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Coercive space–time finite element methods for initial boundary value problems

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Abstract

We propose and analyse new space–time Galerkin–Bubnov type finite element formulations of parabolic and hyperbolic second order partial differential equations in finite time intervals. This approach is based, using Hilbert type transformations, on elliptic reformulations of first and second order time derivatives, for which the Galerkin finite element discretisation results in positive definite and symmetric matrices. For the variational formulation of the heat and of the wave equation we prove related stability conditions in appropriate norms, and we discuss the stability of related finite element discretisations. Numerical results are given which confirm the theoretical results.

1 Introduction

While for the analysis of parabolic and hyperbolic partial differential equations a variety of approaches such as Fourier methods, semigroups, or Galerkin methods is available, see, for example, [19, 23, 24, 27, 37, 39], standard approaches for the numerical solution are based on semi–discretisations where the discretisation in space and time is splitted accordingly, see, e.g., [35] for parabolic partial differential equations, and [7, 8] for hyperbolic problems. More recently, there exist space–time approaches as for example in [1, 25, 26, 29, 32, 36] for parabolic problems, and [3, 5, 12, 15, 40] for hyperbolic equations.

In this work we introduce a new Fourier type method for the analysis of first and second order ordinary differential equations, and we transfer this approach to the corresponding parabolic and hyperbolic partial differential equations. The aim of this work is to provide space–time Galerkin–Bubnov type variational formulations where unique solvability follows from related coercivity estimates. This analysis may then serve not only as basis for the development and numerical analysis of adaptive space–time finite element methods simultaneously in space and time, and for the construction of time–parallel iterative solution strategies, but also for the analysis of related boundary integral equations methods for the heat and wave equation, respectively, and the coupling of finite and boundary element methods.

As a first model problem we consider the Dirichlet boundary value problem for the heat equation,

$$\left.\begin{array}{rcll}\alpha\partial_t u(x,t)-\Delta_x u(x,t) &=& f(x,t) & \text{for } (x,t)\in Q:=\Omega\times(0,T),\\ u(x,t) &=& 0 & \text{for } (x,t)\in\Sigma:=\Gamma\times(0,T),\\ u(x,0) &=& 0 & \text{for } x\in\Omega,\end{array}\right\} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$, $d=1,2,3$, is a bounded domain with, for $d=2,3$, Lipschitz boundary $\Gamma=\partial\Omega$, $\alpha>0$ is a given heat capacity constant, and $f(x,t)$ is a given right–hand side. Note that in the spatially one–dimensional case $d=1$ we have $\Omega=(a,b)$ and $\Gamma=\{a,b\}$.

A variational formulation of (1.1) is to find $u\in L^2(0,T;H^1_0(\Omega))\cap H^1_{0,}(0,T;H^{-1}(\Omega))$ such that

$$\int_0^T\int_\Omega \alpha\,\partial_t u(x,t)v(x,t)\,dx\,dt+\int_0^T\int_\Omega \nabla_x u(x,t)\cdot\nabla_x v(x,t)\,dx\,dt \tag{1.2}$$
$$=\int_0^T\int_\Omega f(x,t)v(x,t)\,dx\,dt$$

is satisfied for all $v\in L^2(0,T;H^1_0(\Omega))$, where we assume $f\in L^2(0,T;H^{-1}(\Omega))$. Note that we use the standard Bochner spaces, where $u\in H^1_{0,}(0,T;H^{-1}(\Omega))$ satisfies $u(x,0)=0$ for $x\in\Omega$. Related to the variational formulation (1.2) we introduce the bilinear form

$$a(u,v):=\int_0^T\int_\Omega\Big[\alpha\,\partial_t u(x,t)v(x,t)+\nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big]\,dx\,dt\,. \tag{1.3}$$

Since (1.2) is a Galerkin–Petrov variational formulation we need to establish an appropriate stability condition to ensure unique solvability, see also [13, 29, 32, 36]. A key ingredient in deriving the stability condition will be the definition of an appropriate norm as it is done in what follows.

For $u\in L^2(0,T;H^1_0(\Omega))\cap H^1_{0,}(0,T;H^{-1}(\Omega))$ we define $w\in L^2(0,T;H^1_0(\Omega))$ as the unique solution of the quasi–static Dirichlet boundary value problem

$$\left.\begin{array}{rcll}-\Delta_x w(x,t) &=& \alpha\partial_t u(x,t)-\Delta_x u(x,t) & \text{for } (x,t)\in\Omega\times(0,T),\\ w(x,t) &=& 0 & \text{for } (x,t)\in\Gamma\times(0,T),\end{array}\right\} \tag{1.4}$$

which is the unique solution of the variational formulation

$$\begin{aligned}\int_0^T\int_\Omega \nabla_x w(x,t)\cdot\nabla_x v(x,t)\,dx\,dt &= \int_0^T\int_\Omega\Big[\alpha\,\partial_t u(x,t)-\Delta_x u(x,t)\Big]v(x,t)\,dx\,dt\\ &=\int_0^T\int_\Omega\Big[\alpha\,\partial_t u(x,t)v(x,t)+\nabla_x u(x,t)\cdot\nabla_x v(x,t)\Big]dx\,dt=a(u,v)\end{aligned}$$

for all $v \in L^2(0,T;H^1_0(\Omega))$. When chosing $v = w$ this gives

$$\begin{aligned}
\|w\|^2_{L^2(0,T;H^1_0(\Omega))} &= \int_0^T \int_\Omega |\nabla_x w(x,t)|^2 \, dx \, dt \\
&= \int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) - \Delta_x u(x,t) \Big] w(x,t) \, dx \, dt \\
&\leq \|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} \|w\|_{L^2(0,T;H^1_0(\Omega))},
\end{aligned}$$

and by duality we also obtain

$$\begin{aligned}
\|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} &= \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega))} \frac{\langle \alpha \partial_t u - \Delta_x u, v \rangle_Q}{\|v\|_{L^2(0,T;H^1_0(\Omega))}} \\
= \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega))} \frac{\langle \nabla_x w, \nabla_x v \rangle_{L^2(Q)}}{\|\nabla_x v\|_{L^2(Q)}} &\leq \|\nabla_x w\|_{L^2(Q)} = \|w\|_{L^2(0,T;H^1_0(\Omega))}.
\end{aligned}$$

In particular we have

$$\|\alpha \partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} = \|w\|^2_{L^2(0,T;H^1_0(\Omega))} = a(u,w) \tag{1.5}$$

as the energy norm for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$. Indeed,

$$\|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} = 0 \quad \text{for} \quad u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$$

implies $u = 0$ since the homogeneous heat equation with zero Dirichlet boundary conditions and zero initial conditions has only the trivial solution. An immediate consequence of (1.5) is the stability estimate

$$\|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} \leq \sup_{0 \neq v \in L^2(0,T;H^1_0(\Omega))} \frac{a(u,v)}{\|v\|_{L^2(0,T;H^1_0(\Omega))}} \tag{1.6}$$

for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$. From (1.5) we further conclude

$$\begin{aligned}
\|w\|^2_{L^2(0,T;H^1_0(\Omega))} &= a(u,w) \\
&= \int_0^T \int_\Omega \Big[\alpha \partial_t u(x,t) w(x,t) + \nabla_x u(x,t) \cdot \nabla_x w(x,t) \Big] dx \, dt \\
&\leq \|\alpha \partial_t u\|_{L^2(0,T;H^{-1}(\Omega))} \|w\|_{L^2(0,T;H^1_0(\Omega))} + \|\nabla_x u\|_{L^2(Q)} \|\nabla_x w\|_{L^2(Q)} \\
&\leq \sqrt{2} \sqrt{\|\alpha \partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla_x u\|^2_{L^2(Q)}} \, \|w\|_{L^2(0,T;H^1_0(\Omega))},
\end{aligned}$$

i.e. for all $u \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ we have

$$\|\alpha \partial_t u - \Delta_x u\|_{L^2(0,T;H^{-1}(\Omega))} \leq \sqrt{2} \sqrt{\|\alpha \partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla_x u\|^2_{L^2(Q)}}. \tag{1.7}$$

For the reverse inequality and for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ we consider

$$\begin{aligned}
\|\alpha\partial_t u - \Delta_x u\|^2_{L^2(0,T;H^{-1}(\Omega))} &= a(u,w) = a(u,u) + a(u,w-u) \\
&= \int_0^T \int_\Omega \Big[\alpha\partial_t u\, u + \nabla_x u \cdot \nabla_x u\Big] dx\, dt + \int_0^T \int_\Omega \nabla_x w \cdot \nabla_x (w-u)\, dxdt \\
&= \frac{\alpha}{2}\, \|u(T)\|^2_{L^2(\Omega)} + \|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)} + \langle \nabla_x u, \nabla_x(w-u)\rangle_{L^2(Q)} \\
&\geq \|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)} - \|\nabla_x u\|_{L^2(Q)} \|\nabla_x(w-u)\|_{L^2(Q)} \\
&\geq \frac{1}{2}\, \Big[\|\nabla_x u\|^2_{L^2(Q)} + \|\nabla_x(w-u)\|^2_{L^2(Q)}\Big] \\
&= \frac{1}{2}\, \Big[\|\nabla_x u\|^2_{L^2(Q)} + \|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))}\Big]\,. \qquad (1.8)
\end{aligned}$$

In particular,

$$\|u\|_{L^2(0,T;H^1_0(\Omega))\cap H^1_{0,}(0,T;H^{-1}(\Omega))} := \sqrt{\|\alpha\partial_t u\|^2_{L^2(0,T;H^{-1}(\Omega))} + \|\nabla_x u\|^2_{L^2(Q)}} \qquad (1.9)$$

defines a norm in $L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ which is equivalent to (1.5), and from (1.6) we now conclude the stability condition

$$\frac{1}{\sqrt{2}}\, \|u\|_{L^2(0,T;H^1_0(\Omega))\cap H^1_{0,}(0,T;H^{-1}(\Omega))} \leq \sup_{0\neq v\in L^2(0,T;H^1_0(\Omega))} \frac{a(u,v)}{\|v\|_{L^2(0,T;H^1_0(\Omega))}} \qquad (1.10)$$

for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$. Note that in [32] the norm (1.9) was used within a finite element analysis of the variational formulation (1.2). In particular, the bilinear form (1.3) is continuous satisfying

$$|a(u,v)| \leq \sqrt{2}\, \|u\|_{L^2(0,T;H^1_0(\Omega))\cap H^1_{0,}(0,T;H^{-1}(\Omega))} \|v\|_{L^2(0,T;H^1_0(\Omega))} \qquad (1.11)$$

for $u \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and $v \in L^2(0,T;H^1_0(\Omega))$.

For $v \in L^2(0,T;H^1_0(\Omega))$ we finally define

$$\widetilde{u}(x,t) := \int_0^t v(x,s)\, ds \quad \text{for } x \in \Omega,\ t \in [0,T].$$

By definition we have $\widetilde{u} \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$. Then,

$$\begin{aligned}
a(\widetilde{u},v) &= \int_0^T \int_\Omega \Big[\alpha\partial_t \widetilde{u}(x,t) v(x,t) + \nabla_x \widetilde{u}(x,t) \cdot \nabla_x v(x,t)\Big]\, dx\, dt \\
&= \int_0^T \int_\Omega \Big[\alpha\, [\partial_t \widetilde{u}(x,t)]^2 + \nabla_x \widetilde{u}(x) \cdot \nabla_x \partial_t \widetilde{u}(x,t)\Big]\, dx\, dt \\
&= \alpha\, \|\partial_t \widetilde{u}\|^2_{L^2(Q)} + \frac{1}{2}\, \|\nabla_x \widetilde{u}(T)\|^2_{L^2(\Omega)} \ >\ 0. \qquad (1.12)
\end{aligned}$$

By using the stability condition (1.10), the continuity (1.11), and surjectivity (1.12) this implies unique solvability of the variational problem (1.2), see, e.g., [6, 13]. The initial Dirichlet boundary value problem (1.1) therefore defines an isomorphism

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega)) \to [L^2(0,T;H^1_0(\Omega))]' \,. \tag{1.13}$$

When considering the variational formulation (1.2) and doing integration by parts in time, this leads to the adjoint variational formulation to find $u \in L^2(0,T;H^1_0(\Omega))$ such that

$$\begin{aligned} -\int_0^T \int_\Omega u(x,t)\alpha\partial_t v(x,t)\,dx\,dt + \int_0^T \int_\Omega \nabla_x u(x,t) \cdot \nabla_x v(x,t)\,dx\,dt \\ = \int_0^T \int_\Omega f(x,t)v(x,t)\,dx\,dt \end{aligned} \tag{1.14}$$

is satisfied for all $v \in L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))$, where the test space now includes the final time condition $v(x,T) = 0$ for $x \in \Omega$, and where we now assume $f \in [L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]'$. As for the primal variational formulation (1.2) we can establish unique solvability of the adjoint variational formulation (1.14), which then implies an isomorphism

$$\mathcal{L} : L^2(0,T;H^1_0(\Omega)) \to [L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;H^{-1}(\Omega))]'. \tag{1.15}$$

Both the primal variational formulation (1.2) and the adjoint variational formulation (1.14) are Galerkin–Petrov formulations where the test space is different from the ansatz space, in particular with respect to time. This motivates to consider variational formulations for the initial boundary value problem (1.1) where ansatz and test spaces are of the same order also in time. Using the isomorphisms (1.13) and (1.15) and some interpolation arguments one expects to consider test and ansatz spaces as subspaces of the anisotropic Sobolev space $H^{1,1/2}(Q)$, e.g., [4, 19, 20, 23, 24]. In the case of an infinite time interval, i.e. $T = \infty$, such an approach was considered analytically in the PhD thesis of M. Fontes, [14], see also [21] for a related numerical analysis using wavelets. However, here we will consider only finite time intervals with $T < \infty$. In the case of time–periodic boundary value problems, a related approach is considered in [22].

Although the numerical analysis of space–time finite element methods for the variational formulation (1.2) is well established, see, e.g., [1, 25, 26, 29, 32, 36], the analysis of boundary integral equations and related boundary element methods for the solution of the heat equation (1.1) relies on Galerkin–Bubnov variational formulations in anisotropic Sobolev trace spaces of $H^{1,1/2}(Q)$, see, e.g., [2, 4]. In particular, instead of the stability condition (1.6) in the finite element analysis, an ellipticity estimate in the boundary element analysis is used. So we are interested in a unified approach to analyse both finite and boundary element methods within one framework, and allowing a numerical analysis also for the coupling of space–time finite and boundary element methods.

In addition to the initial boundary value problem (1.1) of the heat equation we also consider the related model problem for the wave equation,

$$\left.\begin{array}{rcll} \frac{1}{c^2}\partial_{tt}u(x,t) - \Delta_x u(x,t) & = & f(x,t) & \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) & = & 0 & \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) = \partial_t u(x,t)_{|t=0} & = & 0 & \text{for } x \in \Omega. \end{array}\right\} \quad (1.16)$$

A standard approach for a space–time finite element method to solve (1.16) is to consider an equivalent system with first order time derivatives, see, e.g., [3, 12]. Alternatively, one may consider variational formulations of the wave equation in (1.16) using integration by parts also in time, see, e.g., [5, 15, 40]. Here we will consider related variational formulations in suitable subspaces of $H^1(Q)$, and we will prove and discuss stability conditions in appropriate function spaces.

The rest of this paper is organised as follows: In Sect. 2 we consider simple first order ordinary differential equations to motivate the choice of a transformation operator to derive an elliptic and symmetric bilinear form for the first order time derivative. We discuss several properties of the Hilbert type transformation operator and we present some numerical results to illustrate the theoretical results. The results for the first order ordinary differential equations are extended in Sect. 3 to the heat equation in several space dimensions. We prove that the heat partial differential operator with zero Dirichlet boundary and initial conditions defines an isomorphism in certain anisotropic Sobolev spaces, implying a stability condition as required in the numerical analysis of the proposed Galerkin scheme. We comment on the stability of the numerical scheme and present some numerical results. Second order ordinary differential equations are considered in Sect. 4 where we introduce a different transformation operator which is not semi–definite as in the case of first order equations. Hence we have to use different Sobolev norms to establish optimal stability estimates. As for the first order equations we provide a numerical analysis for the finite element discretisation, and we give some numerical results. Finally, in Sect. 5 we consider the space–time variational formulation for the wave equation, we discuss the discretisation scheme, and we provide some numerical results for illustration.

2 First order ordinary differential equations

As a first model problem we consider for $T > 0$ the simple initial value problem

$$\partial_t u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \qquad (2.1)$$

where we aim to derive and to analyse a coercive variational formulation which later will be used for the discretisation of time–dependent partial differential equations which are of first order in time.

2.1 Primal variational formulation

If we define the Sobolev space

$$H^1_{0,}(0,T) := \Big\{ v \in H^1(0,T) \colon\ v(0) = 0 \Big\},$$

then the primal variational formulation of (2.1) is to find $u \in H^1_{0,}(0,T)$ such that

$$\int_0^T \partial_t u(t) v(t)\, dt = \int_0^T f(t) v(t)\, dt \quad \text{for all } v \in L^2(0,T). \tag{2.2}$$

Obviously it is sufficient to assume $f \in L^2(0,T)$ in this case. Recall that

$$\|u\|^2_{H^1_{0,}(0,T)} := \|\partial_t u\|^2_{L^2(0,T)} = \int_0^T [\partial_t u(t)]^2\, dt$$

defines a norm in $H^1_{0,}(0,T)$.

Lemma 2.1 *The bilinear form $a(\cdot,\cdot) : H^1_{0,}(0,T) \times L^2(0,T) \to \mathbb{R}$,*

$$a(u,v) := \int_0^T \partial_t u(t) v(t)\, dt, \tag{2.3}$$

is bounded, i.e.

$$|a(u,v)| \leq \|\partial_t u\|_{L^2(0,T)} \|v\|_{L^2(0,T)} \quad \textit{for all } u \in H^1_{0,}(0,T),\ v \in L^2(0,T), \tag{2.4}$$

and satisfies the stability condition

$$\|\partial_t u\|_{L^2(0,T)} \leq \sup_{0 \neq v \in L^2(0,T)} \frac{a(u,v)}{\|v\|_{L^2(0,T)}} \quad \textit{for all } u \in H^1_{0,}(0,T). \tag{2.5}$$

Moreover, it holds

$$\|v\|_{L^2(0,T)} \leq \sup_{0 \neq u \in H^1_{0,}(0,T)} \frac{a(u,v)}{\|\partial_t u\|_{L^2(0,T)}} \quad \textit{for all } v \in L^2(0,T). \tag{2.6}$$

Proof. The boundedness estimate (2.4) is a direct consequence of the Cauchy–Schwarz inequality. To prove (2.5), for any $0 \neq u \in H^1_{0,}(0,T)$ we choose $\overline{v} = \partial_t u \in L^2(0,T)$ to obtain

$$a(u,\overline{v}) = \int_0^T \partial_t u(t) \overline{v}(t)\, dt = \int_0^T [\partial_t u(t)]^2\, dt = \|\partial_t u\|^2_{L^2(0,T)} = \|\partial_t u\|_{L^2(0,T)} \|\overline{v}\|_{L^2(0,T)},$$

i.e.

$$\|\partial_t u\|_{L^2(0,T)} = \frac{a(u,\overline{v})}{\|\overline{v}\|_{L^2(0,T)}} \leq \sup_{0 \neq v \in L^2(0,T)} \frac{a(u,v)}{\|v\|_{L^2(0,T)}}.$$

Finally, for $0 \neq v \in L^2(0,T)$ we define $\overline{u} \in H^1_{0,}(0,T)$,

$$\overline{u}(t) = \int_0^t v(s)\, ds, \quad \partial_t \overline{u}(t) = v(t) \quad \text{for } t \in [0,T].$$

Then,

$$a(\overline{u}, v) = \int_0^T \partial_t \overline{u}(t) v(t)\, dt = \int_0^T [v(t)]^2\, dt = \|v\|^2_{L^2(0,T)} = \|v\|_{L^2(0,T)} \|\partial_t \overline{u}\|_{L^2(0,T)},$$

implying the stability condition (2.6), i.e.

$$\|v\|_{L^2(0,T)} = \frac{a(\overline{u}, v)}{\|\partial_t \overline{u}\|_{L^2(0,T)}} \leq \sup_{0 \neq u \in H^1_{0,}(0,T)} \frac{a(u,v)}{\|\partial_t u\|_{L^2(0,T)}}.$$

□

As a consequence of Lemma 2.1, see, e.g., [6, Satz 3.6] or [13, Corollary A.45], we conclude unique solvability of the primal variational formulation (2.2), and the bilinear form (2.3) implies, by using the Riesz representation theorem, a bounded and invertible operator

$$B_1 \colon\; H^1_{0,}(0,T) \to L^2(0,T),$$

satisfying

$$\|u\|_{H^1_{0,}(0,T)} \leq \|B_1 u\|_{L^2(0,T)} \quad \text{for all } u \in H^1_{0,}(0,T).$$

2.2 Dual variational formulation

When using integration by parts, instead of the primal variational formulation (2.2) we may consider the dual variational formulation to find $u \in L^2(0,T)$ such that

$$\int_0^T u(t) \partial_t v(t)\, dt = -\int_0^T f(t) v(t)\, dt \quad \text{for all } v \in H^1_{,0}(0,T), \tag{2.7}$$

where

$$H^1_{,0}(0,T) := \Big\{ v \in H^1(0,T) \colon\; v(T) = 0 \Big\}, \quad \|v\|^2_{H^1_{,0}(0,T)} := \int_0^T [\partial_t v(t)]^2\, dt.$$

Now it is sufficient to assume $f \in [H^1_{,0}(0,T)]'$.

Lemma 2.2 *The bilinear form* $a(\cdot,\cdot) : L^2(0,T) \times H^1_{,0}(0,T) \to \mathbb{R}$,

$$a(u,v) := \int_0^T u(t) \partial_t v(t)\, dt,$$

is bounded, i.e.

$$|a(u,v)| \leq \|u\|_{L^2(0,T)} \|\partial_t v\|_{L^2(0,T)} \quad \textit{for all } u \in L^2(0,T),\ v \in H^1_{,0}(0,T), \tag{2.8}$$

and satisfies the stability condition

$$\|u\|_{L^2(0,T)} \leq \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{a(u,v)}{\|\partial_t v\|_{L^2(0,T)}} \quad \textit{for all } u \in L^2(0,T). \tag{2.9}$$

Moreover, it holds

$$\|\partial_t v\|_{L^2(0,T)} \leq \sup_{0 \neq u \in L^2(0,T)} \frac{a(u,v)}{\|u\|_{L^2(0,T)}} \quad \textit{for all } v \in H^1_{,0}(0,T). \tag{2.10}$$

Proof. The estimate (2.8) is again a consequence of the Cauchy–Schwarz inequality. To prove (2.9), for any $0 \neq u \in L^2(0,T)$ we define $\overline{v} \in H^1_{,0}(0,T)$,

$$\overline{v}(t) = -\int_t^T u(s)\, ds, \quad \partial_t \overline{v}(t) = u(t) \quad \text{for } t \in [0,T].$$

Then,

$$a(u,\overline{v}) = \int_0^T u(t)\partial_t\overline{v}(t)\, dt = \int_0^T [u(t)]^2\, dt = \|u\|^2_{L^2(0,T)} = \|u\|_{L^2(0,T)}\|\partial_t\overline{v}\|_{L^2(0,T)},$$

implying the stability condition (2.9). To prove (2.10), for any $0 \neq v \in H^1_{,0}(0,T)$ we choose $\overline{u} = \partial_t v \in L^2(0,T)$ to obtain

$$a(\overline{u},v) = \int_0^T \overline{u}(t)\partial_t v(t)\, dt = \int_0^T [\partial_t v(t)]^2\, dt = \|\partial_t v\|^2_{L^2(0,T)} = \|\overline{u}\|_{L^2(0,T)}\|\partial_t v\|_{L^2(0,T)}.$$

□

As for the primal variational formulation we conclude unique solvability of the dual variational formulation (2.7), which then implies a bounded and invertible operator

$$B_0 \colon\ L^2(0,T) \to [H^1_{,0}(0,T)]',$$

satisfying

$$\|u\|_{L^2(0,T)} \leq \|B_0 u\|_{[H^1_{,0}(0,T)]'} \quad \text{for all } u \in L^2(0,T).$$

2.3 Interpolation of operators

Related to the initial value problem (2.1) we consider the operator $B_1 \colon\ H^1_{0,}(0,T) \to L^2(0,T)$ of the primal formulation (2.2), and the operator $B_0 \colon\ L^2(0,T) \to [H^1_{,0}(0,T)]'$ of the dual formulation (2.7). Hence, using interpolation arguments we may consider for $s \in (0,1)$ an operator

$$B_s \colon\ [H^1_{0,}(0,T), L^2(0,T)]_s \to [L^2(0,T), [H^1_{,0}(0,T)]']_s,$$

and we may ask for a representation of B_s, in particular for $s = \frac{1}{2}$. Recall that the Sobolev space

$$H^{1/2}_{0,}(0,T) := [H^1_{0,}(0,T), L^2(0,T)]_{1/2}$$

is a dense subspace of $H^{1/2}(0,T)$ with norm

$$\|u\|^2_{H^{1/2}_{0,}(0,T)} = \int_0^T [u(t)]^2\, dt + \int_0^T \int_0^T \frac{[u(s)-u(t)]^2}{|s-t|^2}\, ds\, dt + \int_0^T \frac{[u(t)]^2}{t}\, dt.$$

For $B_1 \colon\ H^1_{0,}(0,T) \to L^2(0,T)$ we define the adjoint operator $B_1' \colon\ L^2(0,T) \to [H^1_{0,}(0,T)]'$ via

$$\langle u, B_1' v\rangle_{(0,T)} = \langle B_1 u, v\rangle_{L^2(0,T)} \quad \text{for all } u \in H^1_{0,}(0,T),\ v \in L^2(0,T),$$

where $\langle \cdot, \cdot \rangle_{(0,T)}$ denotes the duality pairing as extension of the inner product in $L^2(0,T)$. Then we introduce

$$A := B_1' B_1 \colon \; H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'.$$

In particular for $u \in H^1_{0,}(0,T)$ we may consider the eigenvalue problem

$$Au = \lambda u \quad \text{in } [H^1_{0,}(0,T)]',$$

i.e. for all $v \in H^1_{0,}(0,T)$ we have

$$\langle Au, v \rangle_{(0,T)} = \langle B_1 u, B_1 v \rangle_{L^2(0,T)} = \int_0^T \partial_t u(t) \partial_t v(t) \, dt = \lambda \int_0^T u(t) v(t) \, dt.$$

Note that this is the variational formulation of an eigenvalue problem with mixed boundary conditions,

$$-\partial_{tt} u(t) = \lambda u(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \quad \partial_t u(T) = 0.$$

Hence we find

$$v_k(t) = \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad \lambda_k = \frac{1}{T^2} \left(\frac{\pi}{2} + k\pi \right)^2, \quad k = 0, 1, 2, 3, \ldots. \tag{2.11}$$

Recall that the eigenfunctions v_k form an orthogonal basis in $L^2(0,T)$ satisfying

$$\int_0^T v_k(t) v_\ell(t) \, dt = \frac{T}{2} \, \delta_{k\ell}, \tag{2.12}$$

and in $H^1_{0,}(0,T)$,

$$\int_0^T \partial_t v_k(t) \partial_t v_\ell(t) \, dt = \lambda_k \int_0^T v_k(t) v_\ell(t) \, dt = \frac{1}{2T} \left(\frac{\pi}{2} + k\pi \right)^2 \delta_{k\ell}.$$

This motivates to consider for $u \in H^1_{0,}(0,T)$

$$u(t) = \sum_{k=0}^\infty u_k \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad u_k = \frac{2}{T} \int_0^T u(t) \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) dt, \tag{2.13}$$

and by Parseval's identity we have

$$\begin{aligned} \|u\|^2_{L^2(0,T)} &= \sum_{k=0}^\infty \sum_{\ell=0}^\infty u_k u_\ell \int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \sin\left(\left(\frac{\pi}{2} + \ell\pi \right) \frac{t}{T} \right) dt \\ &= \frac{T}{2} \sum_{k=0}^\infty u_k^2, \end{aligned} \tag{2.14}$$

as well as

$$\|\partial_t u\|^2_{L^2(0,T)} = \sum_{k=0}^\infty \sum_{\ell=0}^\infty u_k u_\ell \int_0^T \partial_t v_k(t) \partial_t v_\ell(t) \, dt = \frac{1}{2T} \sum_{k=0}^\infty \left(\frac{\pi}{2} + k\pi \right)^2 u_k^2. \tag{2.15}$$

Hence, using interpolation, we define an equivalent norm in $H^{1/2}_{0,}(0,T)$, e.g., [23, 38],

$$\|u\|^2_{H^{1/2}_{0,}(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)u_k^2, \tag{2.16}$$

as well as an inner product,

$$\langle u,v\rangle_{H^{1/2}_{0,}(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)u_k v_k.$$

Analogously, we consider for $w \in H^{1/2}_{,0}(0,T)$,

$$w(t) = \sum_{k=0}^{\infty}\overline{w}_k \cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right), \quad \overline{w}_k = \frac{2}{T}\int_0^T w(t)\cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)\,dt,$$

with the related norm and inner product,

$$\|w\|^2_{H^{1/2}_{,0}(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)\overline{w}_k^2, \quad \langle w,z\rangle_{H^{1/2}_{,0}(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)\overline{w}_k\overline{z}_k.$$

Finally we introduce the dual space $[H^{1/2}_{,0}(0,T)]'$ with the norm

$$\|f\|_{[H^{1/2}_{,0}(0,T)]'} = \sup_{0\neq w\in H^{1/2}_{,0}(0,T)}\frac{\langle f,w\rangle_{(0,T)}}{\|w\|_{H^{1/2}_{,0}(0,T)}}.$$

Lemma 2.3 *For $f \in [H^{1/2}_{,0}(0,T)]'$ we have*

$$\|f\|^2_{[H^{1/2}_{,0}(0,T)]'} = \frac{T^2}{2}\sum_{k=0}^{\infty}\Big(\frac{\pi}{2}+k\pi\Big)^{-1}\overline{f}_k^2 \tag{2.17}$$

with

$$\overline{f}_k = \frac{2}{T}\langle f,w_k\rangle_{(0,T)}, \quad w_k(t) = \cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right).$$

Proof. From the norm definition, using a series representation of $w \in H^{1/2}_{,0}(0,T)$, and with Hölder's inequality we first have

$$\begin{aligned}
\|f\|_{[H^{1/2}_{,0}(0,T)]'} &= \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{\langle f,w\rangle_{(0,T)}}{\|w\|_{H^{1/2}_{,0}(0,T)}} \\
&= \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{\displaystyle\sum_{k=0}^{\infty} \overline{w}_k \langle f, w_k\rangle_{(0,T)}}{\left(\displaystyle\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)\overline{w}_k^2\right)^{1/2}} \\
&= \frac{T}{\sqrt{2}} \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{\displaystyle\sum_{k=0}^{\infty} \overline{w}_k \overline{f}_k}{\left(\displaystyle\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)\overline{w}_k^2\right)^{1/2}} \leq \frac{T}{\sqrt{2}}\left(\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k^2\right)^{1/2},
\end{aligned}$$

i.e.

$$\|f\|^2_{[H^{1/2}_{,0}(0,T)]'} \leq \frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k^2 .$$

On the other hand, if the coefficients $\overline{f}_k$ are given, we define

$$\overline{w}_k^* = \left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k$$

and obtain

$$\begin{aligned}
&\left(\frac{T^2}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k^2\right)^{1/2} \\
&\qquad = \frac{T}{\sqrt{2}} \frac{\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k^2}{\left(\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)^{-1}\overline{f}_k^2\right)^{1/2}} = \frac{\sum_{k=0}^{\infty}\overline{f}_k\overline{w}_k^*\frac{T}{2}}{\left(\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)[\overline{w}_k^*]^2\right)^{1/2}} \\
&\qquad = \frac{\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\overline{f}_k\overline{w}_\ell^*\displaystyle\int_0^T \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)\,\cos\left(\left(\frac{\pi}{2}+\ell\pi\right)\frac{t}{T}\right)\,dt}{\left(\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2}+k\pi\right)[\overline{w}_k^*]^2\right)^{1/2}} \\
&\qquad = \frac{\langle f, w^*\rangle_{(0,T)}}{\|w^*\|_{H^{1/2}_{,0}(0,T)}} \leq \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{\langle f,w\rangle_{(0,T)}}{\|w\|_{H^{1/2}_{,0}(0,T)}} = \|f\|_{[H^{1/2}_{,0}(0,T)]'} .
\end{aligned}$$

This concludes the proof. $\square$

The variational formulation of the initial value problem (2.1) is to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, w\rangle_{(0,T)} = \langle f, w\rangle_{(0,T)} \quad \text{for all } w \in H^{1/2}_{,0}(0,T), \tag{2.18}$$

where $f \in [H^{1/2}_{,0}(0,T)]'$ is given. Note that (2.18) is a Galerkin–Petrov variational formulation with different trial and test spaces. Hence we have to establish an appropriate stability condition which is equivalent to an ellipticity estimate for the bilinear form $\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)}$ with some transformation operator $\mathcal{H}_T\colon\ H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ to be specified.

2.4 Transformation operator

To motivate the particular definition of the operator $\mathcal{H}_T\colon\ H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ we write, by using (2.13),

$$\partial_t u(t) = \frac{1}{T}\sum_{k=0}^\infty u_k\Big(\frac{\pi}{2}+k\pi\Big)\cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)$$

as distributional derivative, i.e. for $w \in H^{1/2}_{,0}(0,T)$ we have

$$\langle \partial_t u, w\rangle_{(0,T)} = \frac{1}{T}\int_0^T \sum_{k=0}^\infty u_k\Big(\frac{\pi}{2}+k\pi\Big)\cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)\, w(t)\,dt.$$

If we define

$$w(t) = (\mathcal{H}_T u)(t) := \sum_{\ell=0}^\infty u_\ell \cos\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right), \tag{2.19}$$

we conclude the ellipticity estimate

$$\begin{aligned}\langle \partial_t u, \mathcal{H}_T u\rangle_{(0,T)} &= \frac{1}{T}\sum_{k=0}^\infty\sum_{\ell=0}^\infty u_k u_\ell\Big(\frac{\pi}{2}+k\pi\Big)\int_0^T \cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)\cos\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right)\,dt \\ &= \frac{1}{2}\sum_{k=0}^\infty\Big(\frac{\pi}{2}+k\pi\Big)u_k^2 \;=\; \|u\|^2_{H^{1/2}_{0,}(0,T)}. \end{aligned}\tag{2.20}$$

Remark 2.1 *$\mathcal{H}_T u \in H^{1/2}_{,0}(0,T)$ as given in (2.19) is the unique solution of the variational problem*

$$\langle \mathcal{H}_T u, z\rangle_{H^{1/2}_{,0}(0,T)} = \langle \partial_t u, z\rangle_{(0,T)} \quad \textit{for all } z \in H^{1/2}_{,0}(0,T).$$

The definition of the transformation operator $\mathcal{H}_T$ therefore coincides with the definition of the optimal test space as used, e.g. in discontinuous Galerkin–Petrov methods [9]. Indeed, for

$$u(t) = \sum_{k=0}^\infty u_k \sin\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}$$

we use the ansatz

$$w(t) = (\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} \overline{w}_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right),$$

and the test function

$$z(t) = \sum_{k=0}^{\infty} \overline{z}_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)$$

to obtain

$$\langle w, z\rangle_{H^{1/2}_{,0}(0,T)} = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right)\overline{w}_k\overline{z}_k = \langle \partial_t u, z\rangle = \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{\pi}{2} + k\pi\right)u_k\overline{z}_k$$

for all $\overline{z}_k$, from which we conclude $\overline{w}_k = u_k$ for $k = 0, 1, 2, \ldots$.

By construction we have $w = \mathcal{H}_T u \in H^{1/2}_{,0}(0,T)$, and $\mathcal{H}_T \colon H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ is norm preserving, i.e.

$$\|\mathcal{H}_T u\|_{H^{1/2}_{,0}(0,T)} = \|u\|_{H^{1/2}_{0,}(0,T)} \quad \text{for all } u \in H^{1/2}_{0,}(0,T).$$

Vice versa, if $w \in H^{1/2}_{,0}(0,T)$ is given,

$$w(t) = \sum_{k=0}^{\infty} \overline{w}_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \overline{w}_k = \frac{2}{T}\int_0^T w(t)\cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\,dt,$$

the inverse transformation operator reads

$$u(t) = (\mathcal{H}_T^{-1} w)(t) = \sum_{k=0}^{\infty} \overline{w}_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right).$$

Next, we are going to prove some properties of the transformation operator $\mathcal{H}_T$. First, we consider a commutation property with the time derivative operator ∂_t.

Lemma 2.4 *For $u \in H^{1/2}_{0,}(0,T)$ we have*

$$\langle \partial_t \mathcal{H}_T u, v\rangle_{(0,T)} = -\langle \mathcal{H}_T^{-1}\partial_t u, v\rangle_{(0,T)} \quad \textit{for all } v \in H^{1/2}_{0,}(0,T).$$

Proof. For an arbitrary $\varphi \in C^\infty([0,T])$ with $\varphi(0) = 0$ we first compute

$$(\mathcal{H}_T\varphi)(t) = \sum_{k=0}^{\infty} \varphi_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \varphi_k = \frac{2}{T}\int_0^T \varphi(t)\sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\,dt,$$

and therefore

$$\partial_t(\mathcal{H}_T\varphi)(t) = -\frac{1}{T}\sum_{k=0}^{\infty}\varphi_k\Big(\frac{\pi}{2}+k\pi\Big)\sin\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)$$

follows. On the other hand,

$$\partial_t\varphi(t) = \frac{1}{T}\sum_{k=0}^{\infty}\varphi_k\Big(\frac{\pi}{2}+k\pi\Big)\cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)$$

implies

$$(\mathcal{H}_T^{-1}\partial_t\varphi)(t) = \frac{1}{T}\sum_{k=0}^{\infty}\varphi_k\Big(\frac{\pi}{2}+k\pi\Big)\sin\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right),$$

i.e.

$$\partial_t\mathcal{H}_T\varphi = -\mathcal{H}_T^{-1}\partial_t\varphi \quad \text{for all } \varphi\in C^\infty([0,T]) \text{ with } \varphi(0)=0.$$

Now the assertion follows by completion. □

Next we prove that $\mathcal{H}_T$ is unitary.

Lemma 2.5 *For $u \in H^{1/2}_{0,}(0,T)$ and $w \in H^{1/2}_{,0}(0,T)$ there holds*

$$\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} = \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}.$$

Proof. For $u \in H^{1/2}_{0,}(0,T)$ and $w \in H^{1/2}_{,0}(0,T)$ we have

$$u(t) = \sum_{k=0}^{\infty} u_k \sin\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right), \quad w(t) = \sum_{\ell=0}^{\infty}\overline{w}_\ell\cos\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right),$$

and

$$(\mathcal{H}_T u)(t) = \sum_{k=0}^{\infty} u_k \cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right), \quad (\mathcal{H}_T^{-1} w)(t) = \sum_{\ell=0}^{\infty}\overline{w}_\ell\sin\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right).$$

Hence we compute

$$\begin{aligned}
\langle \mathcal{H}_T u, w\rangle_{L^2(0,T)} &= \int_0^T (\mathcal{H}_T u)(t) w(t)\, dt \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} u_k\overline{w}_\ell \int_0^T \cos\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)\, \cos\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right)\, dt \\
&= \frac{T}{2}\sum_{k=0}^{\infty} u_k\overline{w}_k \\
&= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} u_k\overline{w}_\ell \int_0^T \sin\left(\Big(\frac{\pi}{2}+k\pi\Big)\frac{t}{T}\right)\, \sin\left(\Big(\frac{\pi}{2}+\ell\pi\Big)\frac{t}{T}\right)\, dt \\
&= \int_0^T u(t)(\mathcal{H}_T^{-1} w)(t)\, dt \;=\; \langle u, \mathcal{H}_T^{-1} w\rangle_{L^2(0,T)}.
\end{aligned}$$

□

Using Lemma 2.4 and Lemma 2.5 we conclude the following symmetry relation.

Corollary 2.6 *For $u, v \in H^{1/2}_{0,}(0,T)$ there holds*

$$\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} = \langle \mathcal{H}_T u, \partial_t v\rangle_{(0,T)} = \langle u, v\rangle_{H^{1/2}_{0,}(0,T)}.$$

Proof. For $\varphi, \psi \in C^\infty([0,T])$ with $\varphi(0) = \psi(0) = 0$ we first have $\mathcal{H}_T\varphi(T) = \mathcal{H}_T\psi(T) = 0$ and therefore

$$\begin{aligned}
\langle \partial_t\varphi, \mathcal{H}_T\psi\rangle_{L^2(0,T)} &= \langle \mathcal{H}_T^{-1}\partial_t\varphi, \psi\rangle_{L^2(0,T)} \\
&= -\langle \partial_t\mathcal{H}_T\varphi, \psi\rangle_{L^2(0,T)} \\
&= -(\mathcal{H}_T\varphi)(t)\psi(t)\Big|_0^T + \langle \mathcal{H}_T\varphi, \partial_t\psi\rangle_{L^2(0,T)} \\
&= \langle \mathcal{H}_T\varphi, \partial_t\psi\rangle_{L^2(0,T)}
\end{aligned}$$

holds. Now the assertion follows by completion. □

The next property of $\mathcal{H}_T$ is required when considering, instead of (2.1), more general differential equations.

Lemma 2.7 *There holds*

$$\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} \geq 0 \quad \textit{for all } v \in H^{1/2}_{0,}(0,T). \tag{2.21}$$

Proof. By using

$$v(t) = \sum_{k=0}^\infty v_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad (\mathcal{H}_T v)(t) = \sum_{\ell=0}^\infty v_\ell \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right),$$

we have

$$\begin{aligned}
\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &= \sum_{k=0}^\infty\sum_{\ell=0}^\infty v_k v_\ell \int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\, \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right)\, dt \\
&= \frac{1}{2}\sum_{k=0}^\infty\sum_{\ell=0}^\infty v_k v_\ell \int_0^T \left[\sin\left((k+\ell+1)\pi\frac{t}{T}\right) + \sin\left((k-\ell)\pi\frac{t}{T}\right)\right]\, dt \\
&= \frac{1}{2}\sum_{k=0}^\infty\sum_{\ell=0}^\infty v_k v_\ell \left[-\frac{T}{(k+\ell+1)\pi}\cos\left((k+\ell+1)\pi\frac{t}{T}\right)\right]_0^T \\
&= \frac{T}{2\pi}\sum_{k=0}^\infty\sum_{\ell=0}^\infty v_k v_\ell \frac{1}{k+\ell+1}\left[1 - (-1)^{k+\ell+1}\right],
\end{aligned}$$

where the second integral is ignored due to symmetry. When splitting k and ℓ into odd and even indices, i.e. $k = 2i, 2i+1,\ \ell = 2j, 2j+1$, this gives

$$\begin{aligned}\langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} &= \frac{T}{\pi}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\frac{v_{2i}v_{2j}}{2i+2j+1} + \frac{v_{2i+1}v_{2j+1}}{2i+2j+3}\right] \\ &= \frac{T}{\pi}\lim_{M\to\infty}\sum_{i=0}^{M}\sum_{j=0}^{M}\left[v_{2i}v_{2j}\int_0^1 x^{2i+2j}\,dx + v_{2i+1}v_{2j+1}\int_0^1 x^{2i+2j+2}\,dx\right] \\ &= \frac{T}{\pi}\lim_{M\to\infty}\left[\int_0^1\left(\sum_{i=0}^{M} v_{2i}x^{2i}\right)^2 dx + \int_0^1\left(\sum_{i=0}^{M} v_{2i+1}x^{2i+1}\right)^2 dx\right] \geq 0. \qquad \square\end{aligned}$$

Remark 2.2 *The matrix H as used in the previous proof, i.e.*

$$H[j,i] = \frac{1}{i+j+1} \quad \textit{for } i,j = 0,1,\ldots,N,$$

is a Hilbert matrix [17] *which is positive definite, but ill conditioned. But for our purpose it is sufficient to use that* (2.21) *is non–negative.*

Next we will have a closer look on the definition of the transformation operator $\mathcal{H}_T$ to see its relation with the well known Hilbert transform, see, e.g., [18].

Lemma 2.8 *The operator $\mathcal{H}_T$ as defined in* (2.19) *allows the integral representation*

$$(\mathcal{H}_T u)(t) = \int_0^T K(s,t)u(s)\,ds$$

as a Cauchy principal value integral where the kernel function is given as

$$K(s,t) = \frac{1}{2T}\left[\frac{1}{\sin\left(\frac{\pi}{2}\frac{s-t}{T}\right)} + \frac{1}{\sin\left(\frac{\pi}{2}\frac{s+t}{T}\right)}\right]. \tag{2.22}$$

Proof. In fact,

$$\begin{aligned}(\mathcal{H}_T u)(t) &= \sum_{k=0}^{\infty} u_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\ &= \sum_{k=0}^{\infty}\frac{2}{T}\int_0^T u(s)\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s}{T}\right)ds\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\ &= \int_0^T u(s)K(s,t)\,ds\end{aligned}$$

with

$$
\begin{aligned}
K(s,t) &= \frac{2}{T}\sum_{k=0}^{\infty} \sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s}{T}\right)\,\cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\
&= \frac{1}{T}\sum_{k=0}^{\infty}\left[\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s-t}{T}\right)+\sin\left(\left(\frac{\pi}{2}+k\pi\right)\frac{s+t}{T}\right)\right].
\end{aligned}
$$

By using

$$
\sum_{k=0}^{\infty}\sin\left(\left(\frac{\pi}{2}+k\pi\right)x\right)=\frac{1}{2}\frac{1}{\sin\left(\frac{\pi}{2}x\right)} \quad \text{for } x\neq 0,2,4,\ldots
$$

we further conclude the representation (2.22). □

Remark 2.3 *For fixed $s,t\in(0,T)$, $s\neq t$, we may consider*

$$
\lim_{T\to\infty} K(s,t) = \frac{1}{\pi}\frac{2s}{(s-t)(s+t)}
$$

to conclude

$$
(\mathcal{H}_\infty u)(t) = \frac{1}{\pi}\int_0^\infty \frac{u(s)}{s-t}\frac{2s}{s+t}\,ds,
$$

where the kernel function shows for $s\to t$ the same behaviour as for the Hilbert transform

$$
(\mathcal{H}u)(t) = \frac{1}{\pi}\int_0^\infty \frac{u(s)}{s-t}\,ds
$$

for which all the previous properties are well known, see, e.g., [18].

2.5 Variational formulations

For the solution of the initial value problem (2.1) we consider the variational formulation (2.18) to find $u\in H^{1/2}_{0,}(0,T)$ such that

$$
\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)} \quad \text{for all } v\in H^{1/2}_{0,}(0,T), \tag{2.23}
$$

where $f\in[H^{1/2}_{,0}(0,T)]'$ is given. Since the bilinear form $\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)}$ is bounded, i.e. for $u,v\in H^{1/2}_{0,}(0,T)$ there holds

$$
|\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)}| \leq \underbrace{\|\partial_t u\|_{[H^{1/2}_{,0}(0,T)]'}}_{=\|B_{1/2}u\|_{[H^{1/2}_{,0}(0,T)]'}} \|\mathcal{H}_T v\|_{H^{1/2}_{,0}(0,T)} = \|u\|_{H^{1/2}_{0,}(0,T)}\|v\|_{H^{1/2}_{0,}(0,T)},
$$

and elliptic, see (2.20), we conclude unique solvability of the variational formulation (2.23).

Remark 2.4 *From the ellipticity estimate* (2.20) *we also conclude the stability condition*

$$\|u\|_{H^{1/2}_{0,}(0,T)} = \frac{\langle \partial_t u, \mathcal{H}_T u\rangle_{(0,T)}}{\|\mathcal{H}_T u\|_{H^{1/2}_{,0}(0,T)}} \leq \sup_{0\neq w\in H^{1/2}_{,0}(0,T)} \frac{\langle \partial_t u, w\rangle_{(0,T)}}{\|w\|_{H^{1/2}_{,0}(0,T)}} \quad \text{for all } u \in H^{1/2}_{0,}(0,T),$$

and from which we conclude unique solvability of the Galerkin–Petrov formulation to find $u \in H^{1/2}_{0,}(0,T)$ *such that*

$$\langle \partial_t u, w\rangle_{(0,T)} = \langle f, w\rangle_{(0,T)} \quad \text{for all } w \in H^{1/2}_{,0}(0,T). \tag{2.24}$$

Next we consider a conforming finite element discretisation for the variational formulation (2.23). For a time interval $(0,T)$ and a discretisation parameter $N \in \mathbb{N}$ we consider nodes

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T,$$

finite elements $\tau_\ell = (t_{\ell-1}, t_\ell)$ of local mesh size $h_\ell = t_\ell - t_{\ell-1}$, $\ell = 1,\ldots,N$, and a related finite element space $S^1_h(0,T)$ of piecewise linear continuous basis functions φ_k, $k = 0,\ldots,N$, with global mesh size $h = \max h_\ell$. Then the finite element discretisation of the variational formulation (2.23) is to find $u_h \in V_h := S^1_h(0,T) \cap H^{1/2}_{0,}(0,T) = \text{span}\{\varphi_k\}_{k=1}^N$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_h\rangle_{(0,T)} \quad \text{for all } v_h \in V_h. \tag{2.25}$$

Using standard arguments, e.g. [31], we conclude unique solvability of (2.25) as well as the a priori error estimates

$$\|u - u_h\|_{H^\sigma_{0,}(0,T)} \leq c h^{s-\sigma} \|u\|_{H^s(0,T)}, \tag{2.26}$$

when assuming $u \in H^s(0,T)$ for some $s \in [1,2]$, and for $\sigma = 0, \frac{1}{2}, 1$. Note that for $\sigma = \frac{1}{2}$ (2.26) is a consequence of Céa's lemma and the approximation property of $S^1_h(0,T)$, while for $\sigma = 0$ we use the Aubin–Nitsche trick, and for $\sigma = 1$ we have to use an inverse inequality, i.e. we have to assume a globally quasi uniform mesh in this case.

The Galerkin–Bubnov finite element formulation (2.25) is equivalent to the linear system of algebraic equations $K_h \underline{u} = \underline{f}$ with a symmetric and positive definite stiffness matrix K_h defined by

$$K_h[j,k] = \langle \partial_t \varphi_k, \mathcal{H}_T \varphi_j\rangle_{L^2(0,T)} \quad \text{for } k,j = 1,\ldots,N.$$

As numerical example we consider the solution $u(t) = \sin\left(\frac{9\pi}{4}t\right)$ for $t \in (0,2) = (0,T)$ where the right-hand side is $f(t) = \frac{9\pi}{4}\cos\left(\frac{9\pi}{4}t\right)$. For the discretisation we consider a sequence of finite element spaces $S^1_h(0,T)$ of uniform mesh size $h = 2/N$, and $N = 2^{j+1}$, $j = 0,\ldots,7$. Since the solution u is smooth, we use $s = 2$ within the error estimate (2.26) to conclude second order convergence in $L^2(0,2)$, and linear convergence in $H^1(0,2)$, respectively. This behaviour is confirmed by the numerical results as given in Table 1. In addition, we present the minimal and maximal eigenvalues of the stiffness matrix K_h as well as the resulting spectral condition number of K_h which behave as expected for a first order differential operator. Note that these results correspond to the Galerkin discretisation of a

N	$\|u-u_h\|_{L^2}$	eoc	$\|\partial_t(u-u_h)\|_{L^2}$	eoc	$\lambda_{\min}(K_h)$	$\lambda_{\max}(K_h)$	$\kappa_2(K_h)$
2	1.00473818	-	7.05949197	-	0.4166	0.9602	2.3
4	0.86127822	0.2	5.88004588	0.3	0.2844	1.1169	3.9
8	0.16924553	2.3	3.66044528	0.7	0.1688	1.1280	6.7
16	0.03246999	2.4	1.82612730	1.0	0.0915	1.1327	12.4
32	0.00748649	2.1	0.90514235	1.0	0.0475	1.1338	23.9
64	0.00183184	2.0	0.45124173	1.0	0.0241	1.1340	47.0
128	0.00045545	2.0	0.22543481	1.0	0.0122	1.1341	93.2
256	0.00011371	2.0	0.11269290	1.0	0.0061	1.1341	185.6

Table 1: Numerical results for the Galerkin–Bubnov formulation (2.25).

hypersingular boundary integral operator in boundary element methods for second order elliptic partial differential equations, see, e.g., [31].
The evaluation of the transformed basis functions $\mathcal{H}_T\varphi_k$ can be done by using the definition (2.19). Although the piecewise linear basis functions φ_k have local support, the transformed basis functions $\mathcal{H}_T\varphi_k$ are global, see Fig. 1, and therefore the stiffness matrix K_h is dense. As in the case of the hypersingular boundary integral operator one may use different techniques such as adaptive cross approximation [28] to accelerate the computations, but this is far behind the scope of this contribution.

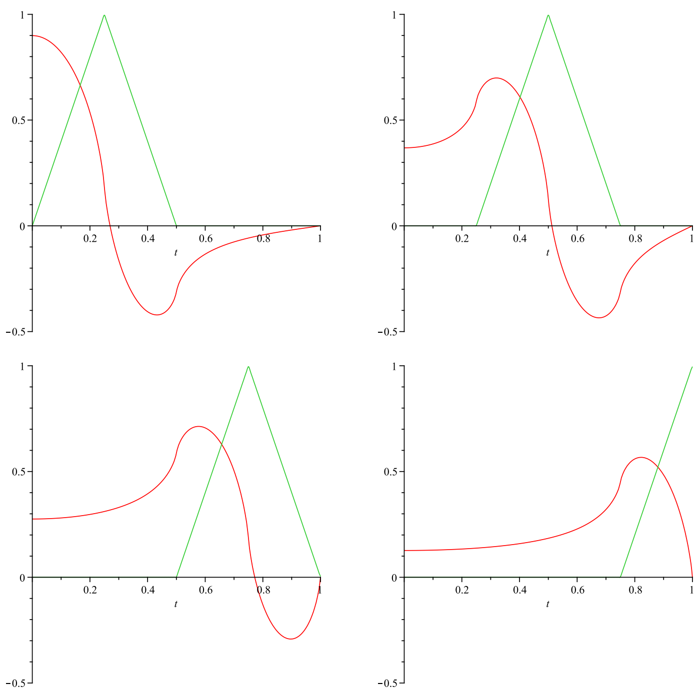

Figure 1: Transformed basis functions $\mathcal{H}_T\varphi_k$, $k = 1,\dots,N$, $N = 4$.

Instead of the initial value problem (2.1) we now consider for $\mu > 0$ the first order linear equation

$$\partial_t u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = 0, \tag{2.27}$$

and the related variational formulation to find $u \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t u, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T v\rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v\rangle_{(0,T)} \quad \text{for all } v \in H^{1/2}_{0,}(0,T), \tag{2.28}$$

where $f \in [H^{1/2}_{,0}(0,T)]'$ is given. When combining (2.20) and (2.21) this gives

$$\langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} + \mu \langle v, \mathcal{H}_T v\rangle_{L^2(0,T)} \geq \langle \partial_t v, \mathcal{H}_T v\rangle_{(0,T)} = \|v\|^2_{H^{1/2}_{0,}(0,T)}$$

for all $v \in H^{1/2}_{0,}(0,T)$, i.e. the bilinear form of the variational problem (2.28) is bounded and elliptic, implying unique solvability of (2.28). For the solution $u \in H^{1/2}_{0,}(0,T)$ of the variational problem (2.28) we have

$$\begin{aligned} \|u\|^2_{H^{1/2}_{0,}(0,T)} &= \langle \partial_t u, \mathcal{H}_T u\rangle_{(0,T)} \leq \langle \partial_t u, \mathcal{H}_T u\rangle_{(0,T)} + \mu \langle u, \mathcal{H}_T u\rangle_{L^2(0,T)} \\ &= \langle f, \mathcal{H}_T u\rangle_{(0,T)} \leq \|f\|_{[H^{1/2}_{,0}(0,T)]'} \|\mathcal{H}_T u\|_{H^{1/2}_{,0}(0,T)}, \end{aligned}$$

implying

$$\|u\|_{H^{1/2}_{0,}(0,T)} \leq \|f\|_{[H^{1/2}_{,0}(0,T)]'}\,. \tag{2.29}$$

For the analysis of the heat equation we also need to have appropriate estimates for the solution u in $L^2(0,T)$.

Lemma 2.9 *Let $u \in H^{1/2}_{0,}(0,T)$ be the unique solution of the variational problem* (2.28), *where $f \in [H^{1/2}_{,0}(0,T)]'$ is given. Then,*

$$\|u\|^2_{L^2(0,T)} \leq \frac{T}{2} \sum_{k=0}^{\infty} \frac{\overline{f}_k^2}{\mu^2 + \frac{1}{T^2}(\frac{\pi}{2} + k\pi)^2}\,, \tag{2.30}$$

where

$$\overline{f}_k := \frac{2}{T}\langle f, w_k\rangle_{(0,T)}, \quad w_k(t) := \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\,.$$

Proof. Let $(f_n)_{n\in\mathbb{N}} \subset L^2(0,T)$ be a sequence with $\lim\limits_{n\to\infty} \|f - f_n\|_{[H^{1/2}_{,0}(0,T)]'} = 0$. We write $f_n \in L^2(0,T)$ as

$$f_n(t) = \sum_{k=0}^{\infty} \overline{f}_{n,k} \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad \overline{f}_{n,k} = \frac{2}{T}\int_0^T f_n(t)\cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt\,. \tag{2.31}$$

Let $u_n \in H^{1/2}_{0,}(0,T)$ be the weak solution of the differential equation (2.27) with right–hand side f_n. It follows analogously to (2.29) that

$$\|u - u_n\|_{H^{1/2}_{0,}(0,T)} \leq \|f - f_n\|_{[H^{1/2}_{,0}(0,T)]'}$$

and therefore $u_n \to u$ in $H^{1/2}_{0,}(0,T)$ and $u_n \to u$ in $L^2(0,T)$ as $n \to \infty$.

Because of $f_n \in L^2(0,T)$ and using (2.31) we have the representation

$$\begin{aligned} u_n(t) &= \int_0^t e^{\mu(s-t)} f_n(s)\, ds \,=\, \sum_{k=0}^\infty \overline{f}_{n,k} \int_0^t e^{\mu s} \cos(a_k s)\, ds\, e^{-\mu t} \\ &= \sum_{k=0}^\infty \frac{\overline{f}_{n,k}}{\mu^2 + a_k^2} \Big[a_k \sin(a_k t) + \mu \cos(a_k t) - \mu e^{-\mu t} \Big], \quad a_k = \frac{1}{T}\Big(\frac{\pi}{2} + k\pi\Big), \end{aligned}$$

and we obtain, when computing all integrals,

$$\|u_n\|^2_{L^2(0,T)} = \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}^2_{n,k}}{\mu^2 + a_k^2} - \frac{1}{2}\mu \Big[1 + e^{-2\mu T}\Big] \left(\sum_{k=0}^\infty \frac{\overline{f}_{n,k}}{\mu^2 + a_k^2} \right)^2 \le \frac{T}{2} \sum_{k=0}^\infty \frac{\overline{f}^2_{n,k}}{\mu^2 + a_k^2}\,.$$

Now the assertion follows as $n \to \infty$. □

Remark 2.5 *From* (2.30) *we immediately conclude the estimate*

$$\|u\|^2_{L^2(0,T)} \le \frac{T^3}{2} \sum_{k=0}^\infty \Big(\frac{\pi}{2} + k\pi\Big)^{-2} \overline{f}^2_k \;=\; \|f\|^2_{[H^1_{,0}(0,T)]'}\,.$$

Moreover, when we assume $f \in L^2(0,T)$, (2.30) *gives*

$$\|u\|^2_{L^2(0,T)} \le \frac{T}{2\mu^2} \sum_{k=0}^\infty \overline{f}^2_k = \frac{1}{\mu^2} \|f\|^2_{L^2(0,T)}, \quad \textit{i.e.} \quad \mu\, \|u\|_{L^2(0,T)} \le \|f\|_{L^2(0,T)}.$$

The Galerkin–Bubnov discretisation of (2.28) is to find $u_h \in V_h$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_{L^2(0,T)} + \mu \langle u_h, \mathcal{H}_T v_h \rangle_{L^2(0,T)} = \langle f, \mathcal{H}_T v_h \rangle_{(0,T)} \quad \text{for all } v_h \in V_h. \tag{2.32}$$

As for the initial value problem (2.1) we have unique solvability of (2.32), but related a priori error estimates depend on μ in general, requiring a sufficient small mesh size h to ensure convergence for large μ.

Remark 2.6 *Instead of the Galerkin–Bubnov variational formulation* (2.28) *we may also consider the Galerkin–Petrov formulation to find $u \in H^{1/2}_{0,}(0,T)$ such that*

$$\langle \partial_t u, w \rangle_{(0,T)} + \mu \langle u, w \rangle_{L^2(0,T)} = \langle f, w \rangle_{(0,T)} \quad \textit{for all } w \in H^{1/2}_{,0}(0,T), \tag{2.33}$$

where the ellipticity of $\langle \partial_t v, \mathcal{H}_T v \rangle_{(0,T)} + \mu \langle v, \mathcal{H}_T v \rangle_{L^2(0,T)}$ implies a related stability estimate from which unique solvability of (2.33) *follows.*

For the finite element discretisation of the Galerkin–Petrov variational formulations (2.24) *and* (2.33) *we have to define a suitable test space $W_h \subset H^{1/2}_{,0}(0,T)$. A first choice is to use $W_h := S^1_h(0,T) \cap H^{1/2}_{,0}(0,T)$. Although the discrete systems are always uniquely solvable, since the stiffness matrices are regular lower triangular, the resulting scheme is never stable when considering* (2.27)*. The construction of a more suitable test space is, in particular when considering partial differential equations such as the heat equation, more challenging.*

3 Heat equation

As model problem for a parabolic partial differential equation we consider the Dirichlet problem for the heat equation,

$$\left.\begin{array}{rcll} \partial_t u(x,t) - \Delta_x u(x,t) & = & f(x,t) & \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) & = & 0 & \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) & = & 0 & \text{for } x \in \Omega, \end{array}\right\} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded domain with, for $d = 2,3$ Lipschitz boundary $\Gamma = \partial\Omega$. To write down a variational formulation we need to have suitable Sobolev spaces. In addition to the eigenfunctions $v_k(t)$ and eigenvalues λ_k as given in (2.11) we consider the eigenfunctions $\phi_i \in H^1_0(\Omega)$ and associated eigenvalues μ_i, $i \in \mathbb{N}$, of the spatial Dirichlet eigenvalue problem

$$-\Delta_x \phi = \mu\phi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \Gamma, \quad \|\phi\|_{L^2(\Omega)} = 1\,.$$

Recall that the eigenfunctions ϕ_i form an orthonormal basis in $L^2(\Omega)$, and an orthogonal basis in $H^1_0(\Omega)$. In addition, we have

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \quad \text{and} \quad \mu_i \to \infty \text{ as } i \to \infty.$$

For a function $u \in L^2(Q)$ we therefore find the representation

$$u(x,t) = \sum_{i=1}^\infty \sum_{k=0}^\infty u_{i,k} v_k(t) \phi_i(x) = \sum_{i=1}^\infty U_i(t)\phi_i(x), \quad U_i(t) = \sum_{k=0}^\infty u_{i,k} v_k(t) \tag{3.2}$$

with the coefficients

$$\begin{array}{rcl} u_{i,k} & = & \displaystyle\frac{2}{T}\int_0^T \int_\Omega u(x,t) v_k(t) \phi_i(x)\, dxdt \\ & = & \displaystyle\frac{2}{T}\int_0^T \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) \int_\Omega u(x,t)\phi_i(x)\, dxdt. \end{array}$$

Note that we have

$$\|u\|^2_{L^2(Q)} = \sum_{i=1}^\infty \|U_i\|^2_{L^2(0,T)} = \frac{T}{2} \sum_{i=1}^\infty \sum_{k=0}^\infty u^2_{i,k}$$

and

$$\begin{array}{rcl} |u|^2_{H^1(Q)} & = & \displaystyle\sum_{i=1}^\infty \left[\|\partial_t U_i\|^2_{L^2(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)} \right] \\ & = & \displaystyle\frac{T}{2} \sum_{i=1}^\infty \sum_{k=0}^\infty \left[\frac{1}{T^2}\left(\frac{\pi}{2} + k\pi\right)^2 + \mu_i \right] u^2_{i,k}\,. \end{array}$$

This motivates to define the norm, for $u \in H^1(Q)$ with $u(\cdot,0) = u_{|\Sigma} = 0$,

$$\begin{aligned} \|u\|^2_{H^{1,1/2}_{0;0,}(Q)} &:= \sum_{i=1}^{\infty} \left[\|U_i\|^2_{H^{1/2}_{0,}(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)} \right] \\ &= \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] u^2_{i,k}, \end{aligned}$$

and to introduce the anisotropic Sobolev space

$$H^{1,1/2}_{0;0,}(Q) := \left\{ u \in L^2(Q) \colon \; \|u\|_{H^{1,1/2}_{0;0,}(Q)} < \infty \right\}.$$

Note that $H^{1,1/2}_{0;0,}(Q) = H^{1/2}_{0,}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$. Analogously, we introduce $H^{1,1/2}_{0;,0}(Q) = H^{1/2}_{,0}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$, which is equipped with the norm

$$\|w\|^2_{H^{1,1/2}_{0;,0}(Q)} = \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right] \overline{w}^2_{i,k},$$

and

$$\overline{w}_{i,k} = \frac{2}{T} \int_0^T \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \int_\Omega w(x,t) \phi_i(x) \, dx dt.$$

Lemma 3.1 *For the dual norm of $f \in [H^{1,1/2}_{0;,0}(Q)]'$ we have*

$$\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'} = \frac{T}{2} \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \left[\frac{1}{T} \left(\frac{\pi}{2} + k\pi \right) + \mu_i \right]^{-1} \overline{f}^2_{i,k} \tag{3.3}$$

with

$$\overline{f}_{i,k} = \frac{2}{T} \langle f, w_k \phi_i \rangle_Q.$$

Proof. From the norm definition, using a series representation of $w \in H^{1,1/2}_{0;,0}(Q)$, and with Hölder's inequality we first have

$$
\begin{aligned}
\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} &= \sup_{0\neq w\in H^{1,1/2}_{0;,0}(Q)} \frac{\langle f, w\rangle_Q}{\|w\|_{H^{1,1/2}_{0;,0}(Q)}} \\
&= \sup_{0\neq w\in H^{1,1/2}_{0;,0}(Q)} \frac{\displaystyle\sum_{i=1}^\infty\sum_{k=0}^\infty \overline{w}_{i,k}\langle f, w_k\phi_i\rangle_Q}{\displaystyle\left(\frac{T}{2}\sum_{i=1}^\infty\sum_{k=0}^\infty \left[\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right]\overline{w}_{i,k}^2\right)^{1/2}} \\
&= \frac{\sqrt{T}}{\sqrt{2}} \sup_{0\neq w\in H^{1,1/2}_{0;,0}(Q)} \frac{\displaystyle\sum_{i=1}^\infty\sum_{k=0}^\infty \overline{w}_{i,k}\overline{f}_{i,k}}{\displaystyle\left(\sum_{i=1}^\infty\sum_{k=0}^\infty \left[\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right]\overline{w}_{i,k}^2\right)^{1/2}} \\
&\leq \frac{\sqrt{T}}{\sqrt{2}} \left(\sum_{i=1}^\infty\sum_{k=0}^\infty \left[\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right]^{-1}\overline{f}_{i,k}^2\right)^{1/2},
\end{aligned}
$$

i.e.

$$
\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'} \leq \frac{T}{2}\sum_{i=1}^\infty\sum_{k=0}^\infty \left[\frac{1}{T}\left(\frac{\pi}{2}+k\pi\right)+\mu_i\right]^{-1}\overline{f}_{i,k}^2.
$$

The lower estimate follows as in the proof of Lemma 2.3, we skip the details. □

According to the previous sections we consider the variational formulation of (3.1) to find $u \in H^{1,1/2}_{0;0,}(Q)$ such that

$$
\langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} = \langle f, v\rangle_Q \tag{3.4}
$$

is satisfied for all $v \in H^{1,1/2}_{0;,0}(Q)$, where $f \in [H^{1,1/2}_{0;,0}(Q)]'$ is given, and $\langle\cdot,\cdot\rangle_Q$ denotes the duality pairing as extension of the inner product in $L^2(Q)$.

Theorem 3.2 *The variational formulation* (3.4) *implies an isomorphism*

$$
\mathcal{L}\colon\ H^{1,1/2}_{0;0,}(Q) \to [H^{1,1/2}_{0;,0}(Q)]',
$$

satisfying

$$
\|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq 2\,\|\mathcal{L}u\|_{[H^{1,1/2}_{0;,0}(Q)]'} \quad \textit{for all } u \in H^{1,1/2}_{0;0,}(Q)\,. \tag{3.5}
$$

Proof. For the solution u of the variational problem (3.4) we use the ansatz (3.2) where $U_i \in H^{1/2}_{0,}(0,T)$ are unknown functions to be determined. When choosing, for a fixed $j \in \mathbb{N}$,

$v(x,t) := V(t)\phi_j(x)$ with $V \in H^{1/2}_{,0}(0,T)$ as test function, the variational formulation (3.4) leads to find $U_j \in H^{1/2}_{0,}(0,T)$ such that

$$\langle \partial_t U_j, V \rangle_{(0,T)} + \mu_j \langle U_j, V \rangle_{L^2(0,T)} = \langle f, V\phi_j \rangle_Q \tag{3.6}$$

is satisfied for all $V \in H^{1/2}_{,0}(0,T)$. It holds

$$|\langle f, V\phi_j \rangle_Q| \le \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|V\phi_j\|_{H^{1,1/2}_{0;,0}(Q)} \le \sqrt{1+\mu_j}\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|V\|_{H^{1/2}_{,0}(0,T)}$$

for all $V \in H^{1/2}_{,0}(0,T)$ and so, $\langle F_j, V \rangle_{(0,T)} := \langle f, V\phi_j \rangle_Q$ fulfils $F_j \in [H^{1/2}_{,0}(0,T)]'$. The unique solvability of (3.6) follows analogously as for (2.27). So, we have for every $j \in \mathbb{N}$ a unique solution $U_j \in H^{1/2}_{0,}(0,T)$ of the variational formulation (3.6), satisfying

$$\begin{aligned} \|U_j\|^2_{H^{1/2}_{0,}(0,T)} &= \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} \\ &\le \langle \partial_t U_j, \mathcal{H}_T U_j \rangle_{(0,T)} + \mu_j \langle U_j, \mathcal{H}_T U_j \rangle_{L^2(0,T)} \\ &= \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q \, . \end{aligned}$$

For $M \in \mathbb{N}$ we define

$$u_M(x,t) = \sum_{j=1}^{M} U_j(t)\phi_j(x),$$

and we conclude

$$\begin{aligned} \|u_M\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega))} &= \sum_{j=1}^{M} \|U_j\|^2_{H^{1/2}_{0,}(0,T)} \le \sum_{j=1}^{M} \langle f, \phi_j \mathcal{H}_T U_j \rangle_Q \\ &= \langle f, \mathcal{H}_T u_M \rangle_Q \\ &\le \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|\mathcal{H}_T u_M\|_{H^{1,1/2}_{0;,0}(Q)} \\ &= \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|u_M\|_{H^{1,1/2}_{0;0,}(Q)} \, . \end{aligned}$$

Hence, using (2.30) for

$$\overline{f}_{i,k} = \frac{2}{T} \langle F_i, w_k \rangle_{(0,T)} = \frac{2}{T} \langle f, \phi_i w_k \rangle_Q$$

we obtain

$$\begin{aligned} \|u_M\|^2_{L^2(0,T;H^1_0(\Omega))} &= \sum_{i=1}^{M} \mu_i \|U_i\|^2_{L^2(0,T)} \\ &\le \frac{T}{2} \sum_{i=1}^{M} \sum_{k=0}^{\infty} \frac{\mu_i}{\mu_i^2 + \frac{1}{T^2(\frac{\pi}{2}+k\pi)^2}} \overline{f}^2_{i,k} \\ &\le T \sum_{i=1}^{M} \sum_{k=0}^{\infty} \frac{1}{\mu_i + \frac{1}{T(\frac{\pi}{2}+k\pi)}} \overline{f}^2_{i,k} \le 2\,\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'}, \end{aligned}$$

where we have used

$$\frac{a}{a^2+b^2} \leq \frac{a+b}{\frac{1}{2}(a+b)^2} = \frac{2}{a+b} \quad \text{for } 0 < a, b \in \mathbb{R}.$$

With this we have

$$\begin{aligned}
\|u_M\|^2_{H^{1,1/2}_{0;0,}(Q)} &= \|u_M\|^2_{H^{1/2}_{0,}(0,T;L^2(Q))} + \|u_M\|^2_{L^2(0,T;H^1_0(\Omega))} \\
&\leq \|f\|_{[H^{1,1/2}_{0;,0}(Q)]'} \|u_M\|_{H^{1,1/2}_{0;0,}(Q)} + 2\,\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'},
\end{aligned}$$

and therefore

$$\|u_M\|_{H^{1,1/2}_{0;0,}(Q)} \leq 2\,\|f\|_{[H^{1,1/2}_{0;,0}(Q)]'}$$

follows for all $M \in \mathbb{N}$. The last inequality yields the bound

$$\begin{aligned}
\|u\|^2_{H^{1,1/2}_{0;0,}(Q)} &= \lim_{M\to\infty} \sum_{i=1}^{M} \left[\|U_i\|^2_{H^{1/2}_{0,}(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)} \right] \\
&= \lim_{M\to\infty} \|u_M\|^2_{H^{1,1/2}_{0;0,}(Q)} \leq 4\|f\|^2_{[H^{1,1/2}_{0;,0}(Q)]'} < \infty,
\end{aligned}$$

and thus, $u \in H^{1,1/2}_{0;0,}(Q)$ with $\lim_{M\to\infty} u_M = u$ in $H^{1,1/2}_{0;0,}(Q)$.

The existence of a solution of the variational formulation (3.4) is proven by inserting the constructed function u into the variational formulation (3.4) and using the approximating sequence $(u_M)_{M\in\mathbb{N}}$.

The uniqueness of a solution of the variational formulation (3.4) is a consequence of the uniqueness of the coefficient functions U_j. □

Corollary 3.3 *As a direct consequence of* (3.5) *we immediately conclude the stability estimate*

$$\frac{1}{2}\,\|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq \sup_{0\neq w\in H^{1,1/2}_{0;,0}(Q)} \frac{\langle \partial_t u, w\rangle_Q + \langle \nabla_x u, \nabla_x w\rangle_{L^2(Q)}}{\|w\|_{H^{1,1/2}_{0;,0}(Q)}} \tag{3.7}$$

for all $u \in H^{1,1/2}_{0;0,}(Q)$.

The variational formulation (3.4) is equivalent to find $u \in H^{1,1/2}_{0;0,}(Q)$ such that

$$\langle \partial_t u, \mathcal{H}_T v\rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v\rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v\rangle_Q \tag{3.8}$$

is satisfied for all $v \in H^{1,1/2}_{0;0,}(Q)$, where the operator $\mathcal{H}_T$ acts only on the time variable t. The stability estimate (3.7) implies the stability estimate

$$\frac{1}{2}\,\|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq \sup_{0\neq v\in H^{1,1/2}_{0;0,}(Q)} \frac{\langle \partial_t u, \mathcal{H}_T v\rangle_Q + \langle \nabla_x u, \nabla_x \mathcal{H}_T v\rangle_{L^2(Q)}}{\|v\|_{H^{1,1/2}_{0;0,}(Q)}} \tag{3.9}$$

for all $u \in H^{1,1/2}_{0;0,}(Q)$, and therefore unique solvability of the variational formulation (3.8) follows.

When using some conforming space–time finite element space $\mathcal{V}_h \subset H^{1,1/2}_{0;0,}(Q)$, the Galerkin variational formulation of (3.8) is to find $u_h \in \mathcal{V}_h$ such that

$$\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T v_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v_h \rangle_Q \qquad (3.10)$$

is satisfied for all $v_h \in \mathcal{V}_h$. Since the related finite element stiffness matrix is positive definite, unique solvability of (3.10) follows for any conforming choice of $\mathcal{V}_h$. However, to perform the temporal transformation $\mathcal{H}_T$ easily, and to be able to present an a priori error analysis, here we will consider a space–time tensor–product finite element space only.

Let $W_{h_x} = \text{span}\{\psi_i\}_{i=1}^M \subset H^1_0(\Omega)$ be some spatial finite element space, e.g., of piecewise linear or bilinear continuous basis functions ψ_i which are defined with respect to some admissible and globally quasi–uniform finite element mesh with mesh size h_x. As before, $V_{h_t} = S^1_{h_t}(0,T) \cap H^{1/2}_{0,}(0,T) = \text{span}\{\varphi_k\}_{k=1}^N$ is the space of piecewise linear functions which are defined with respect to some globally quasi–uniform finite element mesh with mesh size h_t. Hence we can introduce the tensor–product space–time finite element space $\mathcal{V}_h := W_{h_x} \otimes V_{h_t}$.

For a given $v \in H^{1/2}_{0,}(0,T;L^2(\Omega))$ we define the $H^{1/2}_{0,}$ projection $Q^{1/2}_{h_t} v \in L^2(\Omega) \otimes V_{h_t}$ as the unique solution of the variational problem

$$\langle \partial_t Q^{1/2}_{h_t} v, \mathcal{H}_T v_{h_t} \rangle_{L^2(Q)} = \langle \partial_t v, \mathcal{H}_T v_{h_t} \rangle_Q$$

for all $v_{h_t} \in L^2(\Omega) \otimes V_{h_t}$. Moreover, for $v \in L^2(0,T;H^1_0(\Omega))$ we define the H^1_0 projection $Q^1_{h_x} v \in W_{h_x} \otimes L^2(0,T)$ as the unique solution of the variational problem

$$\int_0^T \int_\Omega \nabla_x Q^1_{h_x} v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt = \int_0^T \int_\Omega \nabla_x v(x,t) \cdot \nabla_x v_{h_x}(x,t)\, dx\, dt$$

for all $v_{h_x} \in W_{h_x} \otimes L^2(0,T)$. It turns out that $Q^{1/2}_{h_t} Q^1_{h_x} v \in \mathcal{V}_h$ is well defined when assuming $\partial_t v \in L^2(0,T;H^1_0(\Omega))$ and $\nabla_x v \in H^{1/2}_{0,}(0,T;L^2(\Omega))$, respectively, and that the projection operators $Q^{1/2}_{h_t}$, $Q^1_{h_x}$ and partial derivatives ∂_t, ∇_x commute in space and time [38].

Theorem 3.4 *Let $u \in H^{1,1/2}_{0;0,}(Q)$ and $u_h \in \mathcal{V}_h$ be the unique solutions of the variational problems* (3.8) *and* (3.10)*, respectively. If u is sufficiently regular, and the spatial domain Ω is assumed to be either convex or has a smooth boundary Γ, then there hold the error estimates*

$$\begin{aligned} \|u - u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \;\leq\; & c_1 h_t^{3/2} \|u\|_{H^2(0,T;L^2(\Omega))} + c_2 h_x^{3/2} \|u\|_{H^{1/2}_{0,}(0,T;H^{3/2}(\Omega))} \\ & + c_3 h_t^{1/2} h_x \|\partial_t \nabla_x u\|_{L^2(Q)} + c_4 h_t^{3/2} \|\partial_t \Delta_x u\|_{L^2(Q)} \end{aligned} \qquad (3.11)$$

and

$$\begin{aligned} \|u - u_h\|_{L^2(Q)} \;\leq\; & c_1 h_t^2 \|u\|_{H^2(0,T;L^2(\Omega))} + c_2 h_x^2 \|u\|_{L^2(0,T;H^2(\Omega))} + c_3 h_t h_x \|\partial_t \nabla_x u\|_{L^2(Q)} \\ & + c_4 h_x^2 \|\partial_t u\|_{L^2(0,T;H^2(\Omega))} + c_5 h_t^2 \|\Delta_x u\|_{H^2(0,T;L^2(\Omega))} \,. \end{aligned} \qquad (3.12)$$

Proof. With the norm representation in $H^{1/2}_{0,}(0,T;L^2(\Omega))$, the positivity (2.21), and the Galerkin orthogonality of the variational formulations (3.8) and (3.10), we have for $v_h = Q^{1/2}_{h_t} Q^1_{h_x} u \in \mathcal{V}_h$, using the definitions of the projections $Q^{1/2}_{h_t}$ and $Q^1_{h_x}$, and integration by parts spatially,

$$\begin{aligned}
\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega))} &= \langle \partial_t (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\leq \langle \partial_t (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\quad + \langle \nabla_x (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u), \nabla_x \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_{L^2(Q)} \\
&= \langle \partial_t (u - Q^{1/2}_{h_t} Q^1_{h_x} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\quad + \langle \nabla_x (u - Q^{1/2}_{h_t} Q^1_{h_x} u), \nabla_x \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_{L^2(Q)} \\
&= \langle \partial_t (u - Q^1_{h_x} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\quad + \langle \nabla_x (u - Q^{1/2}_{h_t} u), \nabla_x \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_{L^2(Q)} \\
&= \langle \partial_t (u - Q^1_{h_x} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\quad - \langle \Delta_x (u - Q^{1/2}_{h_t} u), \mathcal{H}_T(u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\leq \|u - Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\quad + \|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]'} \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))},
\end{aligned} \tag{3.13}$$

i.e.

$$\begin{aligned}
&\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\qquad \leq \|u - Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]'}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\|u - u_h\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\quad \leq \|u - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\quad \leq \|u - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\qquad + \|u - Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]'} \\
&\quad \leq \|u - Q^{1/2}_{h_t} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|u - Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\qquad + \left\| \left(I - Q^{1/2}_{h_t}\right)\left(u - Q^1_{h_x} u\right) \right\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\qquad + \|u - Q^1_{h_x} u\|_{H^{1/2}_{0,}(0,T;L^2(\Omega))} + \|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{[H^{1/2}_{,0}(0,T;L^2(\Omega))]'},
\end{aligned}$$

and the energy error estimate (3.11) follows from standard error estimates for the involved projection operators.

With a Poincaré–Friedrichs type inequality and relation (3.13) we also have

$$\begin{aligned}
\frac{1}{c}\,\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|^2_{L^2(Q)} &\leq \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|^2_{H^{1/2}_{0,}(0,T;L^2(\Omega))} \\
&\leq \langle \partial_t (u - Q^1_{h_x} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_Q \\
&\qquad - \langle \Delta_x (u - Q^{1/2}_{h_t} u), \mathcal{H}_T (u_h - Q^{1/2}_{h_t} Q^1_{h_x} u)\rangle_{L^2(Q)} \\
&\leq \|\partial_t (u - Q^1_{h_x} u)\|_{L^2(Q)} \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \\
&\qquad + \|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{L^2(Q)} \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)},
\end{aligned}$$

which implies

$$\|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \leq c\,\|\partial_t (u - Q^1_{h_x} u)\|_{L^2(Q)} + c\,\|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{L^2(Q)},$$

and therefore

$$\begin{aligned}
\|u - u_h\|_{L^2(Q)} &\leq \|u - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)} + \|u_h - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)} \\
&\leq \|u - Q^{1/2}_{h_t} Q^1_{h_x} u\|_{L^2(Q)} + c\,\|\partial_t (u - Q^1_{h_x} u)\|_{L^2(Q)} + c\,\|\Delta_x (u - Q^{1/2}_{h_t} u\|_{L^2(Q)} \\
&\leq \|u - Q^{1/2}_{h_t} u\|_{L^2(Q)} + \|u - Q^1_{h_x} u\|_{L^2(Q)} + \|\Big(I - Q^{1/2}_{h_t}\Big)\Big(u - Q^1_{h_x} u\Big)\|_{L^2(Q)} \\
&\qquad + c\,\|\partial_t (u - Q^1_{h_x} u)\|_{L^2(Q)} + c\,\|\Delta_x (u - Q^{1/2}_{h_t} u)\|_{L^2(Q)}.
\end{aligned}$$

Finally, (3.12) follows again from standard error estimates for the projection operators. □
As numerical example we consider the solution $u(x,t) = \sin\left(\frac{5\pi}{4} t\right) \sin(\pi x)$ for $(x,t) \in Q$ with $Q := (0,1) \times (0,2)$. For a uniform discretisation of the Galerkin variational formulation (3.10) with the tensor–product space–time finite element space $\mathcal{V}_h = W_{h_x} \otimes V_{h_t}$ we use the mesh sizes $h_x = \frac{1}{M}$ and $h_t = \frac{2}{N}$ with $M = N = 2^j$, $j = 1, \ldots, 6$. Since the solution u is smooth, we expect a second order convergence in $L^2(Q)$, see (3.12), and a linear order convergence in $H^1(Q)$. Note that the latter follows by standard arguments when using the $H^1(Q)$ projection and an inverse inequality. The predicted convergence orders are confirmed by the numerical results given in Table 2.

M, N	dof	h_x	h_t	$\|u - u_h\|_{L^2}$	eoc	$\|u - u_h\|_{H^1}$	eoc
2	2	0.500000	1.000000	0.91080532	-	4.48436523	-
4	12	0.250000	0.500000	0.15774388	2.5	1.89083082	1.2
8	56	0.125000	0.250000	0.02936086	2.4	0.84239378	1.2
16	240	0.062500	0.125000	0.00689501	2.1	0.41495910	1.0
32	992	0.031250	0.062500	0.00169574	2.0	0.20679363	1.0
64	4032	0.015625	0.031250	0.00042203	2.0	0.10331237	1.0

Table 2: Convergence rates of the Galerkin–Bubnov formulation (3.10).

Remark 3.1 *Numerical results* [38] *indicate that the stability constant* c_S *of the discrete inf–sup condition*

$$c_S \, \|u_h\|_{H^{1,1/2}_{0;0,}(Q)} \leq \sup_{0 \neq v_h \in \mathcal{V}_h} \frac{\langle \partial_t u_h, \mathcal{H}_T v_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \mathcal{H}_T v_h \rangle_{L^2(Q)}}{\|v_h\|_{H^{1,1/2}_{0;0,}(Q)}}$$

for all $u_h \in \mathcal{V}_h$ *is mesh dependent, i.e.* $c_S = \mathcal{O}(h_t)$*. However, it seems to be possible to derive almost optimal energy error estimates also in this case. Since this is far behind the scope of this paper, this will be discussed elsewhere.*

4 Second order ordinary differential equations

As in (2.1) we consider the initial value problem

$$\partial_{tt} u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0. \tag{4.1}$$

When multiplying the differential equation with a test function w satisfying $w(T) = 0$, integrating over $(0,T)$, and applying integration by parts once, this results in the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$- \int_0^T \partial_t u(t) \partial_t w(t) \, dt = \langle f, w \rangle_{(0,T)} \tag{4.2}$$

is satisfied for all $w \in H^1_{,0}(0,T)$, where $f \in [H^1_{,0}(0,T)]'$ is given. The bilinear form

$$a(u,w) := - \int_0^T \partial_t u(t) \partial_t w(t) \, dt \quad \text{for } u \in H^1_{0,}(0,T), \, w \in H^1_{,0}(0,T) \tag{4.3}$$

is obviously bounded and therefore it remains to establish some stability or ellipticity estimate to ensure unique solvability of the variational formulation (4.2). For this we use the concept of an optimal test function, see Remark 2.1.

Lemma 4.1 *Let* $u \in H^1_{0,}(0,T)$ *be given. The unique solution* $w = \overline{\mathcal{H}}_T u \in H^1_{,0}(0,T)$ *of the variational problem*

$$\int_0^T \partial_t w(t) \partial_t v(t) \, dt = - \int_0^T \partial_t u(t) \partial_t v(t) \, dt \quad \textit{for all } v \in H^1_{,0}(0,T), \tag{4.4}$$

is given as

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right), \quad w_k = \frac{2}{\frac{\pi}{2} + k\pi} \int_0^T \partial_t u(t) \sin\left(\left(\frac{\pi}{2} + k\pi \right) \frac{t}{T} \right) \, dt. \tag{4.5}$$

Proof. When using the ansatz and test functions

$$w(t) = \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad v(t) = \cos\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right)$$

within the variational formulation (4.4) and the orthogonality (2.12), this gives

$$w_\ell = \frac{2}{\frac{\pi}{2} + \ell\pi} \int_0^T \partial_t u(t) \sin\left(\left(\frac{\pi}{2} + \ell\pi\right)\frac{t}{T}\right) dt.$$

□

As for the transformation operator $\mathcal{H}_T \colon H^{1/2}_{0,}(0,T) \to H^{1/2}_{,0}(0,T)$ we state some properties for $\overline{\mathcal{H}}_T \colon H^1_{0,}(0,T) \to H^1_{,0}(0,T)$ as defined in (4.5).

Lemma 4.2 *The operator* $\overline{\mathcal{H}}_T$ *as defined in* (4.5) *is norm preserving satisfying*

$$\|\overline{\mathcal{H}}_T u\|_{H^1_{,0}(0,T)} = \|u\|_{H^1_{0,}(0,T)} \quad \textit{for all } u \in H^1_{0,}(0,T).$$

Proof. Let $w = \overline{\mathcal{H}}_T u$ as defined in (4.5), and when using (2.15) this gives

$$\|w\|^2_{H^1_{,0}(0,T)} = \|\partial_t w\|^2_{L^2(0,T)} = \frac{1}{2T} \sum_{k=0}^{\infty} \left(\frac{\pi}{2} + k\pi\right)^2 w_k^2.$$

On the other hand, for $z = \partial_t u \in L^2(0,T)$ we consider the series representation

$$z(t) = \sum_{k=0}^{\infty} z_k \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right), \quad z_k = \frac{2}{T} \int_0^T z(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt,$$

and with (2.14) we have

$$\|u\|^2_{H^1_{0,}(0,T)} = \|\partial_t u\|^2_{L^2(0,T)} = \|z\|^2_{L^2(0,T)} = \frac{T}{2} \sum_{k=0}^{\infty} z_k^2.$$

Now the assertion follows from

$$\begin{aligned} w_k &= \frac{2}{\frac{\pi}{2} + k\pi} \int_0^T \partial_t u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \\ &= \frac{2}{\frac{\pi}{2} + k\pi} \int_0^T z(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt = \frac{T}{\frac{\pi}{2} + k\pi}\, z_k. \end{aligned}$$

□

Lemma 4.3 *For* $u, v \in H^1_{0,}(0,T)$ *there holds the symmetry relation*

$$a(u, \overline{\mathcal{H}}_T v) = a(v, \overline{\mathcal{H}}_T u).$$

Proof. When using the definition (4.4) this gives

$$\begin{aligned}
a(u, \overline{\mathcal{H}}_T v) &= -\int_0^T \partial_t u(t) \partial_t (\overline{\mathcal{H}}_T v)(t)\, dt \\
&= \int_0^T \partial_t (\overline{\mathcal{H}}_T u)(t) \partial_t (\overline{\mathcal{H}}_T v)(t)\, dt \\
&= \int_0^T \partial_t (\overline{\mathcal{H}}_T v)(t) \partial_t (\overline{\mathcal{H}}_T u)(t)\, dt \\
&= -\int_0^T \partial_t v(t) \partial_t (\overline{\mathcal{H}}_T u)(t)\, dt = a(v, \overline{\mathcal{H}}_T u).
\end{aligned}$$

□

Now we are in the position to prove some ellipticity estimate for the bilinear form (4.3).

Theorem 4.4 *The bilinear form as given in* (4.3) *is elliptic, i.e.*

$$a(u, \overline{\mathcal{H}}_T u) = \|\partial_t u\|^2_{L^2(0,T)} \quad \textit{for all } u \in H^1_{0,}(0,T).$$

Proof. Due to the construction of $w = \overline{\mathcal{H}}_T u$ we have

$$a(u, \overline{\mathcal{H}}_T u) = -\int_0^T \partial_t u(t) \partial_t w(t)\, dt = \int_0^T \partial_t w(t) \partial_t w(t)\, dt = \|\partial_t w\|^2_{L^2(0,T)},$$

and the assertion follows by using Lemma 4.2. □

Lemma 4.5 *For* $u \in H^1_{0,}(0,T)$ *there holds the representation*

$$(\overline{\mathcal{H}}_T u)(t) = u(T) - u(t)\,. \tag{4.6}$$

Proof. For the coefficients in (4.5) we obtain, by using integration by parts,

$$\begin{aligned}
w_k &= \frac{2}{\frac{\pi}{2} + k\pi} \int_0^T \partial_t u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \\
&= \frac{2}{\frac{\pi}{2} + k\pi} \Bigg[u(t) \sin\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right)\Bigg|_0^T \\
&\qquad - \frac{1}{T}\left(\frac{\pi}{2} + k\pi\right) \int_0^T u(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \Bigg] \\
&= \frac{2}{\frac{\pi}{2} + k\pi} u(T) \sin\left(\frac{\pi}{2} + k\pi\right) - \frac{2}{T} \int_0^T u(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt \\
&= u(T)\, \overline{e}_k - \overline{u}_k\,,
\end{aligned}$$

where

$$\overline{u}_k = \frac{2}{T} \int_0^T u(t) \cos\left(\left(\frac{\pi}{2} + k\pi\right)\frac{t}{T}\right) dt$$

and

$$\overline{e}_k = \frac{2}{T}\int_0^T \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right)dt = \frac{2}{\frac{\pi}{2}+k\pi}\sin\left(\frac{\pi}{2}+k\pi\right)$$

are the Fourier coeffecients of the given function $u(t)$ and of the constant function $e(t) \equiv 1$, respectively. Hence we have

$$\begin{aligned}(\overline{\mathcal{H}}_T u)(t) &= \sum_{k=0}^{\infty} w_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\ &= u(T)\sum_{k=0}^{\infty} \overline{e}_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) - \sum_{k=0}^{\infty} \overline{u}_k \cos\left(\left(\frac{\pi}{2}+k\pi\right)\frac{t}{T}\right) \\ &= u(T) - u(t)\,.\end{aligned}$$

□

Remark 4.1 *From the representation* (4.6) *we easily conclude all the properties we have shown before. However, the form*

$$\langle u, \overline{\mathcal{H}}_T u\rangle_{L^2(0,T)} = \int_0^T u(t)\,[u(T)-u(t)]\,dt$$

is indefinite, i.e. a result as (2.21) *for the transformation* $\mathcal{H}_T$ *does not hold for* $\overline{\mathcal{H}}_T$.

For a finite element discretisation of the variational formulation (4.2) we use the same notations as in Section 2. In particular, we have to find $u_h \in V_h := S_h^1(0,T) \cap H^1_{0,}(0,T)$ such that

$$-\langle \partial_t u_h, \partial_t \overline{\mathcal{H}}_T v_h\rangle_{L^2(0,T)} = \langle f, \overline{\mathcal{H}}_T v_h\rangle_{(0,T)} \quad \text{for all } v_h \in V_h. \tag{4.7}$$

As before we have unique solvability of (4.7), the a priori error estimate (2.26) remains valid, where for $\sigma = 1$ this corresponds to the energy error estimate, while for $\sigma = 0$ we have to apply a Nitsche type argument.

For the numerical example we consider the solution $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$ for $t \in (0,T)$ with $T = 2$. The numerical results are given in Table 3, where we observe optimal order of convergence as predicted.
The stiffness matrix of the Galerkin–Bubnov finite element formulation (4.7) is symmetric, see Lemma 4.3, and positive definite, see Theorem 4.4, and its spectral behaviour is as known for finite element discretisations of second order partial differential equations. Moreover, due to (4.6), we have for the piecewise linear basis functions $\varphi_k \in H^1_{0,}(0,T)$, $k = 1,\ldots,N$,

$$(\overline{\mathcal{H}}_T\varphi_k)(t) = -\varphi_k(t) \quad \text{for } k = 1,\ldots,N-1,$$

and

$$(\overline{\mathcal{H}}_T\varphi_N)(t) = \begin{cases} 1 & \text{for } t \in [0,t_{N-1}], \\ \dfrac{T-t}{T-t_{N-1}} & \text{for } t \in (t_{N-1},T], \end{cases}$$

N	$\|u-u_h\|_{L^2}$	eoc	$\|\partial_t(u-u_h)\|_{L^2}$	eoc	$\lambda_{\min}(K_h)$	$\lambda_{\max}(K_h)$	$\kappa_2(K_h)$
4	4.970 –1	-	3.465	-	0.2412	7.06	29.28
8	1.617 –1	1.60	2.088	0.73	0.1362	15.46	113.50
16	4.307 –2	1.90	1.095	0.93	0.0724	31.71	437.70
32	1.094 –2	2.00	0.5542	0.98	0.0374	63.85	1708.66
64	2.746 –3	2.00	0.2780	1.00	0.0189	127.92	6740.68
128	6.872 –4	2.00	0.1391	1.00	0.0095	255.96	26764.98
256	1.718 –4	2.00	0.0696	1.00	0.0048	511.98	106654.71

Table 3: Numerical results for the Galerkin–Bubnov formulation (4.7).

and hence

$$\overline{\mathcal{H}}_T V_h = \operatorname{span}\{\varphi_k\}_{k=0}^{N-1}. \tag{4.8}$$

Instead of (4.1) we now consider the second order linear equation, for $\mu = \nu^2 > 0$,

$$\partial_{tt} u(t) + \mu u(t) = f(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0, \tag{4.9}$$

and the variational formulation to find $u \in H^1_{0,}(0,T)$ such that

$$a(u, \overline{\mathcal{H}}_T v) := -\int_0^T \partial_t u(t) \partial_t (\overline{\mathcal{H}}_T v)(t)\, dt + \mu \int_0^T u(t)(\overline{\mathcal{H}}_T v)(t)\, dt = \langle f, \overline{\mathcal{H}}_T v\rangle_{(0,T)} \tag{4.10}$$

is satisfied for all $v \in H^1_{0,}(0,T)$, where $f \in [H^1_{,0}(0,T)]'$ is given.

Theorem 4.6 *For given $f \in [H^1_{,0}(0,T)]'$ the variational formulation* (4.10) *admits a unique solution $u \in H^1_{0,}(0,T)$ satisfying*

$$\|u\|_{H^1_{0,}(0,T)} \leq c\, \|f\|_{[H^1_{,0}(0,T)]'}.$$

Proof. By using the Riesz representation theorem we can rewrite the variational problem (4.10) as operator equation

$$\mathcal{A}u + \mu \mathcal{C} u = \overline{f},$$

where $\mathcal{A}\colon\ H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$ defined via

$$\langle \mathcal{A}u, v\rangle = -\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} \quad \text{for } u, v \in H^1_{0,}(0,T)$$

is elliptic, and hence, invertible, and $\mathcal{C}\colon\ H^1_{0,}(0,T) \to [H^1_{0,}(0,T)]'$ defined via

$$\langle \mathcal{C}u, v\rangle = \langle u, \overline{\mathcal{H}}_T v\rangle_{L^2(0,T)} \quad \text{for } u, v \in H^1_{0,}(0,T)$$

is compact. Hence we can apply the Fredholm alternative and it remains to ensure the injectivity of $\mathcal{A} + \mu\mathcal{C}$. Let $u \in H^1_{0,}(0,T)$ be a solution of the homogeneous equation $(\mathcal{A} + \mu\mathcal{C})u = 0$, i.e.

$$\langle \partial_t u, \partial_t w\rangle_{L^2(0,T)} = \mu \langle u, w\rangle_{L^2(0,T)} \quad \text{for all } w \in H^1_{,0}(0,T).$$

This is the weak formulation of the eigenvalue problem

$$-\partial_{tt} u(t) = \mu u(t) \quad \text{for } t \in (0,T), \quad u(0) = \partial_t u(0) = 0,$$

which only admits the trivial solution $u \equiv 0$. □

While the result of Theorem 4.6 ensures unique solvability of the variational formulation (4.10), it does not include an explicit dependence on the parameter μ. Hence we will provide a stability estimate from which we can conclude such a result.

Lemma 4.7 *For $u \in H^1_{0,}(0,T)$ there holds the stability estimate*

$$\frac{2}{2+\nu T} \|\partial_t u\|_{L^2(0,T)} \leq \sup_{0 \neq v \in H^1_{,0}(0,T)} \frac{a(u,v)}{\|\partial_t v\|_{L^2(0,T)}} . \tag{4.11}$$

Proof. For given $u \in H^1_{0,}(0,T)$ and suitable chosen $w \in H^2_{,0}(0,T)$ we consider the test function $v := \overline{\mathcal{H}}_T u + w \in H^1_{,0}(0,T)$. Then,

$$\begin{aligned}
a(u,v) &= -\int_0^T \partial_t u(t) \partial_t [-u(t) + w(t)]\, dt + \mu \int_0^T u(t)[u(T) - u(t) + w(t)]\, dt \\
&= \int_0^T [\partial_t u(t)]^2\, dt - \int_0^T \partial_t u(t) \partial_t w(t)\, dt + \mu \int_0^T u(t)[u(T) - u(t) + w(t)]\, dt \\
&= \int_0^T [\partial_t u(t)]^2\, dt - u(t)\partial_t w(t)\Big|_0^T + \int_0^T u(t)\partial_{tt} w(t)\, dt \\
&\qquad + \mu \int_0^T u(t)[u(T) - u(t) + w(t)]\, dt \\
&= \int_0^T [\partial_t u(t)]^2\, dt - u(T)\partial_t w(T) \\
&\qquad + \int_0^T u(t)\Big[\partial_{tt} w(t) + \mu\Big(u(T) - u(t) + w(t)\Big)\Big]\, dt \\
&= \int_0^T [\partial_t u(t)]^2\, dt,
\end{aligned}$$

if

$$\partial_{tt} w(t) + \mu w(t) = \mu[u(t) - u(T)] \quad \text{for } t \in (0,T), \quad w(T) = \partial_t w(T) = 0$$

is satisfied. Using $\mu = \nu^2$ we obtain

$$w(t) = \nu \int_t^T \sin\Big(\nu(s-t)\Big)\, [u(s) - u(T)]\, ds,$$

and therefore

$$\begin{aligned}
\partial_t w(t) &= -\nu^2 \int_t^T \cos\Big(\nu(s-t)\Big)\,[u(s)-u(T)]\,ds \\
&= -\nu \sin\Big(\nu(s-t)\Big)\,[u(t)-u(T)]\Big|_t^T + \nu \int_t^T \sin\Big(\nu(s-t)\Big)\,\partial_s u(s)\,ds \\
&= \nu \int_t^T \sin\Big(\nu(s-t)\Big)\,\partial_s u(s)\,ds
\end{aligned}$$

follows. With

$$\begin{aligned}
[\partial_t w(t)]^2 &= \nu^2 \left[\int_t^T \sin\Big(\nu(s-t)\Big)\,\partial_s u(s)\,ds\right]^2 \\
&\leq \nu^2 \int_t^T \sin^2\Big(\nu(s-t)\Big)\,ds \int_t^T [\partial_s u(s)]^2\,ds \\
&\leq \nu^2 \int_t^T \sin^2\Big(\nu(s-t)\Big)\,ds \int_0^T [\partial_t u(t)]^2\,dt
\end{aligned}$$

we further conclude

$$\begin{aligned}
\int_0^T [\partial_t w(t)]^2\,dt &\leq \nu^2 \int_0^T \int_t^T \sin^2\Big(\nu(s-t)\Big)\,ds\,dt \int_0^T [\partial_t u(t)]^2\,dt \\
&= \nu^2\,\frac{1}{4}\,\frac{\cos^2(\nu T)-1+\nu^2T^2}{\nu^2} \int_0^T [\partial_t u(t)]^2\,dt \\
&\leq \frac{1}{4}\nu^2T^2 \int_0^T [\partial_t u(t)]^2\,dt,
\end{aligned}$$

i.e.

$$\|\partial_t w\|_{L^2(0,T)} \leq \frac{1}{2}\nu T\,\|\partial_t u\|_{L^2(0,T)}\,.$$

With this we finally have

$$\begin{aligned}
\|\partial_t v\|_{L^2(0,T)} &= \|\partial_t w - \partial_t u\|_{L^2(0,T)} \\
&\leq \|\partial_t w\|_{L^2(0,T)} + \|\partial_t u\|_{L^2(0,T)} \leq \left(1+\frac{1}{2}\nu T\right)\|\partial_t u\|_{L^2(0,T)},
\end{aligned}$$

and therefore

$$a(u,v) = \|\partial_t u\|^2_{L^2(0,T)} \geq \frac{2}{2+\nu T}\,\|\partial_t u\|_{L^2(0,T)}\|\partial_t v\|_{L^2(0,T)}$$

follows, which implies the stability condition as stated. □

While Theorem 4.6 implies unique solvability of the variational formulation (4.10) we can use the stability condition (4.11) to conclude a bound for the solution u which explicitely depends on ν.

Corollary 4.8 *For the unique solution $u \in H^1_{0,}(0,T)$ of the variational formulation* (4.10) *there holds*

$$\|\partial_t u\|_{L^2(0,T)} \leq \left(1 + \frac{1}{2}\nu T\right) \|f\|_{[H^1_{,0}(0,T)]'}\,. \tag{4.12}$$

Remark 4.2 *We consider the initial value problem* (4.9) *for $f(t) = \sin(\nu t)$ with the solution*

$$u(t) = \frac{1}{2\nu^2}\Big[\sin(\nu t) - \nu t\cos(\nu t)\Big], \quad \partial_t u(t) = \frac{1}{2}t\sin(\nu t).$$

For this we compute

$$\begin{aligned}\|\partial_t u\|^2_{L^2(0,T)} &= \frac{1}{48}\frac{1}{\nu^3}\Big[2\nu^3T^3 + 3\nu T - 6\nu^2T^2\cos(\nu T)\sin(\nu T) \\ &\qquad -6\nu T\cos^2(\nu T) + 3\cos(\nu T)\sin(\nu T)\Big] \\ &\simeq \frac{1}{24}T^3\end{aligned}$$

as $\nu \to \infty$. On the other hand we determine $w \in H^1_{,0}(0,T)$ as unique solution of the boundary value problem

$$-\partial_{tt}w(t) = f(t) \quad \textit{for}\, t \in (0,T), \quad \partial_t w(0) = w(T) = 0,$$

i.e.

$$\partial_t w(t) = \frac{1}{\nu}\Big[\cos(\nu t) - 1\Big].$$

Hence we compute

$$\|f\|^2_{[H^1_{,0}(0,T)]'} = \|\partial_t w\|^2_{L^2(0,T)} = \frac{1}{2}\frac{1}{\nu^3}\Big[3\nu T + \cos(\nu T)\sin(\nu T) - 4\sin(\nu T)\Big] \simeq \frac{3}{2}\frac{T}{\nu^2}$$

as $\nu \to \infty$. In particular we have

$$\frac{\|\partial_t u\|_{L^2(0,T)}}{\|f\|_{[H^1_{,0}(0,T)]'}} \simeq \frac{1}{6}\nu T$$

as $\nu \to \infty$, which shows that the estimate (4.12) *is sharp with respect to the order of ν and T, respectively.*

While for $f \in [H^1_{,0}(0,T)]'$ the bound (4.12) shows an explicit dependence on $\nu = \sqrt{\mu}$, we can prove an estimate independent of μ when assuming $f \in L^2(0,T)$.

Lemma 4.9 *For given $f \in L^2(0,T)$ the unique solution $u \in H^1_{0,}(0,T)$ satisfies*

$$\|u\|^2_{H^1_{0,}(0,T)} + \mu\,\|u\|^2_{L^2(0,T)} \leq \frac{1}{2}T^2\,\|f\|^2_{L^2(0,T)}\,. \tag{4.13}$$

Proof. For the solution u and its first order derivatitve we find the representations

$$u(t) = \frac{1}{\nu} \int_0^t \sin\Big(\nu(t-s)\Big)\, f(s)\, ds$$

and

$$\partial_t u(t) = \int_0^t \cos\Big(\nu(t-s)\Big)\, f(s)\, ds\,.$$

Hence we compute

$$\begin{aligned}
[\partial_t u(t)]^2 + \nu^2\, [u(t)]^2 &= \left[\int_0^t \cos\Big(\nu(t-s)\Big)\, f(s)\, ds\right]^2 + \left[\int_0^t \sin\Big(\nu(t-s)\Big)\, f(s)\, ds\right]^2 \\
&\leq \int_0^t \cos^2\Big(\nu(t-s)\Big) ds \int_0^t [f(s)]^2\, ds + \int_0^t \sin^2\Big(\nu(t-s)\Big) ds \int_0^t [f(s)]^2\, ds \\
&= t \int_0^t [f(s)]^2\, ds \leq t \int_0^T [f(s)]^2\, ds,
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
\|u\|^2_{H^1_{0,}(0,T)} + \mu\, \|u\|^2_{L^2(0,T)} &= \int_0^T \Big\{[\partial_t u(t)]^2 + \mu [u(t)]^2\Big\}\, dt \\
&\leq \int_0^T t\, dt \int_0^T [f(s)]^2\, ds = \frac{1}{2} T^2\, \|f\|^2_{L^2(0,T)}\,.
\end{aligned}$$

□

Remark 4.3 *As in Remark* 4.2 *we consider problem* (4.9) *for* $f(t) = \sin(\nu t)$ *with the solution* $u(t)$ *and its derivative* $\partial_t u(t) = \frac{1}{2} t \sin(\nu t)$, *i.e.*

$$\|\partial_t u\|^2_{L^2(0,T)} \simeq \frac{1}{24} T^3, \quad \|f\|^2_{L^2(0,T)} = \frac{1}{2}\frac{1}{\nu}\Big[\nu T - \cos(\nu T)\sin(\nu T)\Big] \simeq \frac{1}{2} T.$$

Hence we conclude

$$\frac{\|\partial_t u\|^2_{L^2(0,T)}}{\|f\|^2_{L^2(0,T)}} \simeq \frac{1}{12} T^2,$$

i.e. the estimate (4.13) *is sharp with respect to the order of* T.

The Galerkin–Bubnov finite element formulation of the equivalent variational formulation (4.10) is to find $u_h \in V_h := S^1_h(0,T) \cap H^1_{0,}(0,T)$ such that

$$a(u_h, \overline{\mathcal{H}}_T v_h) = -\langle \partial_t u_h, \partial_t \overline{\mathcal{H}}_T v_h \rangle_{L^2(0,T)} + \mu \langle u_h, \overline{\mathcal{H}}_T v_h \rangle_{L^2(0,T)} = \langle f, \overline{\mathcal{H}}_T v_h \rangle_{(0,T)} \qquad (4.14)$$

is satisfied for all $v_h \in V_h$. Unique solvability and related error estimates follow as for the numerical solution of elliptic operator equations with compact perturbations, which is based on a discrete stability condition.

Theorem 4.10 *Let*

$$h \leq \frac{2\sqrt{3}}{(2+\sqrt{\mu}T)\mu T} \tag{4.15}$$

be satisfied. Then the bilinear form $a(\cdot,\cdot)$ *as defined in* (4.10) *satisfies the stability condition*

$$\frac{4}{(2+\sqrt{\mu}T)^2(2+\mu T)} \|\partial_t u_h\|_{L^2(0,T)} \leq \sup_{0\neq v_h\in V_h} \frac{a(u,\overline{\mathcal{H}}_T v_h)}{\|\partial_t v_h\|_{L^2(0,T)}} \quad \textit{for all } u_h \in V_h. \tag{4.16}$$

Proof. For $u_h \in V_h$ we define $w \in H^1_{0,}(0,T)$ as the unique solution of the variational problem

$$-\int_0^T \partial_t w(t)\partial_t(\overline{\mathcal{H}}_T v)(t)\,dt = -\mu\int_0^T u_h(t)(\overline{\mathcal{H}}_T v)(t)\,dt \quad \text{for all } v \in H^1_{0,}(0,T), \tag{4.17}$$

i.e. $w \in H^1_{0,}(0,T)$ is the weak solution of the initial value problem

$$\partial_{tt} w(t) = -\mu u_h(t) \quad \text{for } t \in (0,T), \quad w(0) = \partial_t w(0) = 0.$$

Then, by using $(\overline{\mathcal{H}}_T v)(t) = v(T) - v(t)$,

$$\begin{aligned} a(u_h, \overline{\mathcal{H}}_T(u_h - w)) &= -\int_0^T \partial_t u_h(t)\partial_t[(\overline{\mathcal{H}}_T u_h)(t) - (\overline{\mathcal{H}}_T w)(t)]\,dt \\ &\qquad +\mu\int_0^T u_h(t)[(\overline{\mathcal{H}}_T u_h)(t) - (\overline{\mathcal{H}}_T w)(t)]\,dt \\ &= \int_0^T \partial_t u_h(t)[\partial_t u_h(t) - \partial_t w(t)]\,dt - \int_0^T \partial_t w(t)[\partial_t u_h(t) - \partial_t w(t)]\,dt \\ &= \int_0^T [\partial_t u_h(t) - \partial_t w(t)]^2\,dt\,. \end{aligned}$$

In addition, let $z \in H^1_{0,}(0,T)$ be the unique solution of the variational formulation such that

$$-\int_0^T \partial_t z(t)\partial_t(\overline{\mathcal{H}}_T v)(t)\,dt = -\int_0^T \partial_t u_h(t)\partial_t(\overline{\mathcal{H}}_T v)(t)\,dt + \mu\int_0^T u_h(t)(\overline{\mathcal{H}}_T v)(t)\,dt \tag{4.18}$$

is satisfied for all $v \in H^1_{0,}(0,T)$. With (4.17) this is equivalent to

$$-\int_0^T \partial_t[z(t) - (u_h(t) - w(t))]\partial_t(\overline{\mathcal{H}}_T v)(t)\,dt = 0 \quad \text{for all } v \in H^1_{0,}(0,T),$$

from which we conclude, recall $u_h(0) = w(0) = z(0) = 0$,

$$z(t) = u_h(t) - w(t),$$

i.e. we have

$$a(u_h, \overline{\mathcal{H}}_T(u_h - w)) = \|\partial_t z\|^2_{L^2(0,T)}.$$

On the other hand, the variational formulation (4.18) gives

$$\begin{aligned}\|\partial_t z\|_{L^2(0,T)} = \frac{a(u_h, \overline{\mathcal{H}}_T z)}{\|\partial_t z\|_{L^2(0,T)}} &\leq \sup_{0 \neq v \in H^1_{0,}(0,T)} \frac{a(u_h, \overline{\mathcal{H}}_T v)}{\|\partial_t v\|_{L^2(0,T)}} \\ &= \sup_{0 \neq v \in H^1_{0,}(0,T)} \frac{\langle \partial_t z, \partial_t v\rangle_{L^2(0,T)}}{\|\partial_t v\|_{L^2(0,T)}} \leq \|\partial_t z\|_{L^2(0,T)},\end{aligned}$$

i.e.

$$\|\partial_t z\|_{L^2(0,T)} = \sup_{0 \neq v \in H^1_{0,}(0,T)} \frac{a(u_h, \overline{\mathcal{H}}_T v)}{\|\partial_t v\|_{L^2(0,T)}} \geq \frac{2}{2 + \nu T} \|\partial_t u_h\|_{L^2(0,T)}$$

when using (4.11). With this we now conclude

$$a(u_h, \overline{\mathcal{H}}_T(u_h - w)) \geq \frac{4}{(2 + \nu T)^2} \|\partial_t u_h\|^2_{L^2(0,T)}.$$

According to (4.17) we define $w_h \in V_h$ as the unique solution of

$$\int_0^T \partial_t w_h(t) \partial_t v_h(t)\, dt = -\mu \int_0^T u_h(t)(\overline{\mathcal{H}}_T v_h)(t)\, dt \quad \text{for all } v_h \in V_h.$$

Then there holds the Galerkin orthogonality

$$\int_0^T [\partial_t w(t) - \partial_t w_h(t)]\, \partial_t v_h(t)\, dt = 0 \quad \text{for all } v_h \in V_h, \tag{4.19}$$

and the error estimate, by using Céa's lemma and standard interpolation error estimates,

$$\begin{aligned}\|\partial_t w - \partial_t w_h\|_{L^2(0,T)} &\leq \inf_{v_h \in V_h} \|\partial_t w - \partial_t v_h\|_{L^2(0,T)} \\ &\leq \|\partial_t (w - I_h w)\|_{L^2(0,T)} \leq \frac{1}{\sqrt{3}}\, h\, \|\partial_{tt} w\|_{L^2(0,T)} = \frac{1}{\sqrt{3}}\, \mu h\, \|u_h\|_{L^2(0,T)}.\end{aligned}$$

With this we have

$$\begin{aligned}a(u_h, \overline{\mathcal{H}}_T(w - w_h)) &= \int_0^T \partial_t u_h(t) \partial_t (w(t) - w_h(t))\, dt + \mu \int_0^T u_h(t)(\overline{\mathcal{H}}_T(w - w_h))(t)\, dt \\ &= \mu \int_0^T u_h(t)(\overline{\mathcal{H}}_T(w - w_h))(t)\, dt \\ &\leq \mu\, \|u_h\|_{L^2(0,T)} \|\overline{\mathcal{H}}_T(w - w_h)\|_{L^2(0,T)}.\end{aligned}$$

Now we define $\psi \in H^1_{0,}(0,T)$ as unique solution of the variational formulation

$$-\int_0^T \partial_t \psi(t) \partial_t (\overline{\mathcal{H}}_T v)(t)\, dt = \int_0^T (\overline{\mathcal{H}}_T(w - w_h))(t)(\overline{\mathcal{H}}_T v)(t)\, dt \quad \text{for all } v \in H^1_{0,}(0,T),$$

i.e.

$$\partial_{tt}\psi(t) = (\overline{\mathcal{H}}_T(w-w_h))(t) \quad \text{for } t \in (0,T), \quad \psi(0) = \partial_t\psi(t) = 0.$$

In particular for $v = w - w_h \in H^1_{0,}(0,T)$ we then conclude

$$\begin{aligned}
\|\overline{\mathcal{H}}_T(w-w_h)\|^2_{L^2(0,T)} &= \int_0^T (\overline{\mathcal{H}}_T(w-w_h))(t)(\overline{\mathcal{H}}_T(w-w_h))(t)\,dt \\
&= -\int_0^T \partial_t\psi(t)\partial_t(\overline{\mathcal{H}}_T(w-w_h))(t)\,dt \\
&= \int_0^T \partial_t\psi(t)[\partial_t w(t) - \partial_t w_h(t)]\,dt \\
&= \int_0^T \partial_t[\psi(t) - I_h\psi(t)][\partial_t w(t) - \partial_t w_h(t)]\,dt \\
&\leq \|\partial_t(\psi - I_h\psi)\|_{L^2(0,T)}\|\partial_t(w-w_h)\|_{L^2(0,T)} \\
&\leq \frac{1}{3}h^2\,\|\partial_{tt}\psi\|_{L^2(0,T)}\|\partial_{tt}w\|_{L^2(0,T)} \\
&= \frac{1}{3}\mu h^2\,\|\overline{\mathcal{H}}_T(w-w_h)\|_{L^2(0,T)}\|u_h\|_{L^2(0,T)},
\end{aligned}$$

i.e.

$$\|\overline{\mathcal{H}}_T(w-w_h)\|_{L^2(0,T)} \leq \frac{1}{3}\mu h^2\|u_h\|_{L^2(0,T)},$$

and therefore, by using $u_h \in H^1_{0,}(0,T)$,

$$a(u_h, \overline{\mathcal{H}}_T(w-w_h)) \leq \frac{1}{3}\,\mu^2h^2\,\|u_h\|^2_{L^2(0,T)} \;\leq\; \frac{1}{6}\,\mu^2h^2T\,\|\partial_t u_h\|^2_{L^2(0,T)}$$

follows. Hence we conclude

$$\begin{aligned}
a(u_h, \overline{\mathcal{H}}_T(u_h-w_h)) &= a(u_h, \overline{\mathcal{H}}_T(u_h-w)) + a(u_h, \overline{\mathcal{H}}_T(w-w_h)) \\
&\geq \left[\frac{4}{(2+\sqrt{\mu}T)^2} - \frac{1}{6}\mu^2h^2T\right]\,\|\partial_t u_h\|^2_{L^2(0,T)} \\
&\geq \frac{2}{(2+\sqrt{\mu}T)^2}\,\|\partial_t u_h\|^2_{L^2(0,T)},
\end{aligned}$$

if

$$\frac{1}{6}\mu^2h^2T \leq \frac{2}{(2+\sqrt{\mu}T)^2}$$

is satisfied, i.e.

$$h^2 \leq \frac{12}{(2+\sqrt{\mu}T)^2\mu^2T}\,.$$

Finally we have

$$\|\partial_t(u_h-w_h)\|_{L^2(0,T)} \leq \|\partial_t u_h\|_{L^2(0,T)} + \|\partial_t w_h\|_{L^2(0,T)}$$

and

$$
\begin{aligned}
\|\partial_t w_h\|^2_{L^2(0,T)} &= -\int_0^T \partial_t w_h(t)\partial_t(\overline{\mathcal{H}}_T w_h)(t)\,dt = -\mu\int_0^T u_h(t)(\overline{\mathcal{H}}_T w_h)(t)\,dt \\
&\le \mu\|u_h\|_{L^2(0,T)}\|\overline{\mathcal{H}}_T w_h\|_{L^2(0,T)} \le \frac{1}{2}\mu T\,\|\partial_t u_h\|_{L^2(0,T)}\|\partial_t w_h\|_{L^2(0,T)},
\end{aligned}
$$

i.e.

$$
\|\partial_t(u_h - w_h)\|_{L^2(0,T)} \le \left(1+\frac{1}{2}\mu T\right)\|\partial_t u_h\|_{L^2(0,T)}.
$$

This concludes the proof. □

For any $w \in H^1_{0,}(0,T)$ we define $w_h = G_h w \in V_h$ as the Galerkin projection satisfying

$$
a(G_h w, \overline{\mathcal{H}}_T v_h) = a(w, \overline{\mathcal{H}}_T v_h) \quad \text{for all } v_h \in V_h,
$$

where the stability condition (4.16) implies

$$
\begin{aligned}
\frac{4}{(2+\sqrt{\mu}T)^2(2+\mu T)}\|\partial_t G_h w\|_{L^2(0,T)} &\le \sup_{0\neq v_h\in V_h}\frac{a(G_h w,\overline{\mathcal{H}}_T v_h)}{\|\partial_t v_h\|_{L^2(0,T)}} \\
&= \sup_{0\neq v_h\in V_h}\frac{a(w,\overline{\mathcal{H}}_T v_h)}{\|\partial_t v_h\|_{L^2(0,T)}} \\
&\le \sup_{0\neq v_h\in V_h}\frac{\|\partial_t w\|_{L^2(0,T)}\|\partial_t v_h\|_{L^2(0,T)} + \mu\|w\|_{L^2(0,T)}\|\overline{\mathcal{H}}_T v_h\|_{L^2(0,T)}}{\|\partial_t v_h\|_{L^2(0,T)}} \\
&\le \left(1+\frac{1}{2}\mu T\right)\|\partial_t w\|_{L^2(0,T)},
\end{aligned}
$$

i.e.

$$
\|\partial_t G_h w\|_{L^2(0,T)} \le \frac{1}{8}(2+\sqrt{\mu}T)^2(2+\mu T)^2\|\partial_t w\|_{L^2(0,T)} \quad \text{for all } w\in H^1_{0,}(0,T). \tag{4.20}
$$

Now we are in a position to state a convergence result for the finite element solution u_h of the variational formulation (4.10).

Theorem 4.11 *Let $u \in H^1_{0,}(0,T)$ and $u_h \in V_h \subset H^1_{0,}(0,T)$ be the unique solutions of the variational formulations* (4.10) *and* (4.14)*, respectively. We assume $u \in H^2(0,T)$ and let* (4.15) *be satisfied. Then there holds the error estimate*

$$
\|\partial_t(u-u_h)\|_{L^2(0,T)} \le \frac{1}{\sqrt{3}}\left[1+\frac{1}{8}(2+\sqrt{\mu}T)^2(2+\mu T)^2\right] h\,\|\partial_{tt}u\|_{L^2(0,T)}. \tag{4.21}
$$

Proof. With $u_h = G_h u$ and $v_h = G_h v_h$ for all $v_h \in V_h$ we have Céa's lemma,

$$
\begin{aligned}
\|\partial_t(u-u_h)\|_{L^2(0,T)} &\le \|\partial_t(u-v_h)\|_{L^2(0,T)} + \|\partial_t G_h(u-v_h)\|_{L^2(0,T)} \\
&\le \left[1+\frac{1}{8}(2+\sqrt{\mu}T)^2(2+\mu T)^2\right]\|\partial_t(u-v_h)\|_{L^2(0,T)}
\end{aligned}
$$

for all $v_h \in V_h$, and the assertion follows from standard interpolation error estimates. $\square$

For the discretisation of the variational formulation (4.14) we use the same notation as in Section 2 to find $u_h \in V_h = \text{span}\{\varphi_k\}_{k=1}^N$ such that

$$-\int_0^T \partial_t u_h(t)\partial_t \varphi_\ell(t)\,dt + \mu \int_0^T u_h(t)\varphi_\ell(t)\,dt = \int_0^T f(t)\varphi_\ell(t)\,dt =: f_\ell \tag{4.22}$$

is satisfied for all $\ell = 0, \ldots, N-1$. Since we consider a uniform discretisation, the stiffness matrix is given by

$$K_h = \frac{1}{h}\begin{pmatrix} 1 & & & & \\ -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \end{pmatrix} + \frac{1}{6}\mu h \begin{pmatrix} 1 & & & & \\ 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \end{pmatrix}$$

Obviously we have unique solvability of (4.22) for all μ independent of h.

Remark 4.4 *For $\ell = 2, \ldots, N-1$ we can write the finite element formulation* (4.22) *as*

$$\left(\frac{1}{h} + \frac{1}{6}\mu h\right) u_{\ell-1} + \left(\frac{2}{3}\mu h - \frac{2}{h}\right) u_\ell + \left(\frac{1}{h} + \frac{1}{6}\mu h\right) u_{\ell+1} = f_\ell, \tag{4.23}$$

which is a kind of a two–step method [16, Chapter III.2]. This method is zero–stable, iff the root condition [16, Chapter III.3] is satisfied. For (4.23) *we therefore conclude the condition*

$$\mu h^2 < 12\,. \tag{4.24}$$

For the numerical example we consider again the solution $u(t) = \sin^2\left(\frac{5}{4}\pi t\right)$ for $t \in (0,T)$ with $T = 2$ and $\mu = 10$. The numerical results are given in Table 4 where we observe linear convergence in the energy norm as predicted in (4.21), and second order convergence in $L^2(0,T)$ which can be proven when applying the Nitsche trick. In Table 5 we present the related numerical results for the case $\mu = 1000$. We observe convergence only when h is sufficiently small. According to (4.24) we note that $\sqrt{12/\mu} \approx 0.1095$. So it remains open to improve assumption (4.15) to ensure the stability condition (4.16). On the other hand, following [40], it is possible to derive and to analyse a stabilised variational formulation for the initial value problem (4.9), see [33]. Using the L^2 projection Q_h^0 on the finite element space $S_h^0(0,T)$ of piecewise constant functions we may consider, instead of (4.14), the perturbed variational problem to find $\widetilde{u}_h \in S_h^1(0,T) \cap H^1_{0,}(0,T)$ such that

$$-\langle \partial_t \widetilde{u}_h, \partial_t w_h \rangle_{L^2(0,T)} + \mu \langle \widetilde{u}_h, Q_h^0 w_h \rangle_{L^2(0,T)} = \langle f, w_h \rangle_{(0,T)} \tag{4.25}$$

is satisfied for all $w_h \in S_h^1(0,T) \cap H^1_{,0}(0,T)$. The stability and error analysis of (4.25) is based on a discrete inf–sup condition [33, Lemma 6], which then results in an optimal energy error estimate [33, Theorem 1].

N	h	$\|u-u_h\|_{L^2}$	eoc	$\|\partial_t(u-u_h)\|_{L^2}$	eoc
4	0.5000000	5.3407e-01	-	3.5365e+00	-
8	0.2500000	1.7632e-01	1.6	2.1021e+00	0.8
16	0.1250000	4.9649e-02	1.8	1.0979e+00	0.9
32	0.0625000	1.2804e-02	2.0	5.5462e-01	1.0
64	0.0312500	3.2263e-03	2.0	2.7800e-01	1.0
128	0.0156250	8.0816e-04	2.0	1.3909e-01	1.0
256	0.0078125	2.0214e-04	2.0	6.9555e-02	1.0
512	0.0039062	5.0541e-05	2.0	3.4779e-02	1.0
1024	0.0019531	1.2636e-05	2.0	1.7390e-02	1.0
2048	0.0009766	3.1589e-06	2.0	8.6948e-03	1.0
4096	0.0004883	7.8972e-07	2.0	4.3474e-03	1.0
8192	0.0002441	1.9737e-07	2.0	2.1737e-03	1.0

Table 4: Numerical results for the Galerkin–Petrov formulation (4.22), $\mu = 10$.

N	h	$\|u-u_h\|_{L^2}$	eoc	$\|\partial_t(u-u_h)\|_{L^2}$	eoc
4	0.5000000	8.0288e+00	-	4.1323e+01	-
8	0.2500000	2.3961e+02	-4.9	2.4811e+03	-5.9
16	0.1250000	4.4282e+01	2.4	1.1065e+03	1.2
32	0.0625000	9.5909e-03	12.2	6.2095e-01	10.8
64	0.0312500	2.7371e-03	1.8	2.8953e-01	1.1
128	0.0156250	7.1356e-04	1.9	1.4072e-01	1.0
256	0.0078125	1.8124e-04	2.0	6.9765e-02	1.0
512	0.0039062	4.5486e-05	2.0	3.4805e-02	1.0
1024	0.0019531	1.1382e-05	2.0	1.7393e-02	1.0
2048	0.0009766	2.8463e-06	2.0	8.6952e-03	1.0
4096	0.0004883	7.1162e-07	2.0	4.3474e-03	1.0
8192	0.0002441	1.7791e-07	2.0	2.1737e-03	1.0

Table 5: Numerical results for the Galerkin–Petrov formulation (4.22), $\mu = 1000$.

5 Wave equation

As model problem for a hyperbolic partial differential equation we consider the Dirichlet problem for the wave equation,

$$\left.\begin{aligned} \partial_{tt}u(x,t) - \Delta_x u(x,t) &= f(x,t) && \text{for } (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) &= 0 && \text{for } (x,t) \in \Sigma := \Gamma \times (0,T), \\ u(x,0) = \partial_t u(x,0) &= 0 && \text{for } x \in \Omega, \end{aligned}\right\} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 1,2,3$, is a bounded domain with, for $d = 2,3$, Lipschitz boundary $\Gamma = \partial\Omega$. According to the previous sections we consider the variational formulation of (5.1) to find $u \in H^{1,1}_{0;0,}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;L^2(\Omega))$ such that

$$-\langle \partial_t u, \partial_t v\rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} = \langle f, v\rangle_{L^2(Q)} \tag{5.2}$$

is satisfied for all $v \in H^{1,1}_{0;,0}(Q) := L^2(0,T;H^1_0(\Omega)) \cap H^1_{,0}(0,T;L^2(\Omega))$. Note that the initial condition $u(\cdot,0) = 0$ is considered in the strong sense, whereas the initial condition $\partial_t u(\cdot,0) = 0$ is incorporated in a weak sense. For $u \in H^{1,1}_{0;0,}(Q)$, an appropriate norm is given by

$$\|u\|^2_{H^{1,1}_{0;0,}(Q)} = \int_0^T \int_\Omega \Big[|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2\Big]\,dx\,dt\,.$$

As in [19] we state the following result on unique solvability of the variational formulation (5.2) when assuming $f \in L^2(Q)$.

Theorem 5.1 *For $f \in L^2(Q)$ there exists a unique solution $u \in H^{1,1}_{0;0,}(Q)$ of the variational formulation* (5.2) *satisfying*

$$\|u\|_{H^{1,1}_{0;0,}(Q)} \leq \frac{1}{\sqrt{2}}\,T\,\|f\|_{L^2(Q)}.$$

Proof. When using the representation (3.2), any $u \in H^{1,1}_{0;0,}(Q)$ can be written as

$$u(x,t) = \sum_{i=1}^\infty \sum_{k=0}^\infty u_{i,k} v_k(t)\phi_i(x) = \sum_{i=1}^\infty U_i(t)\phi_i(x), \tag{5.3}$$

where $v_k(t)$ are the temporal eigenfunctions as given in (2.11), and $\phi_i(x)$ are the spatial $L^2(\Omega)$ orthonormal eigenfunctions of the Laplacian with homogeneous Dirichlet boundary conditions. For the solution of the variational problem (5.2) we now use the ansatz (5.3) where the $U_i \in H^1_{0,}(0,T)$ are unknown functions to be determined. When choosing, for a fixed $j \in \mathbb{N}$, $v(x,t) = V(t)\phi_j(x)$ with $V \in H^1_{,0}(0,T)$ as test function, the variational formulation (5.2) results in finding $U_j \in H^1_{0,}(0,T)$ such that

$$-\int_0^T \partial_t U_j(t)\partial_t V(t)\,dt + \mu_j \int_0^T U_j(t)V(t)\,dt = \int_0^T f_j(t)V(t)\,dt \tag{5.4}$$

is satisfied for all $V \in H^1_{,0}(0,T)$ where

$$f_j(t) = \int_\Omega f(x,t)\phi_j(x)dx$$

are the coefficients of the Fourier expansion

$$f(x,t) = \sum_{j=1}^\infty f_j(t)\phi_j(x).$$

From this we conclude

$$\begin{aligned}\|f\|^2_{L^2(Q)} &= \int_0^T \int_\Omega [f(x,t)]^2\,dx\,dt = \sum_{i=1}^\infty \sum_{j=1}^\infty \int_0^T f_i(t) f_j(t)\,dt \int_\Omega \phi_i(x)\phi_j(x)\,dx \\ &= \sum_{j=1}^\infty \int_0^T [f_j(t)]^2\,dt = \sum_{j=1}^\infty \|f_j\|^2_{L^2(0,T)},\end{aligned}$$

and hence we obtain, by using (4.13),

$$\begin{aligned}\|u\|^2_{H^{1,1}_{0;0,}(Q)} &= \int_0^T \int_\Omega \Big[|\partial_t u(x,t)|^2 + |\nabla_x u(x,t)|^2 \Big]\,dx\,dt \\ &= \sum_{i=1}^\infty \sum_{j=1}^\infty \Big[\int_0^T \partial_t U_i(t) \partial_t U_j(t) dt \int_\Omega \phi_i(x)\phi_j(x)\,dx \\ &\qquad + \int_0^T U_i(t) U_j(t)\,dt \int_\Omega \nabla_x \phi_i(x) \cdot \nabla_x \phi_j(x)\,dx \Big] \\ &= \sum_{i=1}^\infty \Big[\int_0^T |\partial_t U_i(t)|^2 dt + \mu_i \int_0^T |U_i(t)|^2\,dt \Big] \\ &= \sum_{i=1}^\infty \Big[\|U_i\|^2_{H^1_{0,}(0,T)} + \mu_i \|U_i\|^2_{L^2(0,T)} \Big] \\ &\leq \frac{1}{2} T^2 \sum_{i=1}^\infty \|f_i\|^2_{L^2(0,T)} = \frac{1}{2} T^2\, \|f\|^2_{L^2(Q)}.\end{aligned}$$

□

The variational formulation (5.2) is equivalent to find $u \in H^{1,1}_{0;0,}(Q)$ such that

$$-\langle \partial_t u, \partial_t \overline{\mathcal{H}}_T v \rangle_{L^2(Q)} + \langle \nabla_x u, \nabla_x \overline{\mathcal{H}}_T v \rangle_{L^2(Q)} = \langle f, \overline{\mathcal{H}}_T v \rangle_{L^2(Q)} \tag{5.5}$$

is satisfied for all $v \in H^{1,1}_{0;0,}(Q)$, where the transformation operator $\overline{\mathcal{H}}_T$ acts only on the time variable t.

As in the case of the heat equation we consider the tensor–product space–time finite element space $\mathcal{V}_h = W_{h_x} \otimes V_{h_t} \subset H^{1,1}_{0;0,}(Q)$. Then, the Galerkin–Bubnov finite element discretisation of the variational formulation (5.5) is to find $u_h \in \mathcal{V}_h$ such that

$$-\langle \partial_t u_h, \partial_t \overline{\mathcal{H}}_T v_h \rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x \overline{\mathcal{H}}_T v_h \rangle_{L^2(Q)} = \langle f, \overline{\mathcal{H}}_T v_h \rangle_{L^2(Q)} \tag{5.6}$$

is satisfied for all $v_h \in \mathcal{V}_h$. Recall that the transformation $\overline{\mathcal{H}}_T \varphi_k$ is realised by using (4.8). Since we are using a tensor–product space–time finite element space $\mathcal{V}_h = W_{h_x} \otimes V_{h_t}$, we can write

$$u_h(x,t) = \sum_{i=1}^N \sum_{k=1}^M u_{i,k} \varphi_k(t) \psi_i(x) = \sum_{i=1}^N U_{i,h}(t) \psi_i(x), \quad U_{i,h}(t) = \sum_{k=1}^M u_{i,k} \varphi_k(t).$$

By using

$$\widetilde{u}(x,t) = \sum_{i=1}^{N} \widetilde{U}_i(t)\psi_i(x)$$

we can write the intermediate step of the semi–discretisation approach for solving (5.1) as

$$M_h \partial_{tt} \underline{\widetilde{U}}(t) + K_h \underline{\widetilde{U}}(t) = \underline{f}(t) \quad \text{for } t \in (0,T), \quad \underline{\widetilde{U}}(0) = \partial_t \underline{\widetilde{U}}(0) = \underline{0}\,,$$

with the spatial finite element mass matrix M_h, the stiffness matrix K_h, and the load vector $\underline{f}(t)$, i.e. for $i,j = 1,\ldots,N$,

$$\begin{aligned} M_h[j,i] &= \int_\Omega \psi_i(x)\psi_j(x)\,dx, \\ K_h[j,i] &= \int_\Omega \nabla_x\psi_i(x)\cdot\nabla_x\psi_j(x)\,dx, \\ f_j(t) &= \int_\Omega f(x,t)\psi_j(x)\,dx. \end{aligned}$$

By using

$$M_h = L_h L_h^\top, \quad A_h = L_h^{-1} K_h L_h^{-\top}, \quad \underline{W} = L_h^\top \underline{\widetilde{U}}, \quad \underline{g}(t) = L_h^{-1}\underline{f}(t),$$

we further obtain

$$\partial_{tt}\underline{W}(t) + A_h\underline{W}(t) = \underline{g}(t) \quad \text{for } t \in (0,T), \quad \underline{W}(0) = \partial_t\underline{W}(0) = \underline{0}\,.$$

Since A_h is symmetric and positive definite, we conclude the diagonal representation

$$A_h = V_h D_h V_h^\top, \quad D_h = \operatorname{diag}\Big(\lambda_k(A_h)\Big), \quad V_h = \Big(\underline{v}^1,\ldots,\underline{v}^N\Big), \quad A_h\underline{v}^k = \lambda_k\underline{v}^k.$$

By using $\underline{Z}(t) := V_h^\top \underline{W}(t)$ we finally have to solve

$$\partial_{tt}\underline{Z}(t) + D_h\underline{Z}(t) = V_h^\top\underline{g}(t) =: \underline{\widetilde{g}}(t) \quad \text{for } t \in (0,T), \quad \underline{Z}(0) = \partial_t\underline{Z}(0) = \underline{0}\,,$$

which consists of N scalar equations of the form (4.9). The related finite element solution is defined by finding, for $k = 1,\ldots,N$, $z_{k,h_t} \in V_{h_t} = S^1_{h_t}(0,T) \cap H^1_{0,}(0,T)$ such that

$$-\langle \partial_t z_{k,h_t}, \partial_t \overline{\mathcal{H}}_T v_{k,h_t}\rangle_{L^2(0,T)} + \lambda_k(A_h)\langle z_{k,h_t}, \overline{\mathcal{H}}_T v_{k,h_t}\rangle_{L^2(0,T)} = \langle \widetilde{g}_k, \overline{\mathcal{H}}_T v_{k,h_t}\rangle_{(0,T)}$$

for all $v_{k,h_t} \in V_{h_t}$. By construction we have

$$\underline{Z}_h(t) = V_h^\top L_h^\top \underline{U}_h(t)$$

where

$$\underline{U}_h(t) = \Big(U_{1,h}(t),\ldots,U_{N,h}(t)\Big)^\top$$

is the vector of the unknown functions of the approximation $u_h(x,t)$.

Stability and related error estimates for the finite element solutions z_{k,h_t} follow for sufficient small time mesh sizes h_t, see Theorem 4.11. However, as in Remark 4.4 we have stability when the condition (4.24) is satisfied, i.e.

$$\lambda_k(A_h) = \frac{(K_h \underline{v}^k, \underline{v}^k)}{(M_h \underline{v}^k, \underline{v}^k)} = \frac{\|\nabla_x v_h^k\|^2_{L^2(\Omega)}}{\|v_h^k\|^2_{L^2(\Omega)}} \leq \frac{12}{h_t^2} \quad \text{for } k = 1, \ldots, N.$$

With the inverse inequality

$$\|\nabla_x v_h\|^2_{L^2(\Omega)} \leq c_I \, h_x^{-2} \, \|v_h\|^2_{L^2(\Omega)} \quad \text{for all } v_h \in W_h,$$

this condition is satisfied for

$$c_I \, h_x^{-2} \leq 12 \, h_t^{-2} \, .$$

In the particular case $d = 1$ we have $c_I = 12$ and therefore stability follows for

$$h_t \leq h_x \, .$$

When $W_h \subset H_0^1(\Omega)$ is also of tensor–product structure, for example when considering the spatial domain $\Omega = (0,1)^d$, we conclude $c_I = 12d$, and therefore the stability condition

$$h_t \leq \frac{h_x}{\sqrt{d}} \, .$$

As numerical example we consider for $d = 2$ the spatial domain $\Omega = (0,1)^2$, and the exact solution

$$u(x_1, x_2, t) = t^2 \sin(\pi x_1) \sin(\pi x_2) \quad \text{for } (x_1, x_2, t) \in Q = \Omega \times (0,T)$$

with $T = \frac{1}{\sqrt{2}}$. Stability then follows when choosing

$$\frac{h_t}{h_x} = \frac{1}{\sqrt{2}} \approx 0.7071068, \tag{5.7}$$

and we observe optimal orders of convergence, see Table 6. Note that numerical experiments indicate that the stability condition (5.7) is sharp, see [38].

As for the scalar case, and following [40], we can formulate and analyse a stabilised version of the variational formulation (5.6) which is unconditionally stable, and which preserves the optimal order of convergence, see [33].

6 Conclusions

In this paper we have formulated and analysed new non–standard variational formulations for finite element discretisations of parabolic and hyperbolic initial boundary value problems, in particular for the heat and the wave equation. Based on this analysis we can

dof	h_x	h_t	$\|u-u_h\|_{L^2}$	eoc	$|u-u_h|_{H^1}$	eoc
2	0.500000	0.3535534	2.0970e-02	-	3.9813e-01	-
36	0.250000	0.1767767	4.8903e-03	2.1	1.9798e-01	1.0
392	0.125000	0.0883883	1.1986e-03	2.0	9.8593e-02	1.0
3600	0.062500	0.0441942	2.9813e-04	2.0	4.9240e-02	1.0
30752	0.031250	0.0220971	7.4437e-05	2.0	2.4613e-02	1.0
254016	0.015625	0.0110485	1.8603e-05	2.0	1.2305e-02	1.0

Table 6: Numerical results for the Galerkin–Bubnov formulation (5.6) for $Q = (0,1)^2 \times (0, \frac{1}{\sqrt{2}})$, satisfying the CFL condition (5.7).

analyse related boundary integral equations and boundary element methods, where we recover known results in the case of the heat equation [11], but we expect to derive new results in the case of the wave equation. Moreover, using this unified framework it will be possible to analyse the coupling of space–time finite and boundary element methods. While the main focus of this paper was on the stability analysis of space–time variational formulations, much more work is required on the design of computationally efficient methods. This covers the formulation and analysis of inf–sup stable local basis functions for arbitrary space–time finite elements, of efficient and reliable a posteriori error estimators and adaptive schemes, and the construction and analysis of preconditioned parallel iterative solution strategies including domain decomposition methods.

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