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Error Estimates for Neumann Boundary Control Problems with Energy Regularization

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Abstract

A Neumann boundary control problem for a second order elliptic state equation is considered which is regularized by an energy term which is equivalent to the $H^{-1/2}(\Gamma)$-norm of the control. Both the unconstrained and the control constrained cases are investigated. The regularity of the state, control, and co-state variables is studied with particular focus on the singularities due to the corners of the two-dimensional domain. The state and co-state are approximated by piecewise linear finite elements. For the approximation of the control variable we take carefully designed spaces of piecewise linear or piecewise constant functions, such that an inf-sup condition is satisfied. Bounds for the discretization error are proved for all three variables in dependence on the largest interior angle of the domain. Numerical tests suggest that these bounds are optimal in the unconstrained case but too pessimistic in the control constrained case with non-convex domains.

Keywords: optimal control, corner singularities, finite element method, error estimates

AMS Subject Classification: 65N15, 65N30, 49J20

1 Introduction

The numerical solution of Neumann boundary control problems has been studied in a variety of publications [1, 5, 6, 13, 17]. In all of these papers the regularization term contains the $L^2(\Gamma)$-norm of the control. However, for the existence of a weak solution of a second order elliptic state equation in the energy space $H^1(\Omega)$, it is sufficient that the Neumann datum, which is the control variable in our case, is in $H^{-1/2}(\Gamma)$. We call a regularization with Neumann datum in $H^{-1/2}(\Gamma)$ or Dirichlet datum in $H^{1/2}(\Gamma)$ *energy regularization*. This kind of regularization has first been introduced by Lions in his fundamental book [12]. So far, error estimates for control problems with energy regularization have been studied for Dirichlet control by Of/Phan/Steinbach [19, 20].

Depending on the application in mind the energy regularization gives an optimal control which may reflect the physical behavior more properly. As we will see in Figure 3 the optimal control behaves similar along the edges when considering $L^2(\Gamma)$- and $H^{-1/2}(\Gamma)$-regularization, but rather different in corner points. If the angles of all corner points of the computational domain are smaller than $120°$ the control is in $H^2(\Gamma)$ when $L^2(\Gamma)$-regularization for unconstrained problems is used (see [1]). In case of the energy regularization approach we expect only $H^1_{pw}(\Gamma)$-regularity for the control. If angles are larger than $120°$ the spaces $H^2(\Gamma)$ and $H^1(\Gamma)$, respectively, have to be weakened by introducing a weight function. The regularity of state and adjoint state coincide in general for both approaches. As a consequence also the convergence rate of the discrete solution is lower. For the $L^2(\Gamma)$-regularization we know from [1] that the error estimate

$$\|z - z_h\|_{L^2(\Gamma)} \le ch^{\min\{2,1/2+\pi/\omega\}-\varepsilon}, \qquad \varepsilon > 0,$$

holds when the *postprocessing approach* [17] or *variational discretization* [11] is used, where ω is the largest interior angle of the domain and z and z_h the continuous and discrete optimal control. One might expect that the convergence rate for $H^{-1/2}(\Gamma)$-regularization is reduced by one. The proof of this conjecture is the main result of this paper.

In the present paper the Neumann boundary control problem with energy regularization

$$\min_{z\in H^{-1/2}(\Gamma)} \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|z\|^2_{H^{-1/2}(\Gamma)}$$

$$\begin{aligned} \text{s. t.} \qquad -\Delta u + u &= f && \text{in } \Omega, \\ \partial_n u &= z && \text{on } \Gamma, \end{aligned}$$

is studied, where the regularization term is realized with an equivalent formulation using an inverse Steklov-Poincaré operator. We discretize the state by piecewise linears and the control by either piecewise linears or constants on an appropriate boundary mesh and certain modifications at corner points of the domain, and particularly focus on the dependence of the error estimates on the maximal interior angle of the domain. This angle is known to restrict the regularity of elliptic boundary value problems. Moreover, we exploit sharp error estimates for the discretization error on the boundary of the domain which were recently proven by Apel/Pfefferer/Rösch in [1].

In order to formulate the main result of the paper, let (u,z) and (u_h,z_h) denote the continuous and discrete pair of optimal state and control, and ω the largest interior angle of the computational domain. For unconstrained problems we prove the error estimates

$$\begin{aligned} h^{1/2}\|z-z_h\|_{L^2(\Gamma)}+\|z-z_h\|_{H^{-1/2}(\Gamma)} &\le ch^{\min\{3/2,\pi/\omega\}-\varepsilon}, \\ \|u-u_h\|_{H^1(\Omega)} &\le ch^{\min\{1,\lambda\}-\varepsilon}, \end{aligned}$$

for arbitrary $\varepsilon>0$ using certain discretization strategies specified later. For problems involving additional control constraints we will show that the first estimate can be improved and that the convergence rate $\min\{3/2,\pi/\breve\omega,2\pi/\omega-1/2\}$ is achieved where $\breve\omega$ denotes the largest convex angle.

We formulate the optimality conditions for the model problem in Section 2.2 and study the regularity of the state, co-state and control variables in Section 2.3. In Section 3 the discretization strategy and *a priori* error estimates are presented. In particular, we observe that the inf-sup stability of the pair of discrete spaces for state and control is mandatory for both the analysis and the practical realization of the method.

As a by-product of our investigations we observe a special behavior of the unconstrained control in the vicinity of corners of the domain which can also occur for Dirichlet control problems with $L^2(\Gamma)$-regularization [14]. The optimal control is zero in convex corners and becomes infinity in concave corners. This behavior is sometimes questioned, but it is not wrong; it is just the behavior of the optimal control when energy regularization for the Neumann control is used. If it is not desirable due to practical reasons, the modeling of the problem has to be changed. We study one such remedy, namely adding control constraints, in Section 4. Since the control is more regular in this case we are able to improve the convergence order of the discretization error. However, these estimates might still be too pessimistic, as we will see in the numerical experiments from Section 5.

2 The continuous unconstrained optimal control problem

2.1 Formulation of the problem

Before we introduce the model problem, let us summarize some notation. Let $\Omega\subset\mathbb{R}^2$ be a bounded open domain which is assumed to have a polygonal boundary Γ with corner points $\{x^{(j)}\}_{j=1}^d$ enumerated counter-clockwise. We write $\mathcal{C}:=\{1,\ldots,d\}$ in the following. The two edges meeting in $x^{(j)}$ have interior angle $\omega_j\in(0,2\pi)$. The largest angle which has most influence on the regularity, is denoted by $\omega:=\max_j\omega_j$. Moreover, we denote by Γ_j, $j\in\mathcal{C}$, the boundary edges having endpoints $x^{(j)}$ and $x^{(j+1)}$, whereas we set $x^{(d+1)}=x^{(1)}$ by convention. For some $s\in\mathbb{R}_+$ and $p\in[1,\infty]$ we denote the usual Sobolev spaces by $W^{s,p}(\Omega)$ and the corresponding trace spaces by $W^{s-1/p,p}(\Gamma)$. The Hilbertian Sobolev spaces are abbreviated by $H^s(\Omega) := W^{s,2}(\Omega)$ and $H^s(\Gamma) := W^{s,2}(\Gamma)$. For a certain right-hand side $f \in (H^1(\Omega))'$ and Neumann datum $z \in H^{-1/2}(\Gamma) := (H^{1/2}(\Gamma))'$ the state equation is given by

$$-\Delta u + u = f \quad \text{in } \Omega, \qquad \partial_n u = z \quad \text{on } \Gamma, \tag{2.1}$$

and we decompose its solution into $u = u_z + u_f$ such that

$$\begin{aligned} -\Delta u_z + u_z &= 0, & -\Delta u_f + u_f &= f \quad \text{in } \Omega,\\ \partial_n u_z &= z, & \partial_n u_f &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{2.2}$$

From this we find

$$\begin{aligned} \|u_z\|^2_{H^1(\Omega)} &= \int_\Omega \nabla u_z \cdot \nabla u_z \,\mathrm{d}x + \int_\Omega u_z^2 \,\mathrm{d}x = \int_\Gamma \partial_n u_z u_z \,\mathrm{d}s_x\\ &= \int_\Gamma z\, u_z \,\mathrm{d}s_x = \langle z, \mathcal{N} z\rangle_\Gamma =: \|z\|^2_{H^{-1/2}(\Gamma)} \end{aligned} \tag{2.3}$$

for the representation of the energy norm in $H^{-1/2}(\Gamma)$, where

$$\langle \cdot,\cdot\rangle_\Gamma := \langle \cdot,\cdot\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}$$

is the dual pairing. Here, $\mathcal{N} : H^{-1/2}(\Gamma) \ni z \mapsto u_z \in H^{1/2}(\Gamma)$ denotes the inverse Steklov-Poincaré operator which realizes the Neumann-to-Dirichlet map with respect to the homogeneous partial differential equation. In case of $f \equiv 0$ we have $u = u_z$ and the regularization term is equal to the energy norm of the optimal state.

Note, that due to the mapping properties of $\mathcal{N}$ our definition of the norm in $H^{-1/2}(\Gamma)$ is equivalent to the dual Sobolev-Slobodetskii norm (compare e. g. [23, Section 4.1.3]).

We denote by

$$a(u, v) := \int_\Omega [\nabla u(x) \cdot \nabla v(x) + u(x)v(x)] \,\mathrm{d}x$$

the bilinear form related to the operator $-\Delta + I$ and by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. The weak formulations of (2.2) then read: *Find* $u_z, u_f \in H^1(\Omega)$ *such that*

$$a(u_z, v) = \langle z, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega), \tag{2.4}$$

$$a(u_f, v) = \langle f, v\rangle_\Omega \qquad \forall v \in H^1(\Omega), \tag{2.5}$$

where $\langle \cdot,\cdot\rangle_\Omega := \langle \cdot,\cdot\rangle_{(H^1(\Omega))',H^1(\Omega)}$ denotes the dual pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$.

For a given desired state $u_d \in L^2(\Omega)$ and regularization parameter $\alpha > 0$ we consider the Neumann boundary control problem

$$J(u, z) := \frac{1}{2}\|u - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\langle z, \mathcal{N} z\rangle_\Gamma \to \min! \tag{2.6}$$

with the constraint that u and z satisfy equation (2.1). Throughout the paper, $z \in H^{-1/2}(\Gamma)$ denotes the control variable and $u \in H^1(\Omega)$ the state variable. Note, that we will assume higher regularity of the input data later in order to prove optimal error estimates. This optimization problem is used to track the desired state u_d when the Neumann datum is controlled. The operator $\mathcal{N} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is linear, continuous and self-adjoint. As a consequence, the functional $R\colon H^{-1/2}(\Gamma) \to \mathbb{R}$ defined by $R(z) := \frac{1}{2}\langle z, \mathcal{N}z\rangle_\Gamma$ is Fréchet-differentiable with derivative

$$[R'(z)]\,(h) = \langle h, \mathcal{N}z\rangle_\Gamma \qquad \text{for all } h \in H^{-1/2}(\Gamma). \tag{2.7}$$

2.2 Optimality conditions

The aim of this section is to derive an optimality system for the problem (2.6). In what follows, $S : H^{-1/2}(\Gamma) \to H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega)$ denotes the solution operator of the homogeneous state equation, i. e.

$$u_z = Sz \quad :\Longleftrightarrow \quad a(u_z, v) = \langle z, v\rangle_\Gamma \qquad \forall v \in H^1(\Omega). \tag{2.8}$$

This mapping is well-defined, linear and continuous and we may thus rewrite the original problem (2.6) as

$$j(z) := \frac{1}{2}\|Sz + u_f - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\langle z, \mathcal{N}z\rangle_\Gamma \to \min! \quad \text{s. t.} \quad z \in H^{-1/2}(\Gamma). \tag{2.9}$$

The reduced functional j is Fréchet-differentiable and the necessary optimality condition then reads

$$0 = \langle v, j'(z)\rangle_\Gamma = (Sz + u_f - u_d, Sv) + \alpha\,\langle v, \mathcal{N}z\rangle_\Gamma \quad \forall v \in H^{-1/2}(\Gamma). \tag{2.10}$$

Let us summarize the linear and constant part of the optimality condition by introducing the operator $T^\alpha : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and the element $g \in H^{1/2}(\Gamma)$ defined by

$$T^\alpha := S^*S + \alpha\mathcal{N}, \qquad g := S^*(u_f - u_d). \tag{2.11}$$

Then, the optimality condition can be written as $T^\alpha z + g = 0$ in $H^{1/2}(\Gamma)$. Here, S^* denotes the adjoint operator defined by $S^*v = [Pv]_{|\Gamma}$ where $P : (H^1(\Omega))' \to H^1(\Omega)$ is the solution operator of the boundary value problem

$$-\Delta w + w = v \quad \text{in } \Omega, \qquad \partial_n w = 0 \quad \text{on } \Gamma.$$

The operator T^α possesses the following properties:

Lemma 2.1. *The bilinear form defined by $\langle \cdot, T^\alpha \cdot\rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$ is continuous and $H^{-1/2}(\Gamma)$-elliptic, i. e. the inequalities*

$$\begin{aligned}\langle z, T^\alpha v\rangle_\Gamma &\le M\|z\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)},\\ \langle z, T^\alpha z\rangle_\Gamma &\ge \alpha\|z\|^2_{H^{-1/2}(\Gamma)},\end{aligned}$$

hold, for all $z, v \in H^{-1/2}(\Gamma)$ with some constant $M > 0$.

Proof. The continuity follows directly from the definition of T^α, the continuity of $\mathcal{N}$ and S, and the norm equivalence $\|Sv\|_{H^1(\Omega)} = \|v\|_{H^{-1/2}(\Gamma)}$. Then we get

$$\begin{aligned}\langle z, T^\alpha v\rangle_\Gamma &= (Sz, Sv) + \alpha \langle z, \mathcal{N} v\rangle_\Gamma \\ &\le c\left(\|Sz\|_{H^1(\Omega)}\|Sv\|_{H^1(\Omega)} + \|z\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)}\right) \\ &\le c\|z\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

To show the $H^{-1/2}(\Gamma)$-ellipticity we express the $H^{-1/2}(\Gamma)$-norm by the representation (2.3) which leads to

$$\begin{aligned}\langle z, T^\alpha z\rangle_\Gamma &= (Sz, Sz) + \alpha \langle z, \mathcal{N} z\rangle_\Gamma \\ &= \|u_z\|^2_{L^2(\Omega)} + \alpha\|z\|^2_{H^{-1/2}(\Gamma)} \ge \alpha\|z\|^2_{H^{-1/2}(\Gamma)}.\end{aligned}$$

□

In order to find a representation of the optimality condition which does not involve the operators S and S^* explicitly, we introduce the adjoint state $p := P(Sz + u_f - u_d)$ which may be written as the solution of the adjoint equation

$$-\Delta p + p = u - u_d \quad \text{in } \Omega, \qquad \partial_n p = 0 \quad \text{on } \Gamma.$$

Due to the representation $p = P\,(u - u_d)$ the optimality condition (2.10) can be written in the following form:

Theorem 2.2. *The tuple $(u, z) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ solves the model problem (2.6), if and only if there exists some $p \in H^1(\Omega)$ such that*

$$\begin{aligned} a(u_z, v) - \langle z, v\rangle_\Gamma &= 0 && \forall v \in H^1(\Omega), \\ a(p, v) - (u_z, v) &= (u_f - u_d, v) && \forall v \in H^1(\Omega), \\ \langle w, \alpha u_z + p\rangle_\Gamma &= 0 && \forall w \in H^{-1/2}(\Gamma). \end{aligned} \tag{2.12}$$

Note, that we already used the decomposition $u = u_z + u_f$ and $\mathcal{N} z = u_{z|\Gamma}$. The optimality condition does not depend on the control explicitly.

2.3 Regularity in weighted Sobolev spaces

This section is devoted to regularity results for the solution of problem (2.12). We will first give an overview on regularity results for the solution of the boundary value problem

$$\begin{aligned} -\Delta y + y &= f \quad \text{in } \Omega, \\ \partial_n y &= g \quad \text{on } \Gamma, \end{aligned} \tag{2.13}$$

in classical Sobolev spaces, and introduce weighted Sobolev spaces afterwards which allow a better description of the occurring corner singularities.

We define the singular exponents $\lambda_{j,m} := m\pi/\omega_j$ for $j \in \mathcal{C}$ and $m \in \mathbb{N}$, and introduce local polar coordinates (r_j, φ_j) centered at $x^{(j)}$ in such a way that $\varphi_j = 0$ and $\varphi_j = \omega_j$ coincide with the two edges Γ^j and Γ^{j-1} which meet in $x^{(j)}$. Then, it is well-known [9, Section 2.7], that the solution of (2.13) can be decomposed into

$$y(x) = y_R(x) + \sum_{j=1}^{d} \sum_{\substack{m \in \mathbb{N} \\ \lambda_{j,m} < 2-2/q}} c_{j,m}\, \eta_j(r_j)\, r_j^{\lambda_{j,m}} \cos(\lambda_{j,m}\varphi_j), \tag{2.14}$$

with a regular part $y_R \in W^{2,q}(\Omega)$. This decomposition is valid for arbitrary $q \in [1, \infty)$ satisfying $2-2/q \neq \lambda_{j,m}$ for all $j \in \mathcal{C}$ and $m \in \mathbb{N}$. Here, $c_{j,m} \in \mathbb{R}$ are constants (the so-called stress-intensity factors) and η_j smooth cut-off functions with $\eta_j \equiv 1$ in a neighborhood of the corner $x^{(j)}$. Since the singular functions for $m = 1$ are the dominating ones, we write $\lambda_j := \lambda_{j,1}$ and abbreviate the most restrictive one with $\lambda := \min_j \lambda_j$. It is easy to show that the singular parts vanish for the choice

$$q \in \begin{cases} [1, 2/(2-\lambda)), & \text{if } \lambda < 2, \\ [1, \infty), & \text{otherwise,} \end{cases} \tag{2.15}$$

and only in this case the classical shift theorem holds.

In the following lemma a consequence of the regularity results in classical Sobolev spaces is presented.

Lemma 2.3. *Let $f, u_d \in L^q(\Omega)$ for some q satisfying (2.15). Then, the solution of the optimality system (2.12) possesses the regularity*

$$z \in W^{1-1/q,q}(\Gamma_j),\ \forall j \in \mathcal{C}, \quad u \in W^{2,q}(\Omega), \quad p \in W^{2,q}(\Omega).$$

Proof. The Lax-Milgram lemma guarantees a unique solution $u \in H^1(\Omega)$ and hence $u \in L^q(\Omega)$ for arbitrary $q \in [1, \infty)$. Under the assumption that q satisfies (2.15) this implies $p \in W^{2,q}(\Omega)$. Consider the decomposition $u_z = u_0 - \alpha^{-1}p$ with $u_0 \in H_0^1(\Omega)$ solving the equation $-\Delta u_0 + u_0 = \alpha^{-1}(-\Delta p + p) \in L^q(\Omega)$. Standard results then imply $u_0,\ u_z \in W^{2,q}(\Omega)$. Moreover, $u_f \in W^{2,q}(\Omega)$ follows in case of $f \in L^q(\Omega)$ and we thus have $u = u_z + u_f \in W^{2,q}(\Omega)$. By a standard trace theorem we obtain that $z \in W^{1-1/q,q}(\Gamma_j)$ for all $j \in \mathcal{C}$. □

In the numerical experiments we observe that the control exhibits a similar behavior to the optimal control of a Dirichlet control problem with $L^2(\Gamma)$-regularization (see e.g. [14, 20]). More precisely, the control is drawn down to zero at convex corners and tends to ∞ or $-\infty$ at concave corners. In the following we will study this behavior in detail. Let (r, φ) denote polar coordinates centered at some corner $x^{(j)}$ and let B be a vicinity of $x^{(j)}$ containing no other corners. Since p is the solution of a Neumann problem it admits a decomposition as in (2.14), namely

$$p(x) = p_R(x) + cr^{\lambda}\cos(\lambda\varphi), \qquad \text{for } x \in B, \qquad \lambda = \frac{\pi}{\omega_j},$$

with a regular part p_R in $H^2(B)$. Note that we omitted the cut-off function η as introduced in (2.14) which is possible due to local considerations. Further singular terms with exponents $\lambda_k := k\pi/\omega_j$ for $k \geq 2$ are neglected since the corresponding singular functions belong to $H^2(B)$. Due to the homogeneous Neumann conditions we have

$$0 = \partial_n p = \partial_n p_R \pm c\lambda r^{\lambda-1} \sin(\lambda\varphi). \tag{2.16}$$

Since $\sin(\lambda\varphi) = 0$ for $\varphi = 0$ and $\varphi = \omega_j$ we have $\partial_n p_R = 0$. Due to the optimality condition (2.10) we may now write the state in terms of $u_z = u_0 - \alpha^{-1}p$ with some u_0 which vanishes on the boundary. Then, the equation (2.1) can be rewritten in the form

$$-\Delta u_0 + u_0 = \frac{1}{\alpha}(-\Delta p + p) \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{on } \Gamma.$$

Due to the Dirichlet boundary conditions we can decompose u_0 into a regular part $u_{0,R} \in W^{2,q}(B)$ with $q \in (2, (1-\lambda)^{-1})$, and a singular part:

$$u_0(x) = u_{0,R}(x) + cr^\lambda \sin(\lambda\varphi), \qquad x \in B.$$

Exploiting this decomposition and $z = \partial_n u_z = \partial_n u_0$ we obtain by some calculations

$$\begin{aligned} z|_{\varphi=0} &= \partial_n u_{0,R} - c\lambda\, r^{\lambda-1} \cos(0), \\ z|_{\varphi=\omega_j} &= \partial_n u_{0,R} + c\lambda\, r^{\lambda-1} \cos(\lambda\omega_j), \end{aligned}$$

and consequently, using the fact that $\lambda = \pi/\omega_j$,

$$z(x) = \partial_n u_{0,R}(x) - c\lambda r^{\lambda-1}, \qquad x \in \partial B \cap \Gamma. \tag{2.17}$$

The assumption $q > 2$ implies differentiability of the of the state since $u_{0,R} \in W^{2,2+\varepsilon} \hookrightarrow C^1(B)$ and hence the normal derivative is piecewise continuous, i. e. $\partial_n u_{0,R} \in C(\Gamma_j \cap \partial B)$ for $j \in \mathcal{C}$. Due to $u_{0,R} \equiv 0$ on Γ the tangential derivatives on the boundary also vanish and since the normal vector in a corner can be represented as linear combination of the tangential vectors, this implies that

$$\lim_{r\to 0} \partial_n u_{0,R}(r,\varphi) = 0 \qquad \text{for } \varphi \in \{0, \omega_j\}.$$

However, the term $\lambda r^{\lambda-1}$ in (2.17) could either grow unboundedly or could tend to zero, which depends on λ. If $x^{(j)}$ is a reentrant corner we have $\lambda < 1$ and in case of a convex corner $\lambda > 1$. Consequently, there holds

$$\lim_{r\to 0} z(r,\varphi) \to \begin{cases} 0, & \text{if } \omega_j < \pi, \\ \pm\infty, & \text{if } \omega_j > \pi, \end{cases} \quad \text{for } \varphi \in \{0, \omega_j\}.$$

Note that in case of $\omega_j > \pi$ the control tends either to $+\infty$ on both legs, or to $-\infty$, but the case that it tends to $+\infty$ on the one leg and to $-\infty$ on the other one can never occur.

Let us now discuss some improved regularity results. Weighted Sobolev spaces provide a suitable framework to prove sharp finite element error estimates in the presence of singularities.

We denote by $\alpha \in \mathbb{N}_0^2$ a multi-index and write $D^\alpha v = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} v$ as well as $|\alpha| = \alpha_1 + \alpha_2$. Let $\{U_j\}_{j\in\mathcal{C}}$ denote a covering of Ω where U_j contains only the corner $x^{(j)}$ and no other ones. Moreover, we use the notation $r_j(x) := |x - x^{(j)}|$. The weighted space $W^{k,q}_{\vec\beta}(\Omega)$ with $k \in \mathbb{N}_0$, $q \in [1,\infty]$ and a weight vector $\vec\beta \in \mathbb{R}^d$ is defined as the closure of the set $C^\infty(\overline{\Omega}\backslash\{x^{(j)} \colon j \in \mathcal{C}\})$ with respect to the norm

$$\|v\|_{W^{k,q}_{\vec\beta}(\Omega)} := \begin{cases} \left(\displaystyle\sum_{|\alpha|\le k}\sum_{j=1}^{d}\int_{U_j\cap\Omega} r_j^{q\beta_j}|D^\alpha v|^q\right)^{1/q}, & \text{if } q \in [1,\infty), \\ \displaystyle\sum_{|\alpha|\le k}\max_{j\in\mathcal{C}}\operatorname*{ess\,sup}_{x\in U_j\cap\Omega} r_j^{\beta_j}|D^\alpha v|, & \text{if } q = \infty. \end{cases} \tag{2.18}$$

Note, that this definition is similar to the ones from [15, 18] where the presence of only one singular point was assumed. As usual we denote the related semi-norms by $|\cdot|_{W^{k,q}_{\vec\beta}(\Omega)}$ whose only difference to (2.18) is, that the first sum is performed only over all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = k$. The corresponding trace space $W^{k-1/q,q}_{\vec\beta}(\Gamma)$ is the space of functions with finite norm

$$\|v\|_{W^{k-1/q,q}_{\vec\beta}(\Gamma)} := \inf\left\{\|u\|_{W^{k,q}_{\vec\beta}(\Omega)} \colon u|_{\Gamma_j} = v \text{ on } \Gamma_j \text{ for all } j\in\mathcal{C}\right\}. \tag{2.19}$$

Let us first summarize some regularity results in weighted Sobolev spaces. The proof of the following theorem can be adapted from the one of Theorem 8.1.7 in [16], where general polyhedra in 3D were considered. Neglecting the polyhedral corners in this proof leads to:

Theorem 2.4. *Let be given some functions $f \in W^{k-2,2}_{\vec\alpha}(\Omega)$ and $g \in W^{k-3/2,2}_{\vec\alpha}(\Gamma)$ for $k = 2$ or $k = 3$. Then, the solution y of (2.13) is contained in $W^{k,2}_{\vec\alpha}(\Omega)$ if the weights satisfy*

$$\begin{aligned} k-1-\lambda_j < \alpha_j < k-1 \qquad & \text{if } k-1-\lambda_j \ge 0, \\ \alpha_j = 0 \qquad & \text{if } k-1-\lambda_j < 0, \end{aligned} \tag{2.20}$$

for all $j = 1,\ldots,d$. Furthermore, the following a priori *estimate holds:*

$$\|y\|_{W^{k,2}_{\vec\alpha}(\Omega)} \le c\left[\|f\|_{W^{k-2,2}_{\vec\alpha}(\Omega)} + \|g\|_{W^{k-3/2,2}_{\vec\alpha}(\Gamma)}\right].$$

Another important regularity result in weighted $W^{2,\infty}(\Omega)$-spaces is proven in [1]:

Theorem 2.5. *Assume that $g \equiv 0$ and $f \in C^{0,\sigma}(\bar\Omega)$ for some $\sigma \in (0,1)$. Then, the solution y of (2.13) belongs to $W^{2,\infty}_{\vec\beta}(\Omega)$ with a weight vector having components satisfying*

$$\begin{aligned} 2-\lambda_j < \beta_j < 2 \qquad & \text{if } 2-\lambda_j \ge 0, \\ \beta_j = 0 \qquad & \text{if } 2-\lambda_j < 0, \end{aligned}$$

for all $j \in \mathcal{C}$. Furthermore, the a priori *estimate $\|y\|_{W^{2,\infty}_{\vec\beta}(\Omega)} \le \|f\|_{C^{0,\sigma}(\bar\Omega)}$ holds.*

In the following corollary we will transfer these results to the optimal control problem (2.6).

Corollary 2.6. *Let $f, u_d \in L^2(\Omega)$ and let be given a weight vector $\vec{\alpha} \in \mathbb{R}^d$ satisfying*

$$\begin{aligned} k-1-\lambda_j < \alpha_j < k-1 & \qquad \text{if } \lambda_j \le k-1, \\ \alpha_j = 0 & \qquad \text{if } \lambda_j > k-1, \end{aligned}$$

for all $j \in \mathcal{C}$ and $k = 2, 3$. Then the solution of the problem (2.6) possesses the regularity

$$(z, u, p) \in W^{k-3/2,2}_{\vec{\alpha}}(\Gamma) \times W^{k,2}_{\vec{\alpha}}(\Omega) \times W^{k,2}_{\vec{\alpha}}(\Omega)$$

Moreover, if $f, u_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$, and if the vector $\vec{\beta} \in \mathbb{R}^d$ satisfies

$$\begin{aligned} 2-\lambda_j < \beta_j < 2 & \qquad \text{if } \lambda_j \le 2 \\ \beta_j = 0 & \qquad \text{if } \lambda_j > 2, \end{aligned}$$

then the solution of (2.6) satisfies

$$(z, u, p) \in W^{1,\infty}_{\vec{\beta}}(\Gamma) \times W^{2,\infty}_{\vec{\beta}}(\Omega) \times W^{2,\infty}_{\vec{\beta}}(\Omega).$$

Additionally, if the vector $\vec{\gamma} \in \mathbb{R}^d$ satisfies

$$\begin{aligned} 3/2-\lambda_j < \gamma_j < 3/2, & \qquad \text{if } \lambda_j \le 3/2, \\ \gamma_j = 0 & \qquad \text{if } \lambda_j > 3/2, \end{aligned}$$

then, we have $z \in W^{1,2}_{\vec{\gamma}}(\Gamma)$.

Proof. We start again with $u \in H^{k-2}(\Omega)$ for $k = 2, 3$. Theorem 2.4 implies now that the adjoint state belongs to $W^{k,2}_{\vec{\alpha}}(\Omega)$ if $\vec{\alpha}$ satisfies (2.20). As in Lemma 2.3 this is transferred to u_z, and the regularity $u_f \in W^{k,2}_{\vec{\alpha}}(\Omega)$ follows from $f \in H^1(\Omega) \hookrightarrow W^{1,2}_{\vec{\alpha}}(\Omega)$. By a trace theorem the regularity of z can be concluded.

The $W^{2,\infty}_{\vec{\beta}}(\Omega)$-regularity of p follows from Theorem 2.5 and $u \in W^{2,q}(\Omega) \hookrightarrow C^{0,\sigma}(\bar{\Omega})$ which holds due to $\sigma \in (0, 1/2)$ and $q \in (4/3, 2/(2-\lambda)) \neq \emptyset$ (see the regularity result of Lemma 2.3). Due to the optimality condition from Theorem 2.2 we can express the state in the form $u_z = -\alpha^{-1} p + u_0$ with some u_0 satisfying the boundary value problem

$$-\Delta u_0 + u_0 = \frac{1}{\alpha}(u - u_d) \quad \text{in } \Omega, \qquad u_0 = 0 \quad \text{on } \Gamma.$$

This is a homogeneous Dirichlet problem with right-hand side in $C^{0,\sigma}(\Omega)$. We may thus apply a regularity result from [2, Theorem 2.2] and obtain $u_0 \in V^{2,\infty}_{\vec{\beta}}(\Omega)$ which is a weighted Sobolev space with homogeneous weights $r_j^{\beta_j - k + |\alpha|}$ but due to $k - |\alpha| > 0$ the embedding $V^{2,\infty}_{\vec{\beta}}(\Omega) \hookrightarrow W^{2,\infty}_{\vec{\beta}}(\Omega)$ follows. Consequently, we get $u_z \in W^{2,\infty}_{\vec{\beta}}(\Omega)$. The assumption $f \in C^{0,\sigma}(\Omega)$ and Theorem 2.5 yield also that $u_f \in W^{2,\infty}_{\vec{\beta}}(\Omega)$, which leads altogether to $u \in W^{2,\infty}_{\vec{\beta}}(\Omega)$. Furthermore, we have $\nabla u \in W^{1,\infty}_{\vec{\beta}}(\Omega)$ and thus $z \in W^{1,\infty}_{\vec{\beta}}(\Gamma)$. With the Hölder inequality we then obtain $W^{1,\infty}_{\vec{\beta}}(\Gamma) \hookrightarrow W^{1,2}_{\vec{\gamma}}(\Gamma)$ and the last assertion follows. $\square$

3 The discrete unconstrained optimal control problem

3.1 Discretization and general convergence results

In this section we deal with a conforming finite element discretization of the optimality system (2.12). Let us introduce some notation. A family of admissible and quasi-uniform finite element triangulations $\mathcal{T}_h$ with mesh size $h > 0$ and nodes $\{x^i\}_{i=1}^{N_\Omega}$ is considered. We approximate the state and adjoint state variable with continuous and piecewise linear functions, i. e. we search u_h and p_h in the finite-dimensional subspace

$$U_h := \left\{ v_h \in C(\bar{\Omega}) \colon v_h|_T \in \mathcal{P}_1 \quad \forall T \in \mathcal{T}_h \right\}. \tag{3.1}$$

We further search an approximation of the control z_h in the finite-dimensional space $Z_h \subset H^{-1/2}(\Gamma)$. Since multiple choices of Z_h are possible, we want to keep the analysis here as general as possible. In Section 3.3 two choices are investigated in detail: an approximation by piecewise constant functions on a coarser boundary mesh or the dual mesh related to the boundary mesh of $\mathcal{T}_h$, and by piecewise linear and continuous functions. Let be given two bases of U_h and Z_h by

$$U_h = \operatorname{span}\left\{\varphi^i\right\}_{i=1}^{N_\Omega}, \qquad Z_h = \operatorname{span}\left\{\psi^j\right\}_{j=1}^{N_\Gamma}.$$

The discretized optimality system of Theorem 2.2 reads:
Find $u_{z,h}, p_h \in U_h, z_h \in Z_h$ such that

$$\begin{aligned}
a(u_{z,h}, v_h) - \langle z_h, v_h\rangle_\Gamma &= 0 && \forall v_h \in U_h,\\
a(p_h, v_h) - (u_{z,h}, v_h) &= (u_{f,h} - u_d, v_h) && \forall v_h \in U_h,\\
\langle w_h, \alpha u_{z,h} + p_h\rangle_\Gamma &= 0 && \forall w_h \in Z_h,
\end{aligned} \tag{3.2}$$

where $u_{f,h} \in Z_h$ can be computed in advance by

$$a(u_{f,h}, v_h) = \langle f, v_h\rangle_\Omega \qquad \forall v_h \in U_h.$$

The finite-dimensional system (3.2) is just the optimality system of the optimization problem

$$\begin{aligned}
&\min_{z_h \in Z_h} \frac{1}{2}\|u_{z,h} + u_{f,h} - u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\left\langle z_h, \mathcal{N}_h z_h\right\rangle_\Gamma\\
&\text{s. t.} \qquad a(u_{z,h}, v_h) = \langle z_h, v_h\rangle_\Gamma \quad \forall v_h \in U_h,
\end{aligned}$$

where $\mathcal{N}_h z_h := [u_{z,h}]_{|\Gamma}$.

We may represent the unknown functions in terms of vectors by using the isomorphisms

$$\vec{z} \leftrightarrow z_h = \sum_{j=1}^{N_\Gamma} z_j \psi^j, \qquad \vec{u}_z \leftrightarrow u_{z,h} = \sum_{i=1}^{N_\Omega} u_{z,i}\varphi^i, \qquad \vec{p} \leftrightarrow p_h = \sum_{i=1}^{N_\Omega} p_i \varphi^i.$$

Let now A denote the standard finite element stiffness matrix related to the operator $-\Delta + I$, M the mass matrix and $\tilde{M} := (m_{ij}) \in \mathbb{R}^{N\times N_\Gamma}$ a transformation matrix, having entries

$$m_{ij} := \int_\Gamma \varphi^i \, \psi^j.$$

As a consequence the optimality system (3.2) reads in matrix-vector notation

$$\begin{pmatrix} 0 & A & \tilde{M} \\ A & -M & 0 \\ \tilde{M}^\top & \alpha\tilde{M}^\top & 0 \end{pmatrix} \begin{pmatrix} \vec{p} \\ \vec{u}_z \\ -\vec{z} \end{pmatrix} = \begin{pmatrix} 0 \\ \vec{f} \\ 0 \end{pmatrix}. \tag{3.3}$$

The vector $\vec{f}$ corresponds to the right-hand side of the adjoint equation and its components are defined by $f_i = \int_\Omega (u_{f,h} - u_d)\varphi^i$. Note, that the system (3.3) can be transformed into a symmetric one by adding α times the first row to the second one. Solving the system (3.3) leads to the approximate solution $(u_{z,h}, z_h, p_h)$. As we will see later, there exists a unique solution of (3.3) under an additional assumption.

Let us now introduce the finite-element solution operators of S and P, defined by

$$\begin{aligned} S_h : H^{-1/2}(\Gamma) \to U_h, &\qquad u_h = S_h z &&:\Longleftrightarrow &&a(u_h, v_h) = \langle z, v_h\rangle_\Gamma &&\forall v_h \in U_h,\\ P_h : L^2(\Omega) \to U_h, &\qquad p_h = P_h u &&:\Longleftrightarrow &&a(p_h, v_h) = (u, v_h) &&\forall v_h \in U_h. \end{aligned}$$

The adjoint operator to S_h is then defined by $S_h^* u := [P_h u]_{|\Gamma}$. The discrete *Steklov-Poincaré* operator can be written in terms of $\mathcal{N}_h z = [S_h z]_{|\Gamma}$. Similar to (2.10) we may now also write the system (3.2) in the compact form

$$0 = \langle w_h, T_h^\alpha z_h + g_h\rangle_\Gamma \qquad \forall w_h \in Z_h \tag{3.4}$$

with

$$T_h^\alpha := S_h^* S_h + \alpha\mathcal{N}_h \quad \text{and} \quad g_h := S_h^*(u_{f,h} - u_d).$$

The operator T_h^α is an approximation of the operator T^α defined in (2.11). As the properties of T^α summarized in Lemma 2.1 cannot be directly transferred to T_h^α we need an additional condition:

Assumption A *The spaces Z_h and U_h satisfy the* Ladyshenskaya-Babuška-Brezzi *condition, i. e. some $c > 0$ exists such that*

$$\|z_h\|_{H^{-1/2}(\Gamma)} \le c \sup_{v_h \in U_h} \frac{\langle z_h, v_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \quad \textit{for all } z_h \in Z_h.$$

This is a natural assumption for mixed finite element discretizations. As a consequence the discrete counterpart to Lemma 2.1 follows:

Lemma 3.1. *Let Assumption A be satisfied. Then, the bilinear form*

$$\langle \cdot, T_h^\alpha \cdot\rangle_\Gamma : H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to \mathbb{R}$$

is continuous and Z_h-elliptic. Moreover, the equation (3.4) possesses a unique solution $z_h \in Z_h$.

Proof. The continuity can be proven in analogy to Lemma 2.1 since the stability properties of S and S^* also hold for their discrete versions. In order to show the Z_h-ellipticity we take into account Assumption A which leads to

$$\|z_h\|_{H^{-1/2}(\Gamma)} \le c \sup_{v_h \in U_h} \frac{\langle z_h, v_h\rangle_\Gamma}{\|v_h\|_{H^1(\Omega)}} \le c \sup_{v_h \in U_h} \frac{a(S_h z_h, v_h)}{\|v_h\|_{H^1(\Omega)}} \le c\|S_h z_h\|_{H^1(\Omega)}.$$

The Cauchy-Schwarz-inequality, the U_h-ellipticity of the bilinear form $a(\cdot,\cdot)$ and the definition of S_h and $\mathcal{N}_h$ imply

$$\begin{aligned}\|z_h\|^2_{H^{-1/2}(\Gamma)} &\le c\|S_h z_h\|^2_{H^1(\Omega)} \le c\, a(S_h z_h, S_h z_h) \\ &= c\,\langle z_h, \mathcal{N}_h z_h\rangle_\Gamma \le c\,\langle z_h, T_h^\alpha z_h\rangle_\Gamma\,.\end{aligned}$$

The last step follows from $(S_h z_h, S_h z_h) \ge 0$. The existence of a unique solution of (3.4) follows then from the Lax-Milgram lemma. □

In the remainder of this section a general error estimate for the optimal control problem will be proven. Therefore, the overall error between z and z_h is decomposed into separate terms which will be discussed in Sections 3.2 and 3.3 separately.

Theorem 3.2. *Let Assumption A be satisfied. For the solutions $z \in H^{-1/2}(\Gamma)$ and $z_h \in Z_h$ of (2.10) and (3.4), respectively, the following estimate holds:*

$$\begin{aligned}&\|z - z_h\|_{H^{-1/2}(\Gamma)} \\ &\le c\Big[\|(S - S_h)z\|_{L^2(\Omega)} + \|(S^* - S_h^*)(u - u_d)\|_{H^{1/2}(\Gamma)} + \alpha\|(\mathcal{N} - \mathcal{N}_h)z\|_{H^{1/2}(\Gamma)} \\ &\quad + \|u_f - u_{f,h}\|_{L^2(\Omega)} + \inf_{\chi \in Z_h} \|z - \chi\|_{H^{-1/2}(\Gamma)}\Big].\end{aligned} \tag{3.5}$$

Proof. Let $\tilde z_h \in Z_h$ denote the unique solution of

$$\langle v_h, T^\alpha \tilde z_h + g\rangle_\Gamma = 0 \qquad \forall v_h \in Z_h. \tag{3.6}$$

Since $Z_h \subset H^{-1/2}(\Gamma)$ we get with (2.10) the orthogonality

$$\langle v_h, T^\alpha(z - \tilde z_h)\rangle_\Gamma = 0 \qquad \forall v_h \in Z_h. \tag{3.7}$$

As a consequence of the $H^{-1/2}(\Gamma)$-ellipticity and boundedness of $\langle\cdot, T^\alpha\cdot\rangle_\Gamma$ (see Lemma 2.1) and equation (3.7), the Cea-Lemma leads to

$$\|z - \tilde z_h\|_{H^{-1/2}(\Gamma)} \le c \inf_{\chi \in Z_h} \|z - \chi\|_{H^{-1/2}(\Gamma)}. \tag{3.8}$$

Next, an estimate for $w_h := \tilde z_h - z_h$ is derived. We may now apply the Z_h-ellipticity of T_h^α, equation (3.4), (3.6) such as (2.10) which leads to

$$\begin{aligned}\|w_h\|^2_{H^{-1/2}(\Gamma)} &\le \langle w_h, T_h^\alpha(\tilde z_h - z_h)\rangle_\Gamma = \langle w_h, T_h^\alpha \tilde z_h + g_h\rangle_\Gamma \\ &= \langle w_h, (T_h^\alpha - T^\alpha)\,\tilde z_h - g + g_h\rangle_\Gamma \\ &= \langle w_h, (T_h^\alpha - T^\alpha)\,(\tilde z_h - z)\rangle_\Gamma + \langle w_h, (T_h^\alpha - T^\alpha)\,z - g + g_h\rangle_\Gamma\end{aligned} \tag{3.9}$$

The boundedness of T^α and T_h^α together with (3.8) imply

$$\langle w_h, (T_h^\alpha - T^\alpha)\,(\tilde{z}_h - z)\rangle_\Gamma \le c\,\|w_h\|_{H^{-1/2}(\Gamma)}\|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)} \tag{3.10}$$

and we may apply (3.8) again.

Exploiting the definition of T^α and T_h^α yields for the second term in (3.9)

$$\begin{aligned}
&\langle w_h, (T_h^\alpha - T^\alpha)\,z + g - g_h\rangle_\Gamma \\
&= \big\langle w_h, S_h^*(S_h - S)z + (S_h^* - S^*)Sz + \alpha(\mathcal{N}_h - \mathcal{N})z \\
&\quad + (S^* - S_h^*)(u_f - u_d) + S_h^*(u_f - u_{f,h})\big\rangle_\Gamma \\
&\le c\|w_h\|_{H^{-1/2}(\Gamma)}\Big[\|S_h^*(S_h - S)z\|_{H^{1/2}(\Gamma)} + \|(S_h^* - S^*)(u - u_d)\|_{H^{1/2}(\Gamma)} \\
&\quad + \alpha\|(\mathcal{N}_h - \mathcal{N})z\|_{H^{1/2}(\Gamma)} + \|S_h^*(u_{f,h} - u_f)\|_{H^{1/2}(\Gamma)}\Big].
\end{aligned} \tag{3.11}$$

Note, that the operator S_h^* is bounded from $L^2(\Omega)$ to $H^{1/2}(\Omega)$ and thus one can simplify

$$\|S_h^*(S_h - S)z\|_{H^{1/2}(\Gamma)} \le c\|(S_h - S)z\|_{L^2(\Omega)}. \tag{3.12}$$

Inserting the estimates (3.10), (3.11) and (3.12) into (3.9) and dividing by $\|w_h\|_{H^{-1/2}(\Gamma)}$ leads to

$$\begin{aligned}
\|w_h\|_{H^{-1/2}(\Gamma)} \le c\Big[&\|(S - S_h)z\|_{L^2(\Omega)} + \|(S^* - S_h^*)(u - u_d)\|_{H^{1/2}(\Gamma)} \\
&+\alpha\|(\mathcal{N} - \mathcal{N}_h)z\|_{H^{1/2}(\Gamma)} + \inf_{\chi\in Z_h}\|z - \chi\|_{H^{-1/2}(\Gamma)} + \|u_f - u_{f,h}\|_{L^2(\Omega)}\Big].
\end{aligned}$$

This estimate, such as (3.8), together with the triangle inequality

$$\|z - z_h\|_{H^{-1/2}(\Gamma)} \le \|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)} + \|\tilde{z}_h - z_h\|_{H^{-1/2}(\Gamma)}$$

leads to the assertion. □

3.2 Error estimates for the state variable

It remains to prove estimates for the terms on the right-hand side of (3.5). Therefore, we first collect some known convergence results and transfer these to our setting afterwards.

Theorem 3.3. *Let be given some functions f, g and let y denote the solution of the boundary value problem*

$$-\Delta y + y = f \quad \text{in } \Omega, \qquad \partial_n y = g \quad \text{on } \Gamma,$$

and $y_h \in U_h$ its finite-element approximation. Assume that weight vectors $\vec{\alpha}$ and $\vec{\beta}$ are given satisfying

$$\begin{aligned}
&\alpha_j = 1 - \lambda_j + \varepsilon && \text{if } \lambda_j \le 1, && \beta_j = 2 - \lambda_j + \varepsilon && \text{if } \lambda_j \le 2, \\
&\alpha_j = 0 && \text{if } \lambda_j > 1, && \beta_j = 0 && \text{if } \lambda_j > 2,
\end{aligned}$$

for all $j \in \mathcal{C}$. *Then, for* $l = 0, 1$, *the error estimates*

$$
\begin{aligned}
\|y - y_h\|_{H^l(\Omega)} &\le c\, h^{(2-l)\min\{1,\lambda-\varepsilon\}} \|y\|_{W^{2,2}_{\vec{\alpha}}(\Omega)}, && \textit{if } y \in W^{2,2}_{\vec{\alpha}}(\Omega),, && (3.13)\\
\|y - y_h\|_{L^2(\Gamma)} &\le c\, h^{\min\{2,1/2+\lambda\}-\varepsilon} \|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)}, && \textit{if } y \in W^{2,\infty}_{\vec{\beta}}(\Omega), && (3.14)\\
\|y - y_h\|_{H^{1/2}(\Gamma)} &\le c\, h^{\min\{3/2,\lambda\}-\varepsilon} \|y\|_{W^{2,\infty}_{\vec{\beta}}(\Omega)}, && \textit{if } y \in W^{2,\infty}_{\vec{\beta}}(\Omega), && (3.15)
\end{aligned}
$$

hold, for arbitrary $\varepsilon > 0$.

Proof. To prove the first estimate we can apply standard techniques: one can show an estimate in the $H^1(\Omega)$-norm using Cea's Lemma and the local interpolation error estimate on an element $T \subset U_j$

$$\|y - I_h y\|_{H^1(T)} \le c h^{1-\alpha_j} |y|_{W^{2,2}_{\alpha_j}(T)}.$$

Summing up all elements $T \in \mathcal{T}_h$ and inserting the assumptions upon the weights yields a finite element error estimate in $H^1(\Omega)$-norm, namely

$$
\begin{aligned}
\|y - y_h\|_{H^1(\Omega)} &\le c\|y - I_h y\|_{H^1(\Omega)}\\
&\le c h^{1-\max_{j\in\mathcal{C}}\alpha_j} |y|_{W^{2,2}_{\vec{\alpha}}(\Omega)} \le c h^{\min\{1,\lambda\}-\varepsilon} |y|_{W^{2,2}_{\vec{\alpha}}(\Omega)}. \qquad (3.16)
\end{aligned}
$$

The Aubin-Nitsche method leads to the stated estimate in $L^2(\Omega)$. For a detailed proof using Besov spaces we refer to [4].

A proof of the second estimate can be found in [21, Corollary 3.49]. For the third estimate we introduce the nodal interpolant $I_h y$ of y as intermediate function and apply an inverse inequality which follows from standard inverse estimates in $H^1(\Gamma)$ with an interpolation argument. We obtain

$$\|y - y_h\|_{H^{1/2}(\Gamma)} \le \|y - I_h y\|_{H^{1/2}(\Gamma)} + h^{-1/2}\left[\|y - I_h y\|_{L^2(\Gamma)} + \|y - y_h\|_{L^2(\Gamma)}\right]. \qquad (3.17)$$

It remains to prove interpolation error estimates in $H^s(\Gamma)$-norm for $s = 0$ and $s = 1/2$. On an element $E \in \mathcal{E}_h$ with $E \subset \partial U_j$ we get from [21, Lemma 3.28] the local estimate

$$\|y - I_h y\|_{H^s(E)} \le c h^{2-s-\gamma_j} |y|_{W^{2,2}_{\gamma_j}(E)},$$

which holds for $s \in \{0, 1\}$ provided that $\gamma_j \in [0, 1)$. We choose the weight $\gamma_j = \max\{0, 3/2 - \lambda_j + \varepsilon\}$ with arbitrary $\varepsilon \in (0, \lambda - 1/2) (\neq \emptyset)$, and after summation over all $E \in \mathcal{E}_h$ we get the global estimate

$$\|y - I_h y\|_{H^s(\Gamma)} \le c\, h^{\min\{3/2,\lambda-\varepsilon\}+1/2-s} |y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}. \qquad (3.18)$$

By an interpolation argument between $s = 0$ and $s = 1$ we conclude the validity of this estimate also for $s = 1/2$. Furthermore, it was assumed that $y \in W^{2,\infty}_{\vec{\beta}}(\Omega)$. Due to the embedding $W^{0,\infty}_{\vec{\beta}}(\Gamma) \hookrightarrow W^{0,2}_{\vec{\gamma}}(\Gamma)$ which holds for $\beta_j < \gamma_j + 1/2$ and $\beta_j = 0$ if $\gamma_j = 0$, for all $j = 1, \ldots, d$, we get

$$|y|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \le c|y|_{W^{2,\infty}_{\vec{\beta}}(\Omega)}.$$

Inserting (3.18) and (3.14) into (3.17) leads to (3.15). □

We may now apply the finite element error estimates from the previous theorem to the terms on the right-hand side of estimate (3.5) in Theorem 3.2.

Corollary 3.4. *The following estimate holds:*

$$\begin{aligned}\|(S-S_h)z\|_{L^2(\Omega)} + \|(S^*-S_h^*)(u-u_d)\|_{H^{1/2}(\Gamma)}\\ +\alpha\|(\mathcal{N}_h-\mathcal{N})z\|_{H^{1/2}(\Gamma)} + \|u_f-u_{f,h}\|_{L^2(\Omega)} \le ch^{\min\{3/2,\lambda\}-\varepsilon}\end{aligned}$$

Proof. We show that Sz, $S^*(u-u_d)$, u_f and $\mathcal{N}z$ possess the required regularity such that Theorem 3.3 can be applied. Let $\vec{\alpha}$ and $\vec{\beta}$ be defined as in Corollary 2.6 or in Theorem 3.3, respectively. From Corollary 2.6 it is already known that

$$Sz = u \in W^{2,2}_{\vec{\alpha}}(\Omega), \qquad S^*(u-u_d) = p_{|\Gamma} \quad \text{and} \quad p \in W^{2,\infty}_{\vec{\beta}}(\Omega).$$

Lemma 2.4 yields $u_f \in W^{2,2}_{\vec{\alpha}}(\Omega)$ for $f \in L^2(\Omega)$. Furthermore, we exploit $\mathcal{N}z = u_{z|\Gamma}$ and $u_z \in W^{2,\infty}_{\vec{\beta}}(\Omega)$ (see, Corollary 2.6) which also implies the required regularity. □

3.3 Approximation and error estimates for the control variable

In the numerical experiments we only obtained a solution provided that Assumption A holds. The case of piecewise constant controls on the boundary mesh $\mathcal{E}_h$ of $\mathcal{T}_h$ is known to be not inf-sup stable. As a consequence, the solution of (3.3) exhibits oscillations due to the structure of the matrix $\tilde{M}$. An overview over possible pairs U_h and Z_h which satisfy Assumption A can be found in [24, Section 1.2]. We discuss several choices in the following two sections.

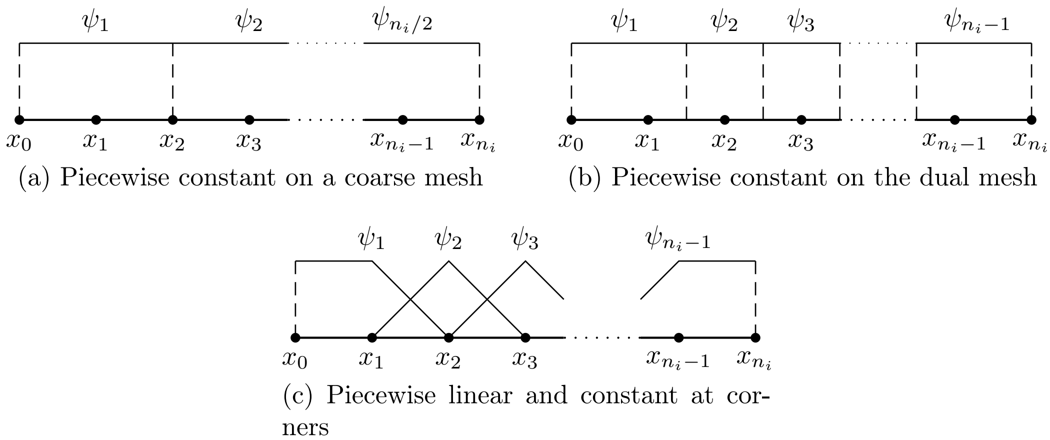

Figure 1: Possible choices for the discretization of the control

3.3.1 Piecewise constant controls on a coarse mesh

In this section we discuss an approximation of the control by piecewise constant functions on a boundary mesh $\mathcal{E}_H$ having mesh size $H > 0$. More precisely, the space Z_h is defined by

$$Z_h := \{v_h \in L^\infty(\Gamma) \colon v_{h|E} \in \mathcal{P}_0 \quad \forall E \in \mathcal{E}_H\}.$$

However, further assumptions are necessary to obtain the validity of Assumption A. Our analysis covers the following two possible choices:

a) The boundary mesh $\mathcal{E}_H$ is assumed to be coarser than the boundary mesh of $\mathcal{T}_h$, i. e. there holds $H/h \geq \gamma$ with some sufficiently large $\gamma > 0$. A proof of Assumption A can be found in [23, Section 11.3]. Certainly, it is not known how large γ has to be, but in the numerical experiments we observed, that $\gamma = 2$ is sufficient in our case. This setting occurs for instance when we refine a given initial mesh $k-1$ times globally to obtain $\mathcal{E}_H$, and refine uniformly once more to obtain $\mathcal{T}_h$ (see Figure 1a).

b) We can also choose $\mathcal{E}_H$ as the dual mesh of the boundary mesh induced by $\mathcal{T}_h$. Therefore, assume that the boundary edge Γ_i, $i \in \mathcal{C}$, coincides with the x-axis and that the boundary nodes of $\mathcal{T}_h$ on Γ_i have coordinates $x^{(i)} = x_0 < x_1 < \ldots < x_{n_i} = x^{(i+1)}$. Then, the elements of the dual mesh $\{E_k\}_{k=1}^{n_i-1}$ lying on Γ_i are defined by

$$\begin{aligned} E_k &:= \left(\frac{1}{2}(x_{k-1} + x_k), \frac{1}{2}(x_k + x_{k+1})\right), \quad \text{for } k = 2, \ldots, n_i - 2, \\ E_1 &:= \left(x_0, \frac{1}{2}(x_1 + x_2)\right), \quad E_{n_i-1} := \left(\frac{1}{2}(x_{n_i-2} + x_{n_i-1}), x_{n_i}\right), \end{aligned}$$

which is also illustrated in Figure 1b. A proof of Assumption A for this choice can be found in [24, Section 1.2]. Due to $H \sim h$ we do not distinguish between h and H in the following.

It remains to prove a best-approximation property of these spaces.

Lemma 3.5. *Let be given some $z \in W^{1,2}_{\vec{\gamma}}(\Gamma)$ with $\gamma_i = 3/2 - \lambda_i + \varepsilon$ if $\lambda_i \leq 3/2$, and $\gamma_i = 0$ otherwise. Then, the error estimate*

$$\inf_{\chi \in Z_h} \|z - \chi\|_{H^{-s}(\Gamma)} \leq c\, h^{\min\{1, \lambda - 1/2 - \varepsilon\} + s} |z|_{W^{1,2}_{\vec{\gamma}}(\Gamma)}.$$

holds for $s \in \{0, 1/2\}$.

Proof. Let $P_h^\partial : L^2(\Gamma) \to Z_h$ denote the $L^2(\Gamma)$-projection onto Z_h. Exploiting the definition of negative norms and the standard estimate

$$\|\varphi - P_h^\partial \varphi\|_{L^2(\Gamma)} \leq c h^{1/2} \|\varphi\|_{H^{1/2}(\Gamma)}$$

we obtain

$$\begin{aligned}\inf_{\chi\in Z_h}\|z-\chi\|_{H^{-1/2}(\Gamma)} &\le c\,\|z-P_h^\partial z\|_{H^{-1/2}(\Gamma)}\\ &= c\sup_{\varphi\in H^{1/2}(\Gamma)}\left\langle z-P_h^\partial z,\varphi\right\rangle_\Gamma/\|\varphi\|_{H^{1/2}(\Gamma)}\\ &= c\sup_{\varphi\in H^{1/2}(\Gamma)}(z-P_h^\partial z,\varphi-P_h^\partial\varphi)/\|\varphi\|_{H^{1/2}(\Gamma)}\\ &\le c\,h^{1/2}\,\|z-P_h^\partial z\|_{L^2(\Gamma)}. \end{aligned}\tag{3.19}$$

It remains to prove a local error estimate since the global projection coincides with the local projection on each $E\in\mathcal{E}_h$. Let p be an arbitrary constant and $\hat E$ denote the reference interval $(0,1)$. Due to the best-approximation property of P_h^∂ in $L^2(\Gamma)$, the embedding $W^{1,2}_\gamma(E)\hookrightarrow L^2(E)$ which holds for arbitrary $\gamma\le 1$ [16, Lemma 6.2.1], and a Deny-Lions type argument using the norm equivalence $\|\cdot\|_{W^{1,2}_\gamma(\hat E)}\sim|\cdot|_{W^{1,2}_\gamma(\hat E)}+|\int_E\cdot\,\mathrm{d}x|$ proved in [1, Lemma 2.2], we obtain for some $E\subset U_j$ with $r_E=0$ the estimate

$$\begin{aligned}\|z-P_h^\partial z\|_{L^2(E)} &\le \|z-p\|_{L^2(E)}\le c\,|E|^{1/2}\|\hat z-p\|_{L^2(\hat E)}\\ &\le c\,|E|^{1/2}\|\hat z-p\|_{W^{1,2}_{\gamma_j}(\hat E)}\le c\,|E|^{1/2}|\hat z|_{W^{1,2}_{\gamma_j}(\hat E)}\\ &\le c\,h^{1-\gamma_j}|z|_{W^{1,2}_{\gamma_j}(E)}. \end{aligned}\tag{3.20}$$

In case of $r_E>0$ we also arrive at (3.20) using the standard estimate

$$\|z-P_h^\partial z\|_{L^2(E)}\le ch|z|_{H^1(E)}$$

and the property $r_j(x)\ge r_E\ge h$ for $x\in E$ with $r_E>0$ which leads to $1\le h^{-\gamma_j}r_j(x)^{\gamma_j}$.

Summation over all $E\in\mathcal{E}_h$ yields

$$\|z-P_h^\partial z\|_{L^2(\Gamma)}\le c\,h^{1-\max_j\gamma_j}|z|_{W^{1,2}_{\vec\gamma}(\Gamma)}.$$

Inserting now $\gamma_j=\max\{0,3/2-\lambda_j+\varepsilon\}$ yields the assertion for $s=0$ and together with (3.19) we conclude the assertion for $s=1/2$. □

3.3.2 Piecewise linear controls

Let now $\mathcal{E}_h$ denote the boundary mesh induced by $\mathcal{T}_h$, i.e. for all $E\in\mathcal{E}_h$ there exists a $T\in\mathcal{T}_h$ such that $E=\partial T\cap\Gamma_i$ for some $i\in\mathcal{C}$. Without loss of generality assume that Γ_i, $i\in\mathcal{C}$, coincides with the x-axis. The boundary edge Γ_i is decomposed into boundary elements $E_k:=(x_{k-1},x_k)\in\mathcal{E}_h$ for $k=1,\ldots,n_i$ where $x^{(i)}=x_0<x_1<\ldots<x_{n_i}=x^{(i+1)}$. The discrete control space can be defined by

$$Z_h^i:=\{v_h\in C(\Gamma_i)\colon v_h|_{E_k}\in\mathcal{P}_1,\ k=2,\ldots,n_i-1,\ \text{and } v_h|_{E_1},v_h|_{E_{n_i}}\in\mathcal{P}_0\},$$

and

$$Z_h:=\{v_h\in L^\infty(\Gamma)\colon v_h|_{\Gamma_i}\in Z_h^i\text{ for all }i\in\mathcal{C}\}.\tag{3.21}$$

Note, that we allow discontinuities at corner points, and we require that the slope of functions in Z_h is zero on edges touching a corner. This property is necessary to ensure the stability condition in Assumption A and a proof can be found in [3], we refer also to [22]. In the following $\{\psi_k\}_{k=1}^{n_i-1}$ is the nodal basis of Z_h^i, i. e. $\psi_k(x_j) = \delta_{k,j}$ for all $j = 1, \ldots, n_i - 1$, which is illustrated in Figure 1c.

Now, we can show a best-approximation property of Z_h.

Lemma 3.6. *The results of Lemma 3.5 also hold for the choice* (3.21).

Proof. Let P_h^∂ denote the $L^2(\Gamma_i)$-projection onto Z_h^i, $i \in \mathcal{C}$. Analogous to the proof of Lemma 3.5 we can show that

$$\|z - P_h^\partial z\|_{H^{-1/2}(\Gamma_i)} \le c \sup_{\varphi \in H^{1/2}(\Gamma_i)} \|z - P_h^\partial z\|_{L^2(\Gamma_i)} \|\varphi - P_h^\partial \varphi\|_{L^2(\Gamma_i)} / \|\varphi\|_{H^{1/2}(\Gamma_i)}. \tag{3.22}$$

Since P_h^∂ is the best-approximation in $L^2(\Gamma_i)$ we replace P_h^∂ by an appropriate interpolation operator onto Z_h^i which is defined locally. Therefore, we introduce the operator $C_h^\partial \colon L^1(\Gamma) \to Z_h^i$ defined by

$$[C_h^\partial z](x) = \sum_{k=1}^{n_i-1} [\Pi_{\sigma_k} v](x_k) \psi_k(x), \qquad \sigma_k := E_k \text{ or } E_{k+1},$$

where Π_{σ_k} denotes the L^2-projection onto the constant functions on σ_k. This quasi-interpolation operator is similar to the operator introduced by Clément [7] and has the advantage that the stability property

$$\|C_h^\partial z\|_{L^2(E)} \le c \|z\|_{L^2(S_E)}, \quad \text{with} \quad S_{E_k} := \operatorname{int}(\overline{E}_k \cup \overline{\sigma}_{k-1} \cup \overline{\sigma}_k), \tag{3.23}$$

holds, which is not the case for the usual Lagrange interpolation operator.

For some $p \in \mathcal{P}_0$ we observe that $p = C_h^\partial p$. Using the triangle inequality and (3.23) we get

$$\|z - C_h^\partial z\|_{L^2(E_k)} \le c \|z - p\|_{L^2(S_{E_k})} \le c h^s |z|_{H^s(S_{E_k})}, \qquad s \in (0, 1], \tag{3.24}$$

for arbitrary $E \in \mathcal{E}_h$, $E \subset \Gamma_i$, where the last step follows from Theorem 4.2 (for $s = 1$) and Proposition 6.1 (for $s \in (0, 1)$) in [8]. An estimate in weighted Sobolev spaces can be deduced from (3.20) and we get

$$\|z - C_h^\partial z\|_{L^2(E_k)} \le c \|z - p\|_{L^2(S_{E_k})} \le c h^{1-\gamma_j} |z|_{W^{1,2}_{\gamma_j}(S_{E_k})}, \quad \text{if } E \in U_j. \tag{3.25}$$

From (3.24) for $s = 1/2$ and (3.25) we conclude the global estimates

$$\begin{aligned} \|z - C_h^\partial z\|_{L^2(\Gamma_i)} &\le c h^{1 - \max_{j \in \mathcal{C}} \gamma_j} |z|_{W^{1,2}_{\vec{\gamma}}(\Gamma_i)}, \\ \|\varphi - C_h^\partial \varphi\|_{L^2(\Gamma_i)} &\le c h^{1/2} |\varphi|_{H^{1/2}(\Gamma_i)}, \end{aligned}$$

where the first estimate yields the assertion for $s = 0$. The assertion for $s = -1/2$ follows after insertion into (3.22). ∎

3.4 Error estimates for the optimal control problem

Now we are in the position to formulate the main result of this paper. Inserting the results of Corollary 3.4 and Lemma 3.5 in case of a piecewise constant control approximation, or Lemma 3.6 in case of continuous and linear controls into Theorem 3.2 yields an error estimate for the control approximation in $H^{-1/2}(\Gamma)$-norm. It is also possible to extend this result to other norms and also to the state variable.

Theorem 3.7. *Let (u, z, p) and (u_h, z_h, p_h) denote the solution of (2.12) and (3.2), respectively. Let $\lambda := \min_j\{\pi/\omega_j\}$ be the singular exponent of the corner with the strongest singularity. Then, the following* a priori *error estimates hold:*

$$\|z - z_h\|_{H^{-1/2}(\Gamma)} \leq ch^{\min\{3/2,\lambda\}-\varepsilon}, \tag{3.26}$$

$$\|z - z_h\|_{L^2(\Gamma)} \leq ch^{\min\{1,\lambda-1/2\}-\varepsilon}, \tag{3.27}$$

$$\|u - u_h\|_{H^1(\Omega)} \leq c\, h^{\min\{1,\lambda\}-\varepsilon}. \tag{3.28}$$

Proof. To obtain the first estimate one has to combine Theorem 3.2, Corollary 3.4 and the Lemmas 3.5 or 3.6. Now, let P_h^∂ denote the $L^2(\Gamma)$-projection onto Z_h. To obtain an $L^2(\Gamma)$-estimate we apply the triangle inequality and the inverse estimate from [23, Lemma 10.4] and arrive at

$$\begin{aligned} &\|z - z_h\|_{L^2(\Gamma)} \\ &\leq c\|z - P_h^\partial z\|_{L^2(\Gamma)} + h^{-1/2}\left(\|z - P_h^\partial z\|_{H^{-1/2}(\Gamma)} + \|z - z_h\|_{H^{-1/2}(\Gamma)}\right). \end{aligned} \tag{3.29}$$

Furthermore, we apply Lemma 3.5 (for piecewise constant controls) or Lemma 3.6 (for piecewise linear controls that are continuous on each Γ_i) which leads to

$$\|z - P_h^\partial z\|_{L^2(\Gamma)} + h^{-1/2}\|z - P_h^\partial z\|_{H^{-1/2}(\Gamma)} \leq ch^{\min\{1,\lambda-1/2-\varepsilon\}}|z|_{W^{1,2}_{\vec{\gamma}}(\Gamma)} \tag{3.30}$$

with $\gamma_j = \max\{0, 3/2 - \lambda_j + \varepsilon\}$ for all $j \in \mathcal{C}$. Inserting now (3.30) together with (3.26) into (3.29) implies the second estimate.

The error in the state variable is obtained by the triangle inequality

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &= \|Sz - S_h z_h\|_{H^1(\Omega)} \\ &\leq c\left(\|(S - S_h)z\|_{H^1(\Omega)} + \|S_h(z - z_h)\|_{H^1(\Omega)}\right). \end{aligned} \tag{3.31}$$

For the first term we take (3.13) and find that the error is bounded by $ch^{\max\{1,\lambda-\varepsilon\}}$. Due to the Lax-Milgram lemma we have the boundedness of S_h from $H^{-1/2}(\Gamma)$ to $H^1(\Omega)$ and using estimate (3.26) we arrive at the third assertion. □

4 The control constrained optimal control problem

Let us now investigate how the results of the foregoing sections change in case of additional control constraints. We consider the model problem (2.6) where we search a control

$$z \in Z_{ad} := \left\{z \in H^{-1/2}(\Gamma)\colon z_a \leq z \leq z_b \text{ on } \Gamma \text{ in sense of } H^{-1/2}(\Gamma)\right\}.$$

For simplification we take constant control bounds $z_a, z_b \in \mathbb{R}$ with $z_a < z_b$. The control constraints in $H^{-1/2}(\Gamma)$, e. g. $z \le z_b$, are defined by

$$\langle z - z_b, v\rangle_\Gamma \le 0 \qquad \forall v \in H^{1/2}(\Gamma) \text{ with } v \ge 0 \text{ a. e. on } \Gamma.$$

The optimality condition in Theorem 2.2 transforms to a system involving a variational inequality

$$\begin{aligned} a(u_z, v) - \langle z, v\rangle_\Gamma &= 0 && \forall v \in H^1(\Omega), \\ a(p, v) - (u_z, v) &= (u_f - u_d, v) && \forall v \in H^1(\Omega), \\ \langle w - z, \alpha u_z + p\rangle_\Gamma &\ge 0 && \forall w \in Z_{ad}. \end{aligned} \tag{4.1}$$

and in a more compact form the optimality condition can be written as

$$\langle w - z, T^\alpha z + g\rangle_\Gamma \ge 0 \qquad \forall w \in Z_{ad}. \tag{4.2}$$

Apparently, we can expect better regularity for the optimal control. For unconstrained problems we had e. g. $\lim_{x\to x^{(j)}} z = \pm\infty$ near concave corners $x^{(j)}$. With box constraints the control z becomes then constant in a neighborhood of such a corner and is hence regular. In what follows we abbreviate by $\hat{\mathcal{C}}$ and $\check{\mathcal{C}}$ the index sets of concave and convex corners, respectively.

In the following the active and inactive sets are denoted by

$$\begin{aligned} \mathcal{A}^+ &:= \{x \in \Gamma\colon (T^\alpha z + g)(x) < 0\}\,, \\ \mathcal{A}^- &:= \{x \in \Gamma\colon (T^\alpha z + g)(x) > 0\}\,, \qquad \mathcal{I} := \Gamma \backslash (\overline{\mathcal{A}^+} \cup \overline{\mathcal{A}^-}). \end{aligned}$$

The optimality condition (4.2) implies that

$$z = \begin{cases} z_b & \text{on } \mathcal{A}^+, \\ z_a & \text{on } \mathcal{A}^-, \\ v \in [z_a, z_b] & \text{on } \mathcal{I}. \end{cases} \tag{4.3}$$

Let us introduce some further notation. Let Ω_R^j, $j \in \mathcal{C}$, denote angular sectors around $x^{(j)}$ with sufficiently small radius R such that no other corner or transition point is contained in that ball. In the same way we define sectors $\tilde{\Omega}_R^j$, $j \in \mathcal{T}$, around the transition points $x_T^{(j)}$ in such a way that no other corner or transition point is contained in that sector. The outer boundaries are denoted by $\Gamma_R^j := \partial\Omega_R^j \cap \Gamma$ and $\tilde{\Gamma}_R^j := \partial\tilde{\Omega}_R^j \cap \Gamma$, respectively.

For our proof we need a structural assumption upon the active set which is in most cases satisfied.

Assumption B *Assume that the control bounds are strictly active in a vicinity of reentrant corners, i. e. there exist some constants $R, \tau > 0$ such that*

$$|(T^\alpha z + g)(x)| > \tau \qquad \textit{for a. a. } x \in \Gamma_R^j,$$

for all $j \in \hat{\mathcal{C}}$. Moreover, the number of transition points $x_T^{(j)}$, $j \in \mathcal{T} := \{1, \dots, d_T\}$, is finite, and transition points can only occur in the interior of a boundary edge Γ_i, $i \in \mathcal{C}$.

Since singularities can also occur at the transition points $x_T^{(j)}$ we introduce the weighted Sobolev spaces $W^{k,q}_\beta(\tilde\Omega^j_R)$, $k \in \mathbb{N}$, $q \in [1,\infty]$ and $\beta \in \mathbb{R}$, in a neighborhood of a transition point defined as the set of functions with finite norm

$$\|v\|_{W^{k,q}_\beta(\tilde\Omega^j_R)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_{\tilde\Omega^j_R} \rho_j(x)^{q\beta} |D^\alpha v(x)|^q \,\mathrm{d}x \right)^{1/q}, & \text{if } q \in [1,\infty), \\ \sum_{|\alpha|\le k} \operatorname*{ess\,sup}_{x\in\tilde\Omega^j_R} \rho_j(x)^\beta |D^\alpha v(x)|, & \text{if } q = \infty, \end{cases} \tag{4.4}$$

for all $j \in \mathcal{T}$, where $\rho_j(x) := |x - x_T^{(j)}|$. The trace spaces $W^{k-1/p,p}_\beta(\tilde\Gamma^j_R)$ are defined in analogy to (2.19).

In the first part of this section we show that the regularity is now improved in comparison to the unconstrained case. However, we also have to show that singularities occurring in a vicinity of the transition points $x_T^{(j)}$ are weak enough such that the convergence rate is not affected by these points. The proof of the following lemma is motivated by a similar observation for Dirichlet control problems in $H^{1/2}(\Gamma)$ [20].

Lemma 4.1. *Let be given $f, u_d \in C^{0,\sigma}(\overline{\Omega})$ with some $\sigma \in (0,1)$. Let (u_z, z, p) be the solution of the optimality system (4.1) and denote by $\vec\alpha, \vec\beta, \vec\gamma \in \mathbb{R}^d$ the vectors defined in Corollary 2.6. Assume that z satisfies Assumption B.*

In a vicinity of the corner points $x^{(j)}$, $j \in \mathcal{C}$, there holds

$$u_z \in W^{2,2}_{\alpha_j}(\Omega^j_R) \cap W^{2,\infty}_{\beta_j}(\Omega^j_R), \quad p \in W^{2,2}_{\alpha_j}(\Omega^j_R) \cap W^{2,\infty}_{\beta_j}(\Omega^j_R), \quad z \in W^{1,2}_{\tilde\gamma_j}(\Gamma^j_R),$$

where $\tilde\gamma_j = 0$ if $j \in \hat{\mathcal{C}}$ and $\tilde\gamma_j = \gamma_j$ if $j \in \check{\mathcal{C}}$. Moreover, in a vicinity of a transition point $x_T^{(j)}$, $j \in \mathcal{T}$, we have the local regularity

$$u_z \in H^2(\tilde\Omega^j_R) \cap W^{2,\infty}_{1/2}(\tilde\Omega^j_R), \quad p \in H^2(\tilde\Omega^j_R) \cap W^{2,\infty}(\tilde\Omega^j_R), \quad z \in H^{1-\varepsilon}(\tilde\Gamma^j_R),$$

with arbitrary $\varepsilon > 0$.

Proof. From standard arguments we conclude from $z \in H^{-1/2}(\Gamma)$ that $u_z \in H^1(\Omega) \hookrightarrow L^q(\Omega)$, $q \in [1,\infty)$, and consequently also $p \in H^{3/2+s}(\Omega) \cap C^{0,\sigma}(\overline{\Omega})$ with some $s \in (0,1/2]$ and $\sigma \in (0,s)$. The state variable u_z satisfies the differential equation

$$-\Delta u_z + u_z = 0 \qquad \text{in } \Omega,$$

as well as the Signorini boundary conditions

$$\begin{aligned} &u_z \ge -\alpha^{-1} p && \text{and} && \partial_n u_z = z_a && \text{on } \mathcal{A}^-, \\ &u_z \le -\alpha^{-1} p && \text{and} && \partial_n u_z = z_b && \text{on } \mathcal{A}^+, \\ &u_z = -\alpha^{-1} p && \text{and} && \partial_n u_z \in [z_a, z_b] && \text{on } \mathcal{I}, \end{aligned} \tag{4.5}$$

stated already in (4.3).

Let (r,φ) denote polar coordinates centered at $x_T^{(j)}$, and without loss of generality let $\varphi=0$ belong to $\mathcal{I}$ and $\varphi=\pi$ to $\mathcal{A}^-$. From [9, Section 2.3] it is known that the solution u_z admits the decomposition

$$u_z(r,\varphi) = u_R(r,\varphi) + Br^\lambda \sin(\lambda\varphi), \qquad \lambda = 1/2,$$

with a regular part $u_R \in W^{2,q}(\tilde{\Omega}_R^j)$ for $q<4$. It is easy to show that $u_z \in C^{0,\sigma}(\overline{\Omega})$ with $\sigma \in (0,\min\{s,1/2\})$. For u_R this follows from an embedding and for the singular part this is a consequence of a direct calculation. From Theorem 2.4 and Theorem 2.5 we then get $p \in W^{2,2}_{\vec{\alpha}}(\Omega)\cap W^{2,\infty}_{\vec{\beta}}(\Omega)$ since $u_z+u_f-u_d \in C^{0,\sigma}(\overline{\Omega})$. The regularity of u_z in the vicinity of a corner $x^{(j)}$, $j\in\mathcal{C}$, follows from the regularity of p using the arguments from the proof of Corollary 2.6. The stated regularity of z in the vicinity of convex corners follows again from embedding and trace theorems. In a vicinity of concave corners we have $z\equiv z_a$ or $z\equiv z_b$ and hence $z\in H^1(\Omega_R^j)$ for $j\in\hat{\mathcal{C}}$.

It remains to investigate how the regularity of p in a vicinity of a transition point is transferred to the state u_z. The definition of the weighted Sobolev space and Assumption B imply $p\in W^{2,\infty}(\tilde{\Omega}_R^j)$ for all $j\in\mathcal{T}$. With the chain rule one obtains the normal derivative

$$\partial_n u_z(r,\varphi) = \partial_n u_R(r,\varphi) \mp B\lambda r^{\lambda-1}\cos(\lambda\varphi)$$

and to fulfill (4.5) we require

$$\begin{aligned} z_a &\overset{!}{\le} \partial_n u_z(r,0) = \partial_n u_R(r,0) - B\lambda r^{\lambda-1},\\ z_a &\overset{!}{=} \partial_n u_z(r,\pi) = \partial_n u_R(r,\pi). \end{aligned}$$

Since $r^{\lambda-1}$ grows unboundedly towards infinity for $r\to 0$, we have to set $B\le 0$ to ensure the first condition. Moreover, we have for $\varphi=\pi$ the inequality

$$u_z(r,\pi) = u_R(r,\pi) + Br^\lambda \overset{!}{\ge} -\alpha^{-1}p.$$

Let us now take the condition $u_z \ge -\alpha^{-1}p$ on $\mathcal{A}^-$ from (4.5) into account. From the trace theorem and the Sobolev embedding theorem we get

$$u_R, p \in W^{2,q}(\tilde{\Omega}_R^j) \hookrightarrow W^{2-1/q,q}(\tilde{\Gamma}_R^j) \hookrightarrow C^1(\tilde{\Gamma}_R^j),$$

which holds for $q>2$. Thus, we can perform a Taylor expansion of $u_R(r,\pi)+\alpha^{-1}p(r,\pi)$ in the point $r=0$ with some intermediate point $\xi\in[0,r]$. Exploiting the fact that $u_R(0,\pi) = -\alpha^{-1}p(0,\pi)$ leads to

$$\begin{aligned} & u_R(r,\pi)+\alpha^{-1}p(r,\pi)+Br^\lambda \ge 0\\ \iff\quad & u_R(0,\pi)+\alpha^{-1}p(0,\pi)+r\partial_r\big(u_R(\xi,\pi)+\alpha^{-1}p(\xi,\pi)\big)+Br^\lambda \ge 0\\ \iff\quad & \partial_r\big(u_R(\xi,\pi)+\alpha^{-1}p(\xi,\pi)\big)+Br^{\lambda-1}\ge 0. \end{aligned}$$

The term $\partial_r \left(u_R(\xi) + \alpha^{-1} p(\xi)\right)$ is bounded since u_R and p are regular and thus the inequality holds in case of $B \geq 0$ only. We already stated the condition $B \leq 0$ and consequently, the boundary conditions (4.5) can only be satisfied in case of $B = 0$. The singular part corresponding to $\lambda = 1/2$ hence vanishes and thus

$$u_S(r, \varphi) := \tilde{B}\, r^{3/2} \sin\left(\frac{3}{2}\varphi\right), \qquad \tilde{B} \in \mathbb{R},$$

is in general the leading singularity. The regular part has the regularity $u_R \in W^{2,\infty}(\tilde{\Omega}_R^j)$ since the singularity corresponding to $\lambda = 5/2$ is contained in that space. However, by a direct calculation one can show $u_S \in W^{2,\infty}_{1/2}(\tilde{\Omega}_R^j) \cap H^2(\tilde{\Omega}_R^j)$ by exploiting the definition of the weighted Sobolev space from (4.4). Moreover, we also get a decomposition of the normal derivative into $z = z_R + z_S$ with

$$z_R := \partial_n u_R \in H^1(\tilde{\Gamma}_R^j), \qquad z_S := \partial_n u_S = \begin{cases} -\tilde{B} r^{1/2}, & \text{on } \mathcal{I}, \\ 0, & \text{on } \mathcal{A}^-. \end{cases}$$

A simple calculation yields then $z_S \in H^{1-\varepsilon}(\tilde{\Gamma}_R^j)$ for arbitrary $\varepsilon > 0$. □

In analogy to the unconstrained case we discretize the optimality condition (4.2) and search a solution in the discrete spaces Z_h and U_h. Throughout this section Z_h is the space of piecewise constant functions as introduced in Section 3.3.1. The choice of piecewise linear controls considered in Section 3.3.2 is also possible, but the proof of Lemma 4.3 is not true for this choice.

The discretized optimality system reads now: *Find $z_h \in Z_{h,ad} := Z_h \cap Z_{ad}$ and $u_{z,h}, p_h \in U_h$ such that*

$$\begin{aligned} a(u_{z,h}, v_h) - \langle z_h, v_h \rangle_\Gamma &= 0 && \forall v_h \in U_h, \\ a(p_h, v_h) - (u_{z,h}, v_h) &= (u_{f,h} - u_d, v_h) && \forall v_h \in U_h, \\ \langle w_h - z_h, \alpha u_{z,h} + p_h \rangle_\Gamma &\geq 0 && \forall w_h \in Z_{h,ad}, \end{aligned} \tag{4.6}$$

where $u_{f,h} \in U_h$ can be computed from the equation

$$a(u_{f,h}, v_h) = \langle f, v_h \rangle_\Omega \qquad \forall v_h \in U_h$$

in advance. As already done in the proof of Theorem 3.2 we introduce the solution $\tilde{z}_h \in Z_{h,ad}$ of the auxiliary problem

$$\langle v_h - \tilde{z}_h, T^\alpha \tilde{z}_h + g \rangle_\Gamma \geq 0 \qquad \text{for all } v_h \in Z_{h,ad}. \tag{4.7}$$

Note that we only approximated the ansatz and test space, but not the operator T^α.

Analogous to Theorem 3.2 we can now show:

Lemma 4.2. *Let $\Gamma_0 \supseteq \tilde{\Gamma} := \{x \in \Gamma\colon \tilde{z}_h(x) \neq z_h(x)\}$. Then the estimate*

$$\begin{aligned}
&\|z - z_h\|_{H^{-1/2}(\Gamma)} \\
&\leq c\Big[\|(S - S_h)z\|_{L^2(\Omega)} + \|(S^* - S_h^*)(u - u_d)\|_{H^{1/2}(\Gamma_0)} \\
&\quad + \alpha\|(\mathcal{N} - \mathcal{N}_h)z\|_{H^{1/2}(\Gamma_0)} + \|u_f - u_{f,h}\|_{L^2(\Omega)} + \|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)}\Big]
\end{aligned} \tag{4.8}$$

holds.

Proof. The arguments applied in the proof of Theorem 3.2 widely coincide with the control-constrained case and we just outline the differences. We apply again the triangle inequality and get

$$\|z - z_h\|_{H^{-1/2}(\Gamma)} \leq \|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)} + \|\tilde{z}_h - z_h\|_{H^{-1/2}(\Gamma)}, \tag{4.9}$$

where $\tilde{z}_h$ is the solution of (4.7). One easily confirms that (3.9) with $w_h := \tilde{z}_h - z_h$ also holds in the control-constrained case when we replace all "=" by "≤". Thus,

$$\|w_h\|^2_{H^{-1/2}(\Gamma)} \leq \langle w_h, (T_h^\alpha - T^\alpha)(\tilde{z}_h - z)\rangle_\Gamma + \langle w_h, (T_h^\alpha - T^\alpha)\, z - g + g_h\rangle_\Gamma\,. \tag{4.10}$$

The estimate (3.10) remains the same and we have

$$\langle w_h, (T_h^\alpha - T^\alpha)(\tilde{z}_h - z)\rangle_\Gamma \leq c\|w_h\|_{H^{-1/2}(\Gamma)}\|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)}. \tag{4.11}$$

Moreover, (3.11) becomes

$$\begin{aligned}
&\langle w_h, (T_h^\alpha - T^\alpha)\, z + g - g_h\rangle_\Gamma \\
&\leq c\|w_h\|_{H^{-1/2}(\Gamma_0)}\Big(\|S_h^*(S_h - S)z\|_{H^{1/2}(\Gamma_0)} + \|(S_h^* - S^*)(u - u_d)\|_{H^{1/2}(\Gamma_0)} \\
&+ \|S_h^*(u_f - u_{f,h})\|_{H^{1/2}(\Gamma_0)} + \alpha\|(\mathcal{N}_h - \mathcal{N})z\|_{H^{1/2}(\Gamma_0)}\Big).
\end{aligned} \tag{4.12}$$

Dividing by $\|w_h\|_{H^{-1/2}(\Gamma)}$ and exploiting stability properties of S_h^* yields the assertion. □

Deriving error estimates for the term $\|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)}$ requires more effort in the control-constrained case than in the unconstrained case where we merely applied the Céa-Lemma (3.8) and inserted the best-approximation properties from Lemma 3.5.

Lemma 4.3. *Let $z \in Z_{ad}$ and $\tilde{z}_h \in Z_{h,ad}$ denote the solutions of (4.2) and (4.7), respectively, where $Z_{h,ad}$ is the space of functions which are feasible and piecewise constant on each $E \in \mathcal{E}_h$. Then, the error estimate*

$$\|z - \tilde{z}_h\|_{H^{-1/2}(\Gamma)} \leq ch^{\min\{3/2, \check{\lambda}\} - \varepsilon}, \quad \textit{with} \quad \check{\lambda} := \min_{j \in \check{\mathcal{C}}} \lambda_j,$$

holds for arbitrary $\varepsilon > 0$, provided that Assumption B holds.

Proof. We take the Céa-type Lemma from [10, Lemma 7.16] and get by using the results of Lemma 2.1

$$\frac{\alpha}{2}\|z-\tilde z_h\|^2_{H^{-1/2}(\Gamma)} \le \inf_{v\in Z_{ad}} \langle v-\tilde z_h, T^\alpha z + g\rangle_\Gamma \\ + \inf_{v_h\in Z_{h,ad}} \left\{\langle v_h - z, T^\alpha z+g\rangle_\Gamma + c\|z-v_h\|^2_{H^{-1/2}(\Gamma)}\right\}.$$

In the present situation the first term on the right-hand side vanishes for the choice $v := \tilde z_h$ (this is possible since $\tilde z_h \in Z_{ad}$). The second term vanishes if we choose

$$v_h \in \tilde Z_{h,ad} := \left\{z_h \in Z_{h,ad}\colon z_h = z_a \text{ on } \mathcal{A}^-,\ z_h = z_b \text{ on } \mathcal{A}^+\right\},$$

since $v_h - z \equiv 0$ on $\mathcal{A}^\pm$ and $T^\alpha z + g \equiv 0$ on $\mathcal{I}$. We consequently get

$$\|z-\tilde z_h\|_{H^{-1/2}(\Gamma)} \le c \inf_{v_h\in\tilde Z_{h,ad}} \|z-v_h\|_{H^{-1/2}(\Gamma)}. \tag{4.13}$$

We insert the $L^2(\Gamma)$-projection onto Z_h as intermediate function and obtain

$$\|z-v_h\|_{H^{-1/2}(\Gamma)} \le \|z-P_h^\partial z\|_{H^{-1/2}(\Gamma)} + \|P_h^\partial z - v_h\|_{H^{-1/2}(\Gamma)}. \tag{4.14}$$

The first term also occurs for unconstrained problems and an estimate is given in Lemma 3.5. However, we can exploit that the term $z - P_h^\partial z$ vanishes in a neighborhood of concave corners which can thus be neglected. This implies

$$\|z-P_h^\partial z\|_{H^{-1/2}(\Gamma)} \le c h^{\min\{3/2,\check\lambda-\varepsilon\}} |z|_{W^{1,2}_{\vec\gamma}(\Gamma)}. \tag{4.15}$$

For the second term in (4.14) we choose

$$v_{h|E} = \begin{cases} [P_h^\partial z]_{|E}, & \text{if } E\subset\mathcal{I},\\ z_a, & \text{if } E\cap\mathcal{A}^-\neq\emptyset,\\ z_b, & \text{if } E\cap\mathcal{A}^+\neq\emptyset.\end{cases}$$

Note that $v_h \in \tilde Z_{h,ad}$ by construction, and that $P_h^\partial z - v_h$ vanishes on all elements

$$E \notin \mathcal{K}_h := \{E\in\mathcal{E}_h\colon \overline{E}\cap\mathcal{A}^\pm\neq\emptyset \ \wedge\ \overline{E}\cap\mathcal{I}\neq\emptyset\}.$$

Due to Assumption B the set $\mathcal{K}_h$ contains a finite number of elements, independent of h. Exploiting the orthogonality property of the projection P_h^∂ we get

$$\begin{aligned} \|P_h^\partial z - v_h\|_{H^{-1/2}(\Gamma)} &= \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E\in\mathcal{K}_h} (P_h^\partial z - v_h, \varphi)_E \\ &= \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E\in\mathcal{K}_h} (P_h^\partial (z - v_h), P_h^\partial\varphi)_E \\ &\le \sup_{\|\varphi\|_{H^{1/2}(\Gamma)}=1} \sum_{E\in\mathcal{K}_h} \|z-v_h\|_{L^2(E)}\|P_h^\partial\varphi\|_{L^2(E)}. \end{aligned} \tag{4.16}$$

Since v_h coincides with z at the endpoint of E which belongs to $\mathcal{A}^\pm$ we get with the Poincaré-Friedrichs inequality

$$\|z - v_h\|_{L^2(E)} \le ch^{1-\varepsilon'}|z|_{H^{1-\varepsilon'}(E)}, \qquad \text{for } E \in \mathcal{K}_h,$$

with arbitrary $\varepsilon' \in (0, 1/2)$, where we exploited the regularity of z stated in Lemma 4.1. For the second term on the right-hand side of (4.16) we apply the Hölder inequality and stability properties of the projection P_h^∂ and get

$$\|P_h^\partial \varphi\|_{L^2(E)} \le ch^{1/2-1/q}\|\varphi\|_{L^q(E)}.$$

Hence, (4.16) becomes

$$\begin{aligned}
&\|P_h^\partial z - v_h\|_{H^{-1/2}(\Gamma)} \\
&\le ch^{3/2-\varepsilon'-1/q} \sum_{E\in\mathcal{K}_h} |z|_{H^{1-\varepsilon'}(E)}\|\varphi\|_{L^q(E)} \\
&\le ch^{3/2-\varepsilon'-1/q} \left(\sum_{E\in\mathcal{K}_h} |z|^2_{H^{1-\varepsilon'}(E)}\right)^{1/2} \left(\sum_{E\in\mathcal{K}_h} 1\right)^{1/2-1/q} \|\varphi\|_{L^q(\Gamma)} \\
&\le ch^{3/2-\varepsilon},
\end{aligned}$$

where we exploited that the number of elements in $\mathcal{K}_h$ is independent of h, the embedding $\|\varphi\|_{L^q(\Gamma)} \le c\|\varphi\|_{H^{1/2}(\Gamma)} = c$, and we chose $\varepsilon' = 1/q = \varepsilon/2$. Inserting this together with (4.15) into (4.14) completes the proof.

□

The control z is in general active in the vicinity of concave corners. In the following lemma we show that this property is transferred also to the discrete solution z_h, and hence we get $z - z_h \equiv 0$ near these corners. This is the key idea for the improved error estimates that we will show in Theorem 4.5.

Lemma 4.4. *Let Assumption B be satisfied. Then, some constant $h_0 > 0$ exists such that*

$$z_h(x) = z_b \quad \textit{or} \quad z_h(x) = z_a \qquad \textit{for all } x \in \Gamma_R^j,$$

provided that $h \le h_0$.

Proof. Without loss of generality we show the assertion for the case that the upper bound is strictly active, i.e. $T^\alpha z + g < -\tau$ within Γ_R^j. The key step is to show uniform convergence of $T_h^\alpha z_h + g_h$ towards $T^\alpha z + g$, i.e.

$$\|(T^\alpha z + g) - (T_h^\alpha z_h + g_h)\|_{L^\infty(\Gamma)} \overset{h\to 0}{\longrightarrow} 0, \tag{4.17}$$

which then implies $T_h^\alpha z_h + g_h < 0$ within Γ_R^j when $h \le h_0$. By element-wise consideration of the discrete optimality condition (4.2) we conclude that $z_h = z_b$ and have shown the assertion.

From the definition (2.11) of T^α and g as well as their discrete analogues (3.5) we get

$$\|(T^\alpha z + g) - (T_h^\alpha z_h + g_h)\|_{L^\infty(\Gamma)} = \|\alpha(u_z - u_{z,h}) + (p - p_h)\|_{L^\infty(\Gamma)}.$$

Let us first derive a pointwise estimate for the state variable. With the triangle inequality and a trace theorem we get

$$\|u_z - u_{z,h}\|_{L^\infty(\Gamma)} \le \|u_z - S_h z\|_{L^\infty(\Omega)} + \|S_h(z - z_h)\|_{L^\infty(\Omega)}. \tag{4.18}$$

For the first term we insert the intermediate function $I_h u_z$, apply the triangle inequality and the discrete Sobolev inequality, and insert the intermediate function u_z which leads to

$$\begin{aligned}
&\|u_z - S_h z\|_{L^\infty(\Omega)} \\
&\le c\|u_z - I_h u_z\|_{L^\infty(\Omega)} + |\ln h|^{1/2}\left(\|u_z - I_h u_z\|_{H^1(\Omega)} + \|u_z - S_h z\|_{H^1(\Omega)}\right) \\
&\le c|\ln h|^{1/2} h^{\min\{1,\lambda\}-\varepsilon}\left(|u_z|_{W^{2,2}_{\vec\alpha}(\Omega)} + |u_z|_{W^{2,\infty}_{\vec\beta}(\Omega)}\right) \to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{4.19}$$

In the last step we applied (3.13) and (3.16), as well as the interpolation error estimate from [21, Corollary 3.30], and exploited the regularity stated in Lemma 4.1.

Moreover, we get with the stability of S_h from $L^2(\Gamma)$ to $L^\infty(\Omega)$, the triangle inequality, and the inverse inequality from [23, Lemma 10.10] the estimate

$$\begin{aligned}
&\|S_h(z - z_h)\|_{L^\infty(\Omega)} \le c\|z - z_h\|_{L^2(\Gamma)} \\
&\le c\|z - P_h^\partial z\|_{L^2(\Gamma)} + h^{-1/2}\left(\|z - P_h^\partial z\|_{H^{-1/2}(\Gamma)} + \|z - z_h\|_{H^{-1/2}(\Gamma)}\right).
\end{aligned} \tag{4.20}$$

With the estimates for the $L^2(\Gamma)$-projection of Lemma 3.5 we immediately get

$$\|z - P_h^\partial z\|_{L^2(\Gamma)} + h^{-1/2}\|z - P_h^\partial z\|_{H^{-1/2}(\Gamma)} \le c h^{\min\{1,\lambda-1/2-\varepsilon\}}. \tag{4.21}$$

The estimate of Lemma 4.2 for $\Gamma_0 = \Gamma$ and the error estimates from Theorem 3.3 and Lemma 4.3 moreover lead to

$$\begin{aligned}
&h^{-1/2}\|z - z_h\|_{H^{-1/2}(\Gamma)} \\
&\le c h^{-1/2}\Big[\|(S - S_h)z\|_{L^2(\Omega)} + \|(S^* - S_h^*)(u - u_d)\|_{H^{1/2}(\Gamma)} \\
&\quad + \alpha\|(\mathcal{N} - \mathcal{N}_h)z\|_{H^{1/2}(\Gamma)} + \|u_f - u_{f,h}\|_{L^2(\Omega)} + \|z - \tilde z_h\|_{H^{-1/2}(\Gamma)}\Big] \\
&\le c\Big(h^{\min\{3/2,2\lambda-1/2\}-\varepsilon}\|u_z\|_{W^{2,2}_{\vec\alpha}} \\
&\quad + h^{\min\{1,\lambda-1/2\}-\varepsilon}(\|p\|_{W^{2,\infty}_{\vec\beta}(\Omega)} + \|u_z\|_{W^{2,\infty}_{\vec\beta}(\Omega)}) + h^{\min\{1,\check\lambda\}-\varepsilon}\Big).
\end{aligned} \tag{4.22}$$

As $\lambda > 1/2$ we conclude from (4.21), (4.22) and (4.20) that

$$\|S_h(z - z_h)\|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad h \to 0. \tag{4.23}$$

and consequently we get with (4.19) and (4.18) the property

$$\|u_z - u_{z,h}\|_{L^\infty(\Gamma)} \to 0 \quad \text{as} \quad h \to 0. \tag{4.24}$$

It remains to show pointwise convergence of the discrete adjoint state. We use the definitions $p = S^*(u_z + u_f - u_d)$ and $p_h = S_h^*(u_{z,h} + u_{f,h} - u_d)$, introduce several intermediate functions and get the equivalent formulation

$$\begin{aligned} p - p_h &= S^*(Sz + u_f - u_d) - S_h^*(S_h z_h + u_{f,h} - u_d) \\ &= (S^* - S_h^*)(Sz + u_f - u_d) + S_h^*(S - S_h)z \\ &\quad + S_h^* S_h(z - z_h) + S_h^*(u_f - u_{f,h}). \end{aligned} \tag{4.25}$$

One easily confirms that

$$\begin{aligned} \|(S^* - S_h^*)(Sz + u_f - u_d)\|_{L^\infty(\Gamma)} &\to 0, \\ \|S_h^*(S - S_h)z\|_{L^\infty(\Gamma)} &\le \|(S - S_h)z\|_{L^2(\Omega)} \to 0, \\ \|S_h^* S_h(z - z_h)\|_{L^\infty(\Gamma)} &\le \|S_h(z - z_h)\|_{L^\infty(\Omega)} \to 0, \\ \|S_h^*(u_f - u_{f,h})\|_{L^\infty(\Gamma)} &\le \|u_f - u_{f,h}\|_{L^2(\Omega)} \to 0, \end{aligned}$$

as $h \to 0$, where we can reuse the arguments used in (4.19) and (4.23) to show the first and third estimates. The second and fourth estimates follow from stability properties of S_h and S_h^* as well as trivial convergence properties of the finite element method. Together with the reformulation (4.25) and the triangle inequality we arrive at

$$\|p - p_h\|_{L^\infty(\Gamma)} \to 0 \quad \text{as} \quad h \to 0.$$

Together with (4.24) the desired property (4.17) follows. □

We are now in the position to improve the error estimates from Theorem 3.7 exploiting the fact that $z - z_h \equiv 0$ in the vicinity of concave corners.

Theorem 4.5. *Let $\check{\lambda} := \min_{j \in \check{\mathcal{C}}} \lambda_j$ be the singular exponent of the largest convex angle of Ω. Then, the error estimates*

$$\begin{aligned} \|z - z_h\|_{H^{-1/2}(\Gamma)} &\le c h^{\min\{3/2, \check{\lambda}, 2\lambda - 1/2\} - \varepsilon}, \\ \|z - z_h\|_{L^2(\Gamma)} &\le c h^{\min\{1, \check{\lambda} - 1/2, 2\lambda - 1\} - \varepsilon}, \\ \|u - u_h\|_{H^1(\Omega)} &\le c h^{\min\{1, \lambda\} - \varepsilon}, \end{aligned}$$

hold, provided that Assumption B is satisfied.

Proof. Due to Lemma 4.4 there exists some $R > 0$ such that $z_h(x) = z(x) \in \{z_a, z_b\}$ for all $x \in \Gamma_R^j$ and $j \in \hat{\mathcal{C}}$. Since $\tilde{z}_h$ behaves like the best-approximation of z (see (4.13)), we also get $\tilde{z}_h(x) \in \{z_a, z_b\}$ for all $x \in \Gamma_R^j$, $j \in \hat{\mathcal{C}}$. In the following we write

$$\Omega_0 := \Omega \setminus \left(\bigcup_{j \in \hat{\mathcal{C}}} \Omega_R^j \right), \qquad \Gamma_0 := \partial\Omega_0 \cap \Gamma.$$

By construction the term $\tilde{z}_h - z_h$ vanishes on $\Gamma \setminus \Gamma_0$ and the assumptions of Lemma 4.2 are satisfied. In order to show the estimate in the $H^{-1/2}(\Gamma)$-norm we have to discuss the four terms on the right-hand side of the estimate in Lemma 4.2.

For the first term we get from (3.13)

$$\|(S - S_h)z\|_{L^2(\Omega)} \le ch^{\min\{2,2\lambda-\varepsilon\}}\|u\|_{W^{2,2}_{\vec{\alpha}}(\Omega)}. \tag{4.26}$$

The same estimate follows for the third term exploiting that $u_f \in W^{2,2}_{\vec{\alpha}}(\Omega)$. For the second term of (4.12) we write $p := S^*(u - u_d)$ and $p^h := S_h^*(u - u_d)$ and therefore

$$\|p - p^h\|_{H^{1/2}(\Gamma_0)} \le \|p - I_h p\|_{H^{1/2}(\Gamma_0)} + h^{-1/2}\left(\|p - I_h p\|_{L^2(\Gamma_0)} + \|p - p^h\|_{L^2(\Gamma_0)}\right) \tag{4.27}$$

using the argument from (3.17). The interpolation error estimate (3.18) leads to

$$\|p - I_h p\|_{H^{1/2}(\Gamma_0)} + h^{-1/2}\|p - I_h p\|_{L^2(\Gamma_0)} \le ch^{\min\{3/2,\check{\lambda}-\varepsilon\}}\|p\|_{W^{2,2}_{\vec{\gamma}}(\Gamma)} \tag{4.28}$$

provided that $\gamma_j = \max\{0, 3/2 - \lambda_j + \varepsilon\}$ for all $j \in \mathcal{C}$. We also exploited that Γ_0 excludes neighborhoods of concave corners. For the finite-element error on the boundary we exploit Lemma 3.12 in [1] which states that if $p \in W^{2,\infty}_{\beta_j}(\Omega^j_R)$ with $\beta_j = \max\{1/2, 2 - \lambda_j + \varepsilon\}$, the error estimate

$$\|p - p^h\|_{L^2(\Gamma^j_R)} \le c\left[h^{\min\{2,1/2+\lambda_j\}-\varepsilon}|p|_{W^{2,\infty}_{\beta_j}(\Omega^j_{2R})} + \|p - p^h\|_{L^2(\Omega^j_{2R})}\right] \tag{4.29}$$

holds. In [1, Equation (3.33)] an estimate on the regular part of the boundary

$$\Omega^{reg}_R := \Omega \setminus \left(\bigcup_{j\in\mathcal{C}} \Omega^j_R\right), \qquad \Gamma^{reg}_R := \partial\Omega^{reg}_R \cap \Gamma,$$

is proved which reads in our situation

$$\|p - p^h\|_{L^2(\Gamma^{reg}_R)} \le ch^{2-\varepsilon}|p|_{W^{2,\infty}(\Omega^{reg}_{R/2})} + \|p - p^h\|_{L^2(\Omega)}. \tag{4.30}$$

Furthermore, we use a standard $L^2(\Omega)$ estimate to get

$$\|p - p^h\|_{L^2(\Omega)} \le ch^{\min\{2,2\lambda-\varepsilon\}}\left(\|u\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)}\right). \tag{4.31}$$

Combining the estimates (4.29), (4.30) and inserting (4.31) leads to

$$\begin{aligned}
&\|p - p^h\|^2_{L^2(\Gamma_0)} \\
&\le \sum_{j\in\check{\mathcal{C}}} \|p - p^h\|^2_{L^2(\Gamma^j_R)} + \|p - p^h\|^2_{L^2(\Gamma^{reg}_R)} \\
&\le c\Big[h^{2\min\{1/2+\check{\lambda},2\}-\varepsilon}\sum_{j\in\check{\mathcal{C}}}|p|^2_{W^{2,\infty}_{\beta_j}(\Omega^j_{2R})} + h^{4-\varepsilon}|p|^2_{W^{2,\infty}(\Omega^{reg}_{R/2})} + \|p - p^h\|^2_{L^2(\Omega)}\Big] \\
&\le ch^{2\min\{2,1/2+\check{\lambda},2\lambda\}-\varepsilon}.
\end{aligned} \tag{4.32}$$

From Lemma 4.1 it is known that p possesses the regularity required in (4.32). Inserting now (4.28) and (4.32) into (4.27) leads to the estimate

$$\|p - p^h\|_{H^{1/2}(\Gamma_0)} \le c h^{\min\{3/2,\check{\lambda},2\lambda-1/2\}-\varepsilon}. \tag{4.33}$$

It remains to estimate the fourth term on the right-hand side of (4.12). Additional singularities occur now in a neighborhood of the transition points. The optimal state u_z is in $W^{2,\infty}$ only on the set

$$\Omega_R^{reg} := \Omega \setminus \left(\bigcup_{j\in\mathcal{C}} \Omega_R^j \cup \bigcup_{j\in\mathcal{T}} \tilde{\Omega}_R^j \right), \qquad \Gamma_R^{reg} := \partial\Omega_R^{reg} \cap \Gamma,$$

but it possesses the regularity $u_z \in W^{2,\infty}_{1/2}(\tilde{\Omega}_R^j)$ in a vicinity of the transition points between active and inactive set (see Lemma 4.1), and $u_z \in W^{2,\infty}_{\beta_j}(\Omega_R^j)$ in a vicinity of corners with $\beta_j = \max\{1/2, 2-\lambda_j+\varepsilon\}$. Thus, estimate (4.29) can be applied again and we obtain for $u_{z|\Gamma} = \mathcal{N} z$ and $u^h_{z|\Gamma} = \mathcal{N}_h z$ the estimate

$$\begin{aligned}
&\|u_z - u_z^h\|^2_{L^2(\Gamma_0)} \\
&= \sum_{j\in\check{\mathcal{C}}} \|u_z - u_z^h\|^2_{L^2(\Gamma_R^j)} + \sum_{j\in\mathcal{T}} \|u_z - u_z^h\|^2_{L^2(\tilde{\Gamma}_R^j)} + \|u_z - u_z^h\|^2_{L^2(\Gamma_R^{reg})} \\
&\le c\Big[h^{2\min\{1/2+\check{\lambda},2\}-\varepsilon} \sum_{j\in\check{\mathcal{C}}} |u_z|^2_{W^{2,\infty}_{\beta_j}(\Omega_R^j)} + h^{4-\varepsilon} \sum_{j\in\mathcal{T}} |u_z|^2_{W^{2,\infty}_{1/2}(\Omega_R^j)} \\
&\quad + h^{4-\varepsilon} |u_z|^2_{W^{2,\infty}(\Omega_{R/2}^{reg})} + \|u_z - u_z^h\|^2_{L^2(\Omega)} \Big] \\
&\le c h^{2\min\{1/2+\check{\lambda},2\lambda,2\}-\varepsilon}.
\end{aligned} \tag{4.34}$$

In analogy to (4.28) we also get

$$\|u_z - I_h u_z\|_{H^{1/2}(\Gamma_0)} + h^{-1/2} \|u_z - I_h u_z\|_{L^2(\Gamma_0)} \le c h^{\min\{3/2,\check{\lambda}-\varepsilon\}} \|u_z\|_{W^{2,2}_{\vec{\gamma}}(\Gamma)}$$

and with an argument like (4.27) this implies

$$\|u_z - u_z^h\|_{H^{1/2}(\Gamma_0)} \le c h^{\min\{3/2,\check{\lambda},2\lambda-1/2\}-\varepsilon}. \tag{4.35}$$

The estimates (4.26), (4.33) and (4.35) together with Lemma 4.3 and Lemma 4.2 yield the desired estimate in the $H^{-1/2}(\Gamma)$-norm. The estimate for the control in $L^2(\Gamma)$ and for the state in $H^1(\Omega)$ follow with the same arguments like in the proof of Theorem 3.7. □

5 Numerical results

In order to confirm the theoretically predicted convergence rates we constructed a benchmark example on the family of domains

$$\Omega^\omega := (-1,1)^2 \setminus \{(r\cos\varphi, r\sin\varphi) \colon r \ge 0,\ \varphi \in [0, 2\pi-\omega]\} \quad \text{for } \omega \in \left[\frac{\pi}{2}, 2\pi\right).$$

These domains have largest interior angle ω and the smallest singular exponent is hence $\lambda := \pi/\omega$ which defines the regularity of the solution. An example for an initial mesh of $\Omega^{7\pi/4}$ is illustrated in Figure 2. The data of the model problem were chosen as follows. The

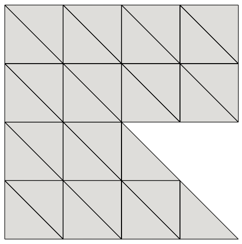

Figure 2: Initial mesh for the domain $\Omega^{7\pi/4}$.

desired state is given by $y_d := x_1^2 + x_2^2$, the right-hand side by $f \equiv 0$ and the regularization parameter $\alpha = 0.01$ is chosen. The computed optimal control and its corresponding state are plotted in Figure 3 for both $L^2(\Gamma)$- and $H^{-1/2}(\Gamma)$-regularization.

The error norms $\|z - z_h\|_{L^2(\Omega)}$ and $\|u - u_h\|_{H^1(\Omega)}$ were computed approximately by comparison with a solution on a finer mesh with $h = 2^{-9}$. To improve the accuracy of the reference solution the fine mesh is further refined locally in the vicinity of the singular corner $\mathbf{c} := (0\ 0)^\top$ such that the mesh property

$$h_T \sim \begin{cases} h^{1/\mu}, & \text{if } r_T = 0, \\ h r_T^{1-\mu}, & \text{if } r_T > 0, \end{cases} \qquad \forall T \in \mathcal{T}_h,$$

holds, where $r_T := \operatorname{dist}(\mathbf{c}, T)$. In the presented experiments the refinement parameter $\mu = 0.5$ was chosen. The global and local refinement was realized by a newest-vertex bisection strategy. The results of our computation on the domain $\Omega^{3\pi/2}$ are summarized in Table 1.

As discussed in Section 4 one can expect better error estimates when control constraints are active in the vicinity of reentrant corners. Thus, the model problem described above was computed with the additional constraint

$$z \in Z_{ad} := \{z \in H^{-1/2}(\Gamma)\colon -1 \leq z \leq 1\}. \tag{5.1}$$

The numerically computed convergence rates are presented in Table 2 for the domain $\Omega^{3\pi/2}$. It is observed that the proven error estimates from Theorem 4.5 seem to be too pessimistic since the rate $2\lambda - 1 = 1/3$ is expected, but the rate one is obtained in the experiment.

In Figure 4 the convergence rates of the discrete control in $L^2(\Gamma)$ for the computation on the domains Ω^ω with $\omega \in \{\pi/2, 3\pi/4, 5\pi/4, 3\pi/2, 7\pi/4\}$ are presented. The experimentally determined convergence rates are computed from the error norms corresponding to the meshes with $h = 2^{-6}$ and $h = 2^{-7}$. Again, the result of Theorem 3.7 for unconstrained problems is confirmed, but the numerical results for constrained problems are better than predicted in Theorem 4.5.

h	# DOF	Piecewise constant control $\|u-u_h\|_{H^1(\Omega)}$	Piecewise constant control $\|z-z_h\|_{L^2(\Gamma)}$	Piecewise linear control $\|u-u_h\|_{H^1(\Omega)}$	Piecewise linear control $\|z-z_h\|_{L^2(\Gamma)}$
2^{-2}	113	0.43728 (0.93)	1.56294 (0.65)	0.42620 (0.97)	1.60564 (0.61)
2^{-3}	417	0.27009 (0.70)	1.30371 (0.26)	0.26396 (0.69)	1.34722 (0.25)
2^{-4}	1601	0.16410 (0.72)	1.10581 (0.24)	0.16090 (0.71)	1.13909 (0.24)
2^{-5}	6273	0.09991 (0.72)	0.95184 (0.22)	0.09819 (0.71)	0.97890 (0.22)
2^{-6}	24833	0.06125 (0.71)	0.82659 (0.20)	0.06025 (0.70)	0.84954 (0.20)
2^{-7}	98817	0.03774 (0.70)	0.71919 (0.20)	0.03714 (0.70)	0.73915 (0.20)
2^{-8}	394241	0.02330 (0.70)	0.62404 (0.20)	0.02292 (0.70)	0.64130 (0.20)

Table 1: Numerical experiment without control constraints on an L-shaped domain indicating the absolute values of the computed error norms with the corresponding experimentally computed convergence rates in parentheses.

h	# DOF	Piecewise Constant Control $\|u-u_h\|_{H^1(\Omega)}$	Piecewise Constant Control $\|z-z_h\|_{L^2(\Gamma)}$	Piecewise Linear Control $\|u-u_h\|_{H^1(\Omega)}$	Piecewise Linear Control $\|z-z_h\|_{L^2(\Gamma)}$
2^{-2}	113	0.25044 (0.96)	0.67491 (0.71)	0.24991 (0.97)	0.76283 (0.82)
2^{-3}	417	0.12798 (0.97)	0.31461 (1.10)	0.12939 (0.95)	0.36988 (1.04)
2^{-4}	1601	0.06478 (0.98)	0.18287 (0.78)	0.06437 (1.01)	0.20800 (0.83)
2^{-5}	6273	0.03255 (0.99)	0.10157 (0.85)	0.03193 (1.01)	0.11023 (0.92)
2^{-6}	24833	0.01589 (1.03)	0.05030 (1.01)	0.01573 (1.02)	0.05432 (1.02)
2^{-7}	98817	0.00767 (1.05)	0.02423 (1.05)	0.00765 (1.04)	0.02634 (1.04)
2^{-8}	394241	0.00343 (1.16)	0.01234 (0.97)	0.00343 (1.16)	0.01314 (1.00)

Table 2: Numerical experiment with control constraints on an L-shaped domain indicating the values of the computed error norms with the corresponding experimentally computed convergence rates in parentheses.

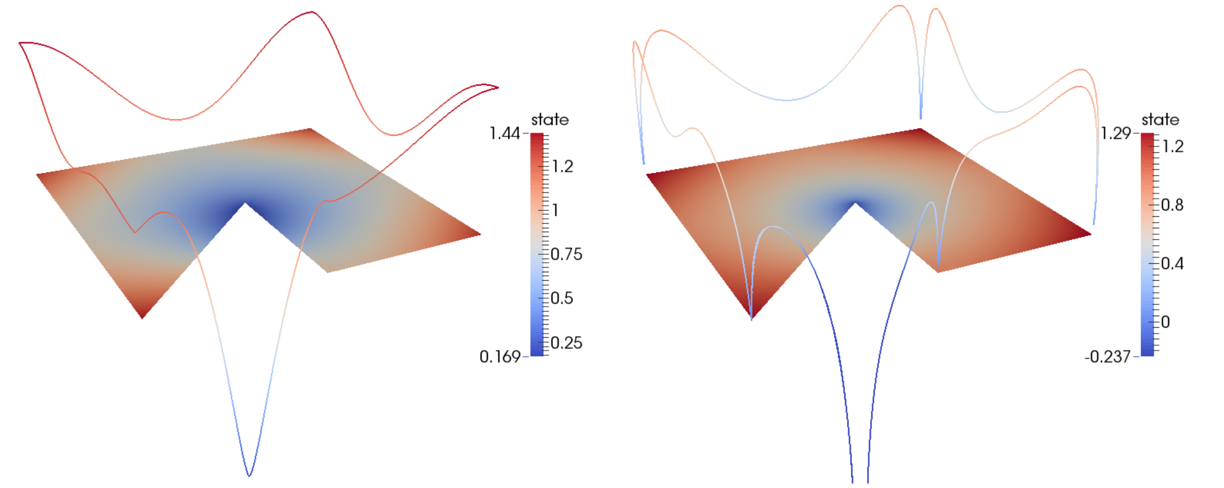

Figure 3: Left: solution of the optimal control problem in $L^2(\Gamma)$, right: solution of the optimal control problem in $H^{-1/2}(\Gamma)$; solid surface is the state, the curve on the boundary the optimal control.

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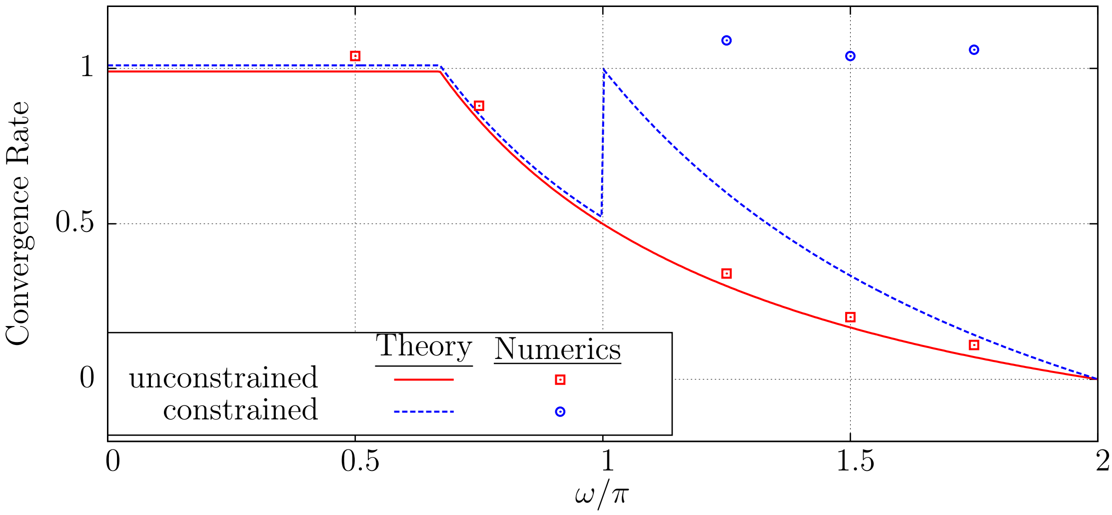

Figure 4: Illustration of the convergence rates for $\|z - z_h\|_{L^2(\Gamma)}$ for several angles; solid lines: theoretically predicted rate, quadrilaterals and circles: corresponding convergence rates obtained in the experiments.

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