To avoid such an inconvenience, let us use the symmetric representation which is obtained by inserting the representation (2.6) into the second boundary integral equation (2.5):

$$\begin{aligned} \partial_n u &= \Big(\frac{1}{2}I + K'\Big)\partial_n u + Du + N_1 f \\ &= \Big[\Big(\frac{1}{2}I + K'\Big)V^{-1}\Big(\frac{1}{2}I + K\Big) + D\Big]u + N_1 f - \Big(\frac{1}{2}I + K'\Big)V^{-1}N_0 f. \end{aligned} \tag{2.7}$$

The representations (2.6) and (2.7) involve two different representations of the Steklov–Poincaré operator

$$S = V^{-1}\Big(\frac{1}{2}I + K\Big) = \Big(\frac{1}{2}I + K'\Big)V^{-1}\Big(\frac{1}{2}I + K\Big) + D, \tag{2.8}$$

which coincide in the continuous case, but which may yield different representations when considering Galerkin approximations of the underlying boundary integral operators. In particular, the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is self–adjoint and $H^{1/2}(\Gamma)$–elliptic, i.e. S admits the same ellipticity estimate as the hypersingular boundary integral operator D,

$$\langle Sv, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma). \tag{2.9}$$

Furthermore, by equating (2.6) and (2.7) one obtains

$$-V^{-1}N_0 f = N_1 f - \Big(\frac{1}{2}I + K'\Big)V^{-1}N_0 f =: -Nf \quad \text{on } \Gamma \tag{2.10}$$

and

$$N_1 f = \left(-\frac{1}{2}I + K'\right) V^{-1}N_0 f \quad \text{on } \Gamma.$$

As a consequence of (2.10), the normal derivative $\partial_n u$ can take the following form

$$\partial_n u = Su - Nf \quad \text{on } \Gamma. \tag{2.11}$$

Let us now return to our problem where we have to find the unknown Dirichlet datum $u_{|\Gamma_N}$ and the unknown Neumann datum $\partial_n u_{|\Gamma_D}$. There exists a wide range of different

boundary integral formulations to solve the mixed boundary value problem (2.1), see, e.g. [23]. Here we consider the formulation by using the Dirichlet to Neumann map (2.11) to find $u \in H^{1/2}(\Gamma)$ such that

$$u = g_D \text{ on } \Gamma_D, \quad \partial_n u = Su - Nf = g_N \text{ on } \Gamma_N. \tag{2.12}$$

Let $\widetilde{g}_D \in H^{1/2}(\Gamma)$ be a suitable extension of the given Dirichlet datum $g_D \in H^{1/2}(\Gamma_D)$ and satisfying $\widetilde{g}_{D|\Gamma_D} = g_D$. Then we have to find $\widehat{u} := u - \widetilde{g}_D \in \widetilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle S\widehat{u}, v\rangle_{\Gamma_N} = \langle g_N + Nf - S\widetilde{g}_D, v\rangle_{\Gamma_N} \tag{2.13}$$

is satisfied for all $v \in \widetilde{H}^{1/2}(\Gamma_N)$. Since the Steklov–Poincaré operator $S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is bounded and $\widetilde{H}^{1/2}(\Gamma_N)$–elliptic, see, e.g., [23, p. 149], the unique solvability of (2.13) is therefore shown. Before moving to the Galerkin discretization of the problem (2.12), we determine the eigensystems of different operators involved in the equation.

2.2 Eigensystems of integral operators

In general it is not possible to obtain the eigenfunctions of boundary integral operators for arbitrary boundaries [2]. In the particular case of a two–dimensional circular domain $\Omega = B_R(O)$ of radius R and centered at the origin it is possible to give an explicit representation of the eigenfunctions of the single and double layer boundary integral operators, and of the hypersingular boundary integral operator as well. The parametrization of $\Gamma = \partial B_R(O)$ is given by

$$\Gamma := \left\{ x \in \mathbb{R}^2 : x(\tau) = R \begin{pmatrix} \cos 2\pi\tau \\ \sin 2\pi\tau \end{pmatrix},\ 0 \le \tau < 1 \right\}. \tag{2.14}$$

By using the parametrization (2.14), the boundary integral operators can be written as follows:

$$\begin{aligned}
(Vt)(\tau) &= R \int_0^1 K_0(2R\alpha\,|\sin\pi(\tau - s)|)\, t(s)\, ds, \\
(Ku)(\tau) &= -\alpha R \int_0^1 K_1(2R\alpha|\sin\pi(\tau - s)|)\,|\sin\pi(\tau - s)|\, u(s)\, ds, \\
(Du)(\tau) &= -\alpha^2 \int_0^1 \Big[RK_0(2R\alpha|\sin\pi(\tau - s)|)\sin^2\pi(\tau - s) + \frac{K_1(2R\alpha|\sin\pi(\tau - s)|)}{2\alpha R\sin\pi(\tau - s)} \Big] u(s) ds.
\end{aligned}$$

Lemma 2.2 *Let Γ be given as in* (2.14)*. The Fourier functions*

$$v_n(s) = e^{\mp i2\pi ns} \quad \textit{for } n = 0, 1, \ldots$$

are eigenfunctions of the single layer boundary integral operator V, of the double layer boundary integral operator K, and of the hypersingular boundary integral operator D, associated to the eigenvalues

$$\begin{aligned}
\lambda_{V,n} &= R\, K_n(\alpha R)\, I_n(\alpha R), \\
\lambda_{K,n} &= -\frac{1}{2} + (\alpha R)\, I_n'(\alpha R)\, K_n(\alpha R) = \frac{1}{2} + (\alpha R)\, I_n(\alpha R)\, K_n'(\alpha R), \\
\lambda_{D,n} &= -(\alpha R)^2\, I_n'(\alpha R)\, K_n'(\alpha R)
\end{aligned}$$

for $n = 0, 1, \ldots$ respectively, where I_n and K_n denote the first kind and the second kind modified Bessel functions, respectively, while I_n' and K_n' represent the first derivatives of the modified Bessel functions I_n and K_n, respectively.

Proof. The derivation of the eigenvalues follows as in the case of the Helmholtz equation, see, e.g., [2, 3, 14], by using the wave number $k = i\alpha$. In addition, we used the relations between modified Bessel functions and Bessel functions [1] to obtain the desired result. ■

Note that the Fourier functions v_n are also eigenfunctions to the double layer integral operator $\frac{1}{2}I + K$ and of the Steklov–Poincaré operator S associated to the eigenvalues

$$\begin{aligned}
\lambda_{\frac{1}{2}I+K,n} &= (\alpha R)\, I_n'(\alpha R)\, K_n(\alpha R), \\
\lambda_{S,n} &= \frac{\lambda_{\frac{1}{2}I+K,n}^2}{\lambda_{V,n}} + \lambda_{D,n} = \frac{\alpha^2 R\, (I_n'(\alpha R))^2\, K_n(\alpha R)}{I_n(\alpha R)} - (\alpha R)^2\, I_n'(\alpha R)\, K_n'(\alpha R)
\end{aligned}$$

for $n = 0, 1, \ldots$, respectively.

3 Boundary element methods for the mixed boundary value problem

In this section we describe the standard Galerkin boundary element method to solve the boundary integral equation (2.13) numerically with the help of one–periodic B–splines of order $\nu \geq 0$. First, we consider the one–periodic parametrization of the boundary Γ as given in (2.14), further we divide the interval $[0, 1)$ into $N > \nu + 1$ subintervals of mesh size $h = 1/N$ and define as follows

$$[0, 1) = \bigcup_{\ell=1}^{N} [s_\ell, s_{\ell+1}) \quad \text{with } s_\ell = (\ell - 1)h \quad \text{for } \ell = 1, \ldots, N + 1.$$

Moreover, we introduce the N–dimensional subspace $\mathbb{H}_N^\nu$ of one–periodic functions, see, e.g., [20, 22], i.e.,

$$\mathbb{H}_N^\nu = \operatorname{span}\Big(\phi_1^{(\nu)}(s), \ldots, \phi_N^{(\nu)}(s)\Big),$$

where $\phi_k^{(\nu)}(s)$, $k = 1, \ldots, N$ are the B–splines of order ν defined by the following recurrence formulae

$$\phi_1^{(0)}(s) = \begin{cases} 1 & \text{for } -h/2 \le s < h/2, \\ 0 & \text{for } -1/2 \le s < -h/2, \quad h/2 \le s < 1/2, \end{cases} \tag{3.1}$$

$$\phi_1^{(\nu)}(s) = \frac{1}{h} \int_{-1/2}^{1/2} \phi_1^{(\nu-1)}(\tau)\, \phi_1^{(0)}(s-\tau)\, d\tau \quad \text{for } \nu = 1, 2, \ldots, \tag{3.2}$$

and

$$\phi_k^{(\nu)}(s) = \phi_1^{(\nu)}(s-(k-1)h) \quad \text{for } k = 1, 2, \ldots, N, \quad \phi_k^{(\nu)}(s+m) = \phi_k^{(\nu)}(s) \quad \text{for } m \in \mathbb{Z}.$$

In addition, we will use the Fourier series representation of the basis functions $\phi_\ell^{(\nu)}$, i.e.,

$$\phi_\ell^{(\nu)}(t) = \sum_{k \in \mathbb{Z}} c_\ell^\nu(k) e^{i2\pi kt}, \quad \ell = 1, \ldots, N, \quad \nu = 0, 1, \ldots,$$

where the Fourier coefficients are given by [20]

$$c_1^\nu(k) = \begin{cases} h & \text{if } k = 0, \\ \dfrac{\sin^{\nu+1}(\pi k h)}{h^\nu (\pi k)^{\nu+1}} & \text{if } k \neq 0, \end{cases}$$

and

$$c_\ell^\nu(k) = c_1^\nu(k) e^{-i2\pi k(\ell-1)h} \quad \text{for } \ell = 1, \ldots, N.$$

The Galerkin boundary element formulation of the boundary integral equation (2.13) is to find $\widehat{u}_h \in \mathbb{H}_N^\nu \cap \widetilde{H}^{1/2}(\Gamma_N)$ such that

$$\langle S\widehat{u}_h, v_h \rangle_{\Gamma_N} = \langle g_N - S\widetilde{g}_D, v_h \rangle_{\Gamma_N} + \langle Nf, v_h \rangle_{\Gamma_N} \quad \text{for all } v_h \in \mathbb{H}_N^\nu \cap \widetilde{H}^{1/2}(\Gamma_N). \tag{3.3}$$

By using (2.9) and the Lax–Milgram lemma we conclude the unique solvability of the Galerkin formulation (3.3). In addition, Cea's lemma and the approximation property of $\mathbb{H}_N^\nu$ [23, Theorem 10.9] yield the following error estimate

$$\|\widehat{u} - \widehat{u}_h\|_{H^{1/2}(\Gamma)} \le c h^{\nu+1/2} |\widehat{u}|_{H^{\nu+1}(\Gamma)}, \tag{3.4}$$

when assuming that $\widehat{u} \in H^{\nu+1}(\Gamma)$.

The boundary element formulation (3.3) is equivalent to the algebraic system of linear equations

$$S_h \underline{\widehat{u}} = \underline{f}_1 + \underline{f}_2, \tag{3.5}$$

where the stiffness matrix is defined by

$$S_h[i,j] = \langle S\phi_j^{(\nu)}, \phi_i^{(\nu)} \rangle_{\Gamma_N} \quad \text{for } i, j = 1, \ldots, M := \dim \mathbb{H}_N^\nu \cap \widetilde{H}^{1/2}(\Gamma_N),$$

and the right hand side

$$f_{1i} = \langle g_N - S\widetilde{g}_D, \phi_i^{(\nu)} \rangle_{\Gamma_N}, \quad f_{2i} = \langle Nf, \phi_i^{(\nu)} \rangle_{\Gamma_N} \quad \text{for } i = 1, \ldots, M.$$

Remember that $Nf = V^{-1} N_0 f$.

Note that for this particular case, the Galerkin discretization of the Steklov–Poincaré operator can be carried out explicitly by means of its eigenfunctions defined above. This is only possible for the scalar Yukawa case. But, since we are interested in using the discrete Fourier matrix as a preconditioning matrix, we will define and use in this work a symmetric approximation of the continuous Steklov–Poincaré operator S. To this end let us first define the Galerkin discretizations of the boundary integral operators V, K, K' and D as follows:

$$V_h[i,j] = \langle V\phi_j^{(\nu-1)}, \phi_i^{(\nu-1)} \rangle_\Gamma, \quad D_h[i,j] = \langle D\phi_j^{(\nu)}, \phi_i^{(\nu)} \rangle_\Gamma, \quad \widehat{K}_h[i,j] = \langle (\frac{1}{2}I + K)\phi_j^{(\nu)}, \phi_i^{(\nu-1)} \rangle_\Gamma,$$

for $i, j = 1, \ldots, N$. These matrices can be computed explicitly and in an efficient way by using the following results [24]:

Lemma 3.1 *Let Γ be given as in (2.14). The discrete single and double layer integral operators V_h and $\widehat{K}_h$, and the discrete hypersingular integral operator D_h are circulant matrices. Moreover, V_h and D_h are symmetric and positive definite. In addition, their eigenvalues are given respectively by*

$$\lambda_{V_{h,j}} = \begin{cases} hR\, I_0(\alpha R)\, K_0(\alpha R) \quad \textit{for } j = 1, \\ hR \left(\dfrac{\sin \pi s}{\pi}\right)^{2\nu} \left(\displaystyle\sum_{k=0}^{\infty} \frac{I_{(j-1+kN)}(\alpha R) K_{(j-1+kN)}(\alpha R)}{(k+s)^{2\nu}} \right. \\ \qquad \left. + \displaystyle\sum_{k=1}^{\infty} \frac{I_{(1-j+kN)}(\alpha R) K_{(1-j+kN)}(\alpha R)}{(k-s)^{2\nu}} \right) \textit{for } j = 2, \ldots, N, \end{cases}$$

$$\lambda_{\hat{K}_{h,j}} = \begin{cases} h(\alpha R)\, I_1(\alpha R)\, K_0(\alpha R) \quad \textit{for } j = 1, \\ h(\alpha R) \left(\dfrac{\sin \pi s}{\pi}\right)^{2\nu+1} \left(\displaystyle\sum_{k=0}^{\infty} \frac{I'_{(j-1+kN)}(\alpha R) K_{(j-1+kN)}(\alpha R)}{(k+s)^{2\nu+1}} (-1)^{k(2\nu-1)} \right. \\ \qquad \left. + \displaystyle\sum_{k=1}^{\infty} \frac{I'_{(1-j+kN)}(\alpha R) K_{(1-j+kN)}(\alpha R)}{(k-s)^{2\nu+1}} (-1)^{(k+1)(2\nu-1)} \right) \quad \textit{for } j = 2, \ldots, N, \end{cases}$$

$$\lambda_{D_{h,j}} = \begin{cases} h(\alpha R)^2\, I_1(\alpha R)\, K_1(\alpha R) \quad \textit{for } j = 1, \\ -h(\alpha R)^2 \left(\dfrac{\sin \pi s}{\pi}\right)^{2\nu+2} \left(\displaystyle\sum_{k=0}^{\infty} \frac{I'_{(j-1+kN)}(\alpha R) K'_{(j-1+kN)}(\alpha R)}{(k+s)^{2\nu+2}} \right. \\ \qquad \left. + \displaystyle\sum_{k=1}^{\infty} \frac{I'_{(1-j+kN)}(\alpha R) K'_{(1-j+kN)}(\alpha R)}{(k-s)^{2\nu+2}} \right) \quad \textit{for } j = 2, \ldots, N \end{cases}$$

for $s = (j-1)/N$. I_n and K_n represent the first and second kind of modified Bessel functions, respectively, while I'_n and K'_n represent their first derivative, respectively.

The most important property of the circulant matrices [4, 6, 20, 22] is that they are diagonalizable and easily invertible if the inverse exists,

$$V_h = \frac{1}{N} Q D_V Q, \qquad \widehat{K}_h = \frac{1}{N} F D_{\widehat{K}} F^*, \qquad D_h = \frac{1}{N} Q D_D Q,$$

where $F \in \mathbb{C}^{N\times N}$ and $Q \in \mathbb{R}^{N\times N}$ are the Fourier matrices given by

$$F[k,\ell] = e^{i2\pi(k-1)(\ell-1)h}, \quad Q[k,l] = \cos[2\pi(k-1)(\ell-1)h] + \sin[2\pi(k-1)(\ell-1)h]$$

for $k,\ell = 1,\ldots,N$ and D_V, $D_{\widehat{K}}$ as well as D_D are diagonal matrices which are defined by the eigenvalues of V_h, $\widehat{K}_h$, and D_h respectively.

The Steklov–Poincaré operator S can now be approximated by

$$\widetilde{S}_h := \widehat{K}_h^\top V_h^{-1} \widehat{K}_h + D_h. \tag{3.6}$$

Remark that from Lemma 3.1 the entries of the discrete approximate Steklov–Poincaré operator $\widetilde{S}_h$ can be computed explicitly and exactly. Moreover, its eigenvalues are given by

$$\lambda_{\widetilde{S}_{h,j}} := \frac{\lambda^2_{\widehat{K}_{h,j}}}{\lambda_{V_{h,j}}} + \lambda_{D_{h,j}} \quad \text{for } j = 1,\ldots,N.$$

The discretization of the boundary integral equation (2.13) can be written as follows

$$\widetilde{S}_h \underline{\widetilde{u}} = \underline{f}_1 + \underline{f}_2. \tag{3.7}$$

Note that $\widetilde{S}_h$ defines a positive definite approximation of the exact Galerkin matrix S_h which arises from the direct Galerkin approximation of the Steklov–Poincaré operator S. The additional error can be analyzed by using the Strang lemma, see, e.g., [23]. Moreover, $\widetilde{S}_h$ is defined with respect to the whole boundary $\Gamma = B_R(0)$, but it has to be applied only to $\underline{\widetilde{u}} \in \mathbb{R}^M \leftrightarrow \widetilde{u}_h \in \mathbb{H}^\nu_N \cap \widetilde{H}^{1/2}(\Gamma_N)$ which can be realized easily within iterative solvers.

It remains to describe the evaluation of the vector $\underline{f}_2$ which results from the Newton potential Nf. This will be done again by using circulant matrices which are efficient for a matrix–vector multiplication, and for an efficient memory storage. For $\ell = 1,\ldots,M$ we have

$$f_{2,\ell} = \langle Nf, \phi^{(\nu)}_\ell \rangle_{\Gamma_N} = \langle V^{-1} N_0 f, \phi^{(\nu)}_\ell \rangle_{\Gamma_N} = \langle w, \phi^{(\nu)}_\ell \rangle_{\Gamma_N}, \tag{3.8}$$

where $w = V^{-1} N_0 f \in H^{-1/2}(\Gamma)$ is the unique solution of the variational problem

$$\langle Vw, z\rangle_\Gamma = \langle N_0 f, z\rangle_\Gamma \quad \text{for all } z \in H^{-1/2}(\Gamma), \tag{3.9}$$

and with the Galerkin boundary element formulation to find $w_h \in \mathbb{H}^{\nu-1}_N$ such that

$$\langle Vw_h, z_h\rangle_\Gamma = \langle N_0 f, z_h\rangle_\Gamma \quad \text{for all } z_h \in \mathbb{H}^{\nu-1}_N. \tag{3.10}$$

Hence we need to determine the Newton potential $N_0 f$ as the right hand side of (3.10). In particular, we have to compute for $\ell = 1, \ldots, N$

$$N_0 f[\ell] = \int_\Gamma \phi_\ell^{(\nu-1)}(x) \int_\Omega U^*(x,y) f(y) dy ds_x. \tag{3.11}$$

With this we can define an approximate right hand side $\widetilde{\underline{f}}_2$ where the additional error can be analyzed again by using the Strang lemma, and which does not disturb the optimal order of convergence, see, e.g., [18, 23].

4 Evaluation of Newton potentials for the scalar Yukawa equation

In what follows we discuss an efficient evaluation of the Newton potential (3.11) in the particular case when considering piecewise constant basis functions, i.e. $\nu = 1$. To compute the discrete Newton potential, the order of integration is interchanged first. In particular, for a B–spline of order zero $\phi_\ell^{(0)}$ we have

$$N_0 f[\ell] = \int_\Omega f(y) \int_{\tau_\ell} U^*(x,y) ds_x dy.$$

Further, a special volume mesh is considered in the domain Ω, see [24] for the detailed procedure. First, the domain Ω is split into rings, and on each ring suitable meshes are constructed in an adaptive way. For simplicity, we assume the mesh size of the volume elements near the boundary to be equal to the mesh size of the boundary elements. Additionally, from the ring close to the boundary to the inner rings the mesh size is reduced with a certain rate, see [24]. We then obtain

$$N_0 f[\ell] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} \int_{T_{jk}} f(y) \int_{\tau_\ell} U^*(x,y) ds_x dy, \tag{4.1}$$

where M_r and N are the number of rings and the number of elements on each ring, respectively. Note that N is also the number of boundary elements and $\overline{\Omega} = \cup_{j=1}^{M_r} \cup_{k=1}^{N} T_{jk}$. Then, an approximation of (4.1) can be given by applying the simple mid point rule

$$\widetilde{N}_0 f[\ell] = \sum_{j=1}^{M_r} \sum_{k=1}^{N} |T_{jk}| \, f(y_{jk}) \int_{\tau_\ell} U^*(x, y_{jk}) ds_x, \tag{4.2}$$

where $|T_{jk}|$ and y_{jk} are the volume and the center of mass of the element T_{jk}, respectively. Note that the remaining boundary integral corresponds to the discretization of the single layer potential by using the collocation method which can be computed easily. From (4.2)

the vector $\widetilde{N}_0 f$ of the approximate Newton potential can be written in matrix form as follows

$$\underline{\widetilde{N}}_0 f := \sum_{j=1}^{M_r} A_j \underline{f}_j, \tag{4.3}$$

where for $j = 1, \ldots, M_r$, $\underline{f}_j$ and A_j represent the piecewise constant approximation of the function f and the matrix obtained by computing the remaining boundary integral on the j^{th} ring, respectively. Moreover, we have for $j = 1, \ldots, M_r$

$$\underline{f}_j[k] = |T_{jk}|\, f(y_{jk}) \quad \text{for } k = 1, \ldots, N \tag{4.4}$$

and

$$A_j[\ell, k] = \int_{\tau_\ell} U^*(x, y_{jk}) ds_x \quad \text{for } \ell, k = 1, \ldots, N. \tag{4.5}$$

Since the meshes on the boundary and on each ring are uniform, i.e. of the same size, and since the fundamental solution $U^*(\cdot, \cdot)$ is invariant with respect to rotations, we obtain for $j = 1, \ldots, M_r$

$$A_j[\ell + 1, k + 1] = A_j[\ell, k] \quad \text{for } \ell, k = 1, \ldots, N, \tag{4.6}$$

which means that, for $j = 1, \ldots, M_r$ the matrices A_j are circulant. This reduces the effort to generate the matrices A_j from quadratic to linear, i.e. from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ as well as the matrix–vector multiplication effort from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log(N))$, see, e.g., [20].

5 Numerical results

For the numerical examples we presented here, we consider the domain Ω to be a disc centered at $c = (1.0, 1.0)^\top$ with radius $r = 1.0$. First we consider the Dirichlet boundary value problem

$$u(x) - \Delta u(x) = -4 + x_1^2 + x_2^2 \quad \text{for } x \in \Omega, \quad u(x) = x_1^2 + x_2^2 \quad \text{for } x \in \Gamma = \partial\Omega, \tag{5.1}$$

where the exact solution is $u(x) = x_1^2 + x_2^2$. The boundary element discretization of the boundary value problem (5.1) results in the linear system

$$V_h \underline{t} = (\frac{1}{2} M_h + K_h)\underline{g} - \underline{\widetilde{N}_0} f,$$

which is solved by using the Fast Fourier transform (FFT) since the matrix V_h is circulant. The numerical results are given in Table 1 below.

The first, the second and the third column contain the level of refinement, the number N of boundary elements, and the number of volume elements in each ring and the number M_r of rings, respectively. In addition, the time in seconds is given for the evaluation of the Newton potential $\underline{\widetilde{N}_0} f$, and the setup and solving the linear system. In columns 5 and 6, the L_2 norm of the error $t - t_h$ and the estimated order of convergence are given

refinement			CPU time (sec)	L_2 error	
Level	N	M_r	setup and solve	$\|t - t_h\|_{L^2(\Gamma)}$	eoc
3	32	8	0.00	2.84 –1	
4	64	16	0.00	1.42 –1	1.00
5	128	32	0.01	7.10 –2	1.00
6	256	64	0.05	3.55 –2	1.00
7	512	128	0.20	1.78 –2	1.00
Theory					1

Table 1: L_2 error, order of convergence and CPU time, Dirichlet problem.

respectively. According to the standard error estimate for the Dirichlet problem, see, e.g. [23] a linear convergence is observed. As N and M_r increase both by a factor of 2, the times for the evaluation of $\underline{\tilde{N}_0} f$, setup and solving the linear system increase by a factor of 4 which corresponds to a quadratic order as expected. Note that a standard evaluation would be of cubic order.

In the second example, we consider the mixed boundary value problem (2.1) where the Neumann boundary Γ_N is the boundary of the first sector of the disc, that is between 0 and $\pi/2$, and the Dirichlet boundary is $\Gamma_D = \Gamma \setminus \Gamma_N$. As in the first example we chose the given data such that $u(x) = x_1^2 + x_2^2$ is the exact solution. The mixed boundary value problem (2.1) is equivalent to the linear system (3.7) where the stiffness matrix is as given in (3.6). Since the Galerkin discretization $\widetilde{S}_h$ of the approximate Steklov-Poincaré operator is symmetric and positive definite we solve the linear system (3.7) by using the preconditioned conjugate gradient algorithm, with the preconditioning matrix [23]

$$C_D := \overline{M}_h \overline{V}_h^{-1} \overline{M}_h,$$

where $\overline{M}_h$ and $\overline{V}_h$ are the mass matrix and the discrete single layer matrix, respectively, defined by using piecewise linear B–splines. Moreover, $\overline{V}_h$ is circulant, symmetric and positive definite, and its eigenvalues are given by as in Lemma 3.1 where we have to set $\nu = 2$. Therefore, $\overline{V}_h$ is diagonalizable. Hence, all matrix–vector multiplications with the approximate stiffness matrix $\widetilde{S}_h$ as with the preconditioning matrix C_D^{-1} can be performed by utilizing the Fast Fourier Transform (FFT). In Table 2 we present the numerical results, where we have used the same number of ring as in the first example. For the same reason as in the first example the overall setup time for the linear system increases by a factor of 4. The L_2 norms of the errors $u - u_h$ and $t - t_h$ are given, and we observe a quadratic and linear convergence, as expected.

6 Linear elasticity of Yukawa type

In this section, we consider the inhomogeneous Dirichlet problem given by

$$s^2 \underline{u}(x) - \mu \Delta \underline{u}(x) - (\lambda + \mu) \mathrm{graddiv}\, \underline{u}(x) = \underline{f}(x) \text{ for } x \in \Omega, \quad \underline{u}(x) = \underline{g}(x) \text{ for } x \in \Gamma, \quad (6.1)$$

N	Iterations PCG	L_2 error $\|u-u_h\|_{L^2(\Gamma)}$	eoc	L_2 error $\|t-t_h\|_{L^2(\Gamma)}$	eoc	CPU time (sec) setup and solve
32	4	1.23 –2		2.71 –1		0.00
64	8	3.04 –3	2.01	1.35 –1	1.00	0.02
128	12	7.57 –4	2.01	6.77 –2	1.00	0.08
256	12	1.89 –4	2.00	3.39 –2	1.00	0.34
512	12	4.71 –5	2.00	1.69 –2	1.00	1.36
Theory			2		1	

Table 2: L_2 error, order of convergence and CPU time, mixed problem.

where μ and λ represent the Lamé constants and s may come from a time discretization of the elastodynamic wave equation. The fundamental solution of the partial differential operator in (6.1) is given for $r = |x-y|$ and $r_i = x_i - y_i$, $i = 1, 2$ by

$$\begin{aligned} U^*_{ij}(x,y) &= \frac{1}{2\pi s^2}\left\{\left(k_2^2\,K_0(k_2r) + \frac{1}{r}\Big[k_2K_1(k_2r) - k_1K_1(k_1r)\Big]\right)\delta_{ij}\right. \\ &\left. -\frac{r_ir_j}{r^2}\left(\Big[k_2^2K_0(k_2r) - k_1^2K_0(k_1r)\Big] + \frac{2}{r}\Big[k_2K_1(k_2r) - k_1K_1(k_1r)\Big]\right)\right\} \end{aligned}$$

with

$$k_1^2 = \frac{s^2}{\lambda+2\mu}, \quad k_2^2 = \frac{s^2}{\mu}.$$

Related boundary integral equations for the boundary value problem (6.1) can be derived as in the scalar Yukawa case, or similar to linear elastostatics, e.g., [23], see also [24]:

$$V\underline{t} = (\frac{1}{2}I + K)\underline{u} - N_0\underline{f} \quad \text{on } \Gamma.$$

In particular, the single layer integral operator can be written in the matrix form as follows

$$V = \begin{pmatrix} V_{11}^1 & 0 \\ 0 & V_{22}^1 \end{pmatrix} + \begin{pmatrix} V_{11}^2 & V_{12} \\ V_{12} & -V_{11}^2 \end{pmatrix},$$

where

$$\begin{aligned} (V_{11}^1 w_1)(x) &= \frac{1}{2\pi s^2}\int_\Gamma \left[k_2^2\,K_0(k_2r) + \frac{[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right] w_1(y)\,ds_y, \\ (V_{22}^1 w_2)(x) &= \frac{1}{2\pi s^2}\int_\Gamma \left[k_1^2\,K_0(k_1r) - \frac{[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right] w_2(y)\,ds_y, \\ (V_{11}^2 w_1)(x) &= -\frac{1}{2\pi s^2}\int_\Gamma \left[[k_2^2K_0(k_2r) - k_1^2K_0(k_1r)] + \frac{2[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right] \frac{r_1^2}{r^2} w_1(y) ds_y, \\ (V_{12} w_2)(x) &= -\frac{1}{2\pi s^2}\int_\Gamma \left[[k_2^2K_0(k_2r) - k_1^2K_0(k_1r)] + \frac{2[k_2K_1(k_2r) - k_1K_1(k_1r)]}{r}\right] \frac{r_1r_2}{r^2} w_2(y) ds_y. \end{aligned}$$

Proposition 6.1 [24] *Let Γ be given as in* (2.14). *The Fourier functions $v_n(\tau) = e^{\mp i2\pi n\tau}$ for $n = 0, 1, \ldots$ are the eigenfunctions of the boundary integral operators V_{11}^1 and V_{22}^1, respectively, i.e.*

$$\begin{aligned}(V_{11}^1 v_n)(\tau) &= \frac{1}{2s^2}\Big[\Lambda_n(k_1,k_2) + 2\,\Gamma_{n,n-1,n+1}(k_1,k_2)\Big]\, v_n(\tau),\\ (V_{22}^1 v_n)(\tau) &= \frac{1}{2s^2}\Big[\Lambda_n(k_1,k_2) - 2\,\Gamma_{n,n-1,n+1}(k_1,k_2)\Big]\, v_n(\tau),\end{aligned}$$

where

$$\Gamma_{m,n,k}(k_1,k_2) = \lambda_m^D(k_1) - \lambda_m^D(k_2) + \frac{1}{4}\Big[k_2^2\lambda_n^V(k_2) - k_1^2\lambda_n^V(k_1) + k_2^2\lambda_k^V(k_2) - k_1^2\lambda_k^V(k_1))\Big],$$

and

$$\Lambda_n(k_1,k_2) = k_1^2\lambda_n^V(k_1) + k_2^2\lambda_n^V(k_2), \quad k_1^2 = \frac{s^2}{\lambda+2\mu}, \quad k_2^2 = \frac{s^2}{\mu}.$$

In addition, the following relations hold

$$\Gamma_{m,n,k}(k_1,k_2) = \Gamma_{-m,-n,-k}(k_1,k_2) = \Gamma_{m,k,n}(k_1,k_2), \quad \Lambda_n(k_1,k_2) = \Lambda_{-n}(k_1,k_2),$$

where λ_n^V and λ_n^D, $n = 0, 1, \ldots$ are the eigenvalues of the single layer integral operator and of the hypersingular integral operator for the scalar Yukawa problem, respectively, as given in Lemma 2.2.

In a straightforward manner as in Sect. 4, the evaluation of the Newton potential takes the form

$$(N_0\underline{f})_\ell[i] = \sum_{j=1}^{M_r}\sum_{k=1}^{N}\sum_{m=1}^{2}\int_{T_{jk}} f_m(y)\int_{\tau_i} U_{\ell m}^*(x,y)ds_x dy \quad \text{for } i = 1,\ldots,N;\ \ell = 1,2, \tag{6.2}$$

where T_{jk} is the k^{th} element on the j^{th} ring, and $\overline{\Omega} = \cup_{j=1}^{M_r}\cup_{k=1}^{N} T_{jk}$. As before, T_{jk} is an isoparametric triangle or an isoparametric quadrangle whether it is on the last inner ring or others rings. If we approximate the functions f_1 and f_2 on each ring by piecewise constant functions, respectively, the approximation of the vector $\underline{N}_0\underline{f}$ denoted by $\widetilde{\underline{N}}_0\underline{f}$ can then be written in the matrix form as follows

$$\widetilde{\underline{N}}_0\underline{f} = \sum_{j=1}^{M_r}\left[\begin{pmatrix} A_{11}^{1j} & 0\\ 0 & A_{22}^{1j}\end{pmatrix} + \begin{pmatrix} A_{11}^{2j} & A_{12}^{j}\\ A_{12}^{j} & -A_{11}^{2j}\end{pmatrix}\right]\begin{pmatrix}\underline{f}_1^j\\ \underline{f}_2^j\end{pmatrix},$$

where for $j = 1, \ldots, M_r$

$$\begin{aligned}A_{\ell\ell}^{1j}[i,k] &= \int_{\tau_i} U_{\ell\ell}^{*1}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N,\ \ell = 1,2,\\ A_{11}^{2j}[i,k] &= \int_{\tau_i} U_{11}^{*2}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N,\\ A_{12}^{j}[i,k] &= \int_{\tau_i} U_{12}^{*}(x,y_{jk})ds_x \quad \text{for } i,k = 1,\ldots,N\end{aligned}$$

and

$$\underline{f}_\ell^j[k] = |T_{jk}|\, f_\ell(y_{jk}) \quad \text{for } k = 1, \ldots, N,\ \ell = 1, 2,$$

where $|T_{jk}|$ and y_{jk} represent the volume and the center of mass of the element T_{jk}, respectively. Note that for $j = 1, \ldots, M_r$ the matrices A_{11}^{1j} and A_{22}^{1j} are derived from the boundary integral operators V_{11}^1 and V_{22}^1, respectively. Hence, A_{11}^{1j} and A_{22}^{1j} are circulant matrices, see Proposition 6.1. Finally, by using the Fast Fourier Transform (FFT) we obtain

$$\widetilde{\underline{N}}_0\underline{f} = \sum_{j=1}^{M_r} \begin{pmatrix} F^{-1}\left(F(\underline{C}_1^j)F(\underline{f}_1^j)\right) + A_{11}^{2j}\underline{f}_1^j + A_{12}^j\underline{f}_2^j \\ F^{-1}\left(F(\underline{C}_2^j)F(\underline{f}_2^j)\right) + A_{12}^j\underline{f}_1^j - A_{11}^{2j}\underline{f}_2^j \end{pmatrix},$$

where for $j = 1, \ldots, M_r$, $\underline{C}_1^j$ and $\underline{C}_2^j$ are the first columns of the circulant matrices A_{11}^{1j} and A_{22}^{1j}, respectively. Note that on each ring the approximate Newton potential vector $\widetilde{\underline{N}}_0\underline{f}$ is given in terms of a matrix–vector multiplication like for the scalar Yukawa problem presented in Sect. 4, but the matrix in this case has a block structure, the first block is made from circulant matrices A_{11}^{1j} and A_{22}^{1j}, therefore the matrix–vector multiplication of this first part can be speed up by applying the Fast Fourier Transform (FFT), while the second block has a symmetric structure, this permits us to compute only two matrices A_{11}^{2j} and A_{12}^j.

For the numerical test, we consider the domain $\Omega = B_r(c)$ to be the same disc as in the first example. We take $\lambda := 115.3846$, $\mu := 76.9231$ and $s := 10.0$. In addition, we chose the given data such that $\underline{u}(x) = (x_1^2 + x_2, x_2^2 - x_1)^\top$ is the exact solution. The resulting linear system is solved by a CG method and the results are given in Table 3.

refinement			CPU time (sec)		Iter.	L_2 error	
Level	N	M_r	setup	solve	CG	$\|t - t_h\|_{L^2(\Gamma)}$	eoc
3	32	8	0.29	0.00	8	137.499	
4	64	16	1.54	0.00	8	68.7058	1.00
5	128	32	8.86	0.00	7	34.3466	1.00
6	256	64	56.7	0.00	6	17.1724	1.00
7	512	128	401.1	0.02	6	8.58611	1.00
Theory							1

Table 3: L_2 error, order of convergence and CPU time, linear elasticity.

In columns 4 and 5, the times in seconds are given for the evaluation of the Newton potential $\widetilde{\underline{N}}_0\underline{f}$, setup and solving the linear system, respectively. In column 6, the number of iterations for the CG method is given, while in columns 7 and 8, the L_2 norm of the error $t - t_h$ and the estimated order of convergence are given, respectively. As expected, a linear convergence is observed. As N and M_r increase both by a factor of 2, the time for the evaluation of the first part of $\widetilde{\underline{N}}_0\underline{f}$, increases by a factor of 4 due to the circulant property of the matrices involved. But, the time for the evaluation of the second part of $\widetilde{\underline{N}}_0\underline{f}$, increases by a factor of 8, i.e. cubic.

7 Conclusions

In this work we have presented an efficient method for the fast evaluation of volume or Newton potentials in boundary element methods, in the particular case of a circular domain. This is of interest, e.g., when considering time–dependent boundary value problems and explicit or implicit time discretization schemes. Applications in mind involve dynamic contact problems with friction [24]. Extensions to three–dimensional problems using the presented approach seem to be possible for rotational–symmetric domains, as considered in [16, 21].

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