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Abstract

In this paper we review, analyse and discuss several boundary integral formulations for a stable solution of exterior boundary value and transmission problems for the Helmholtz equation. Based on the characterisation of spurious modes, which correspond to eigensolutions of the interior Laplace equation, direct, indirect, combined and regularised boundary integral equations are considered for the solution of the exterior Dirichlet boundary value problem. In addition to established approaches, such as Burton/Miller, Brakhage/Werner, or the CHIEF method, we also include a discussion of more recent results which can be applied also in the more general case of Lipschitz domains. For a stable boundary integral formulation for the solution of transmission problems, we rely on the use of both boundary integral equations of the direct approach, and on suitable linear combinations. Here we restrict ourselves to single trace formulations, including the rather standard Steklov–Poincaré operator formulation. This contribution reviews the mathematical analysis of stable boundary integral formulations for the solution of Helmholtz boundary value and transmission problems, and it will provide a foundation for the error and stability analysis of related Galerkin boundary element methods.

1 Introduction

Boundary integral equation methods and related boundary element methods are well established tools for the solution of direct and inverse acoustic and electromagnetic scattering problems. But there is a variety of several boundary integral formulations around, e.g., direct or indirect approaches, and first or second kind Fredholm integral equations, just to name a few. Moreover, although exterior boundary value problems for the Helmholtz equation admit, due to the imposed radiation condition, unique solutions, a naive use of boundary integral equations may result in formulations, which may neither be solvable, or the solution may not be unique. These so–called spurious modes are related to eigensolutions of the, in the case of acoustics, interior Laplace equation. Stable boundary integral formulations, which are robust for all wave numbers, are based on complex linear combinations of the standard boundary integral equations, e.g., the direct approach of Burton and Miller [5], the indirect formulation of Brakhage and Werner [2], or the combined Helmholtz integral equation formulation of Schenck [25]. While in most cases the classical analysis, either in spaces of continuous or Hölder continuous functions, or for square integrable functions, relies on the compactness of certain boundary integral operators, sufficient smooth surfaces had to be assumed. More recent work, see, e.g. [3, 4, 10, 11], on regularised combined boundary integral equations also allow the consideration of Lipschitz surfaces. While the focus of this contribution is on the formulation of boundary integral equations which are stable for all wave numbers, we do not analyse the behaviour of solutions in the high–frequency regime, see [6] for the related state of the art. In addition to exterior boundary value problems we also consider the formulation of stable boundary integral equations for the solution of transmission problems, see, e.g., [14], by appropriate combinations of boundary integral equations for both the interior and exterior subproblem. The resulting formulations can be classified into single and multiple trace formulations [7, 23], which also include the minimal coupling formulation of Mitzner [18]. Probably more common, and similar to the Laplace case, is the Steklov–Poincaré operator formulation which is based on the solution of local Dirichlet boundary value problems to enhance the Neumann transmission condition.

In this paper we aim to present a unified approach to the analysis of boundary integral equations for the solution of acoustic boundary value and transmission problems. In Sect. 2 we recall the definition and we summarise the mapping properties of all boundary integral operators under consideration. A particular focus is on the characterisation of the kernels, and their relations with eigensolutions of the interior Laplace equation. Stable boundary integral formulations for the solution of the exterior Dirichlet boundary value problem are considered in Sect. 3. We start with the direct approach, where the standard boundary integral equations turn out to be solvable for all wave numbers, but the solution may not be unique. If the eigensolution is known, an appropriate orthogonality condition results in a stabilised variational formulation. Otherwise one may consider the evaluation of the interior representation formula to describe the unique solution of the exterior problem, what is known as combined Helmholtz integral equation formulation (CHIEF). Next, complex linear combinations of both boundary integral equations of the direct approach are considered, which correspond to the method of Burton and Miller which originally considered the exterior Neumann boundary value problem. Finally we consider the boundary integral equation system of the exterior Calderon projector which turns out to be injective in the unknown Neumann datum. In fact, the symmetric representation of the exterior Steklov–Poincaré operator is well defined for all wave numbers. In addition to direct formulations we also discuss the use of indirect methods, and several complex linear combinations such as the approach of Brakhage and Werner, and regularised boundary integral equations which are applicable also in the case of Lipschitz surfaces. In Sect. 4 we consider the solution of free–space transmission problems, as a model problem we consider the scattering at an interface between two media of different density. It turns out that in total four boundary integral equations and two transmission conditions can be used to determine the four unknown Cauchy data of the direct approach. The elimination of the Cauchy data of the exterior problem results in single trace formulations, while the elimination of the local Neumann data results in Steklov–Poincaré operator formulations, as known from the Laplace equation. Finally we discuss certain combined formulations including the minimal coupling formulation of Mitzner. We do not discuss multiple trace formulations, for this we refer to [7].

This review may provide a foundation for the error and stability analysis of Galerkin boundary element methods, and for the construction and analysis of preconditioned iterative solution procedures for an efficient solution of boundary value and transmission problems for the Helmholtz equation with moderate wave numbers. However, almost all approaches and methodologies as presented here can be carried over to the case of electromagnetic scattering problems, to the coupling with different physical fields, and for a stable coupling of finite and boundary element methods.

2 Boundary integral equations

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and for a wave number $\kappa \in \mathbb{R}$ we consider either the interior Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \tag{2.1}$$

or the exterior Helmholtz problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c := \mathbb{R}^3 \backslash \overline{\Omega}, \tag{2.2}$$

where for the exterior problem we have to include the Sommerfeld radiation condition

$$\lim_{r \to \infty} \int_{|x|=r} \left| \frac{\partial}{\partial n_x} u(x) - i\kappa u(x) \right|^2 ds_x = 0, \tag{2.3}$$

and n_x is the exterior normal vector. Recall [8, Remark 3.4] that any solution u of the Helmholtz equation (2.2) satisfying the radiation condition (2.3) automatically satisfies

$$u(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty.$$

In addition we consider boundary or transmission conditions on $\Gamma = \partial\Omega$.

In the direct approach, any solution of the interior Helmholtz equation (2.1) is given by the representation formula

$$u(x) = \int_\Gamma U^*_\kappa(x,y) \frac{\partial}{\partial n_y} u(y) ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y) u(y) ds_y \quad \text{for } x \in \Omega, \tag{2.4}$$

where the Helmholtz fundamental solution is given by

$$U_\kappa^*(x,y) = \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|}.$$

In the representation formula (2.4) we have used the single layer potential

$$(\widetilde{V}_\kappa w)(x) = \int_\Gamma U_\kappa^*(x,y)w(y)ds_y \quad \text{for } x \in \mathbb{R}^3\backslash\Gamma, \quad \widetilde{V}_\kappa : H^{-1/2}(\Gamma) \to H^1(\Omega),$$

and the double layer potential

$$(W_\kappa v)(x) = \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y)v(y)ds_y \quad \text{for } x \in \mathbb{R}^3\backslash\Gamma, \quad W_\kappa : H^{1/2}(\Gamma) \to H^1(\Omega).$$

By definition, both the single and double layer potentials are solutions of the interior and exterior Helmholtz equation, i.e. for $x \in \mathbb{R}^3\backslash\Gamma$ we have

$$\Delta(\widetilde{V}_\kappa w)(x) + \kappa^2(\widetilde{V}_\kappa w)(x) = 0, \quad \Delta(W_\kappa v)(x) + \kappa^2(W_\kappa v)(x) = 0.$$

For the interior and exterior Dirichlet traces of the single layer potential $\widetilde{V}_\kappa w$ we find

$$\gamma_0^{\text{int}}(\widetilde{V}_\kappa w)(x) = \gamma_0^{\text{ext}}(\widetilde{V}_\kappa w)(x) = \int_\Gamma U_\kappa^*(x,y)w(y)ds_y =: (V_\kappa w)(x) \quad \text{for } x \in \Gamma,$$

where $V_\kappa := \gamma_0 \widetilde{V}_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the single layer boundary integral operator. For the interior and exterior Neumann traces of the single layer potential $\widetilde{V}_\kappa w$ we obtain, in the sense of $H^{-1/2}(\Gamma)$, i.e. for all $v \in H^{1/2}(\Gamma)$,

$$\langle \gamma_1^{\text{int}} \widetilde{V}_\kappa w, v\rangle_\Gamma = \langle \frac{1}{2}w + K_\kappa' w, v\rangle_\Gamma, \quad \langle \gamma_1^{\text{ext}} \widetilde{V}_\kappa w, v\rangle_\Gamma = \langle -\frac{1}{2}w + K_\kappa' w, v\rangle_\Gamma. \tag{2.5}$$

Here, $K_\kappa' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the adjoint double layer boundary integral operator

$$(K_\kappa' w)(x) = \int_\Gamma \frac{\partial}{\partial n_x} U_\kappa^*(x,y)w(y)ds_y \quad \text{for } x \in \Gamma.$$

In particular, from (2.5) we conclude the jump relation of the normal derivative of the single layer potential $\widetilde{V}_\kappa w$, i.e.

$$[\gamma_1 \widetilde{V}_\kappa w]_{|\Gamma} := \gamma_1^{\text{ext}} \widetilde{V}_\kappa w - \gamma_1^{\text{int}} \widetilde{V}_\kappa w = -w \quad \text{in } H^{-1/2}(\Gamma).$$

For the interior and exterior Dirichlet traces of the double layer potential $W_\kappa v$ we find for almost all $x \in \Gamma$

$$\gamma_0^{\text{int}}(W_\kappa v)(x) = -\frac{1}{2}v(x) + (K_\kappa v)(x), \quad \gamma_0^{\text{ext}}(W_\kappa v)(x) = \frac{1}{2}v(x) + (K_\kappa v)(x), \tag{2.6}$$

where $K_\kappa : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the double layer boundary integral operator

$$(K_\kappa v)(x) = \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y) v(y) ds_y \quad \text{for } x \in \Gamma.$$

From (2.6) we now conclude the jump relation of the double layer potential $W_\kappa v$, i.e.

$$[\gamma_0 W_\kappa v]_{|\Gamma} := \gamma_0^{\text{ext}}(W_\kappa v)(x) - \gamma_0^{\text{int}}(W_\kappa v)(x) = v(x) \quad \text{for } x \in \Gamma.$$

Finally, the interior and exterior Neumann traces of the double layer potential $W_\kappa v$ define the hypersingular boundary integral operator $D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$,

$$(D_\kappa v)(x) = -\frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y) v(y) ds_y \quad \text{for } x \in \Gamma.$$

By using integration by parts we obtain a representation of the hypersingular boundary integral operator D_κ which involves tangential rather than normal derivatives [16, 18], i.e. for $x \in \Gamma$ we have

$$(D_\kappa v)(x) = -\int_\Gamma \Big[\Big(n_x \times \nabla_x U^*_\kappa(x,y) \Big) \cdot \Big(n_y \times \nabla v(y) \Big) + \kappa^2 (n_x \cdot n_y) U^*_\kappa(x,y) v(y) \Big] ds_y. \quad (2.7)$$

Note that $n_y \times \nabla v(y)$, $y \in \Gamma$, describes the surface curl of a function given on the boundary. By considering the weak form of the relation (2.7), and using once again integration by parts, the bilinear form of the hypersingular boundary integral operator D_κ can be written as [19]

$$\begin{aligned} \langle D_\kappa u, v\rangle_\Gamma \; = \; & \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|} \Big(n_y \times \nabla u(y) \Big) \cdot \Big(n_x \times \nabla v(x) \Big) \, ds_y ds_x \\ & - \frac{\kappa^2}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|} u(y) v(x) \, (n_y \cdot n_x) \, ds_y ds_x. \end{aligned}$$

From the representation formula (2.4) to describe solutions of the interior Helmholtz equation (2.1) we find, by considering the interior Dirichlet and Neumann traces of the single and double layer potentials, the system of boundary integral equations

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_\kappa & V_\kappa \\ D_\kappa & \frac{1}{2}I + K'_\kappa \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} =: \mathcal{C} \begin{pmatrix} u \\ t \end{pmatrix} \quad \text{on } \Gamma. \quad (2.8)$$

Note that we have used $t(x) := \frac{\partial}{\partial n_x} u(x)$, $x \in \Gamma$. From the projection property $\mathcal{C} = \mathcal{C}^2$ of the interior Calderon operator $\mathcal{C}$ as defined in (2.8) we find, as in the case of the Laplace equation [26], the well known relations [5]

$$K_\kappa V_\kappa = V_\kappa K'_\kappa, \quad V_\kappa D_\kappa = \frac{1}{4} I - K_\kappa^2. \quad (2.9)$$

The mapping properties of all boundary integral operators as introduced above are well established, see, e.g., [8, 9, 13, 17, 20, 24, 26]. In particular, the single layer boundary integral operator $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive satisfying

$$\langle (V_\kappa + C_V)w, w\rangle_\Gamma = \langle Vw, w\rangle_\Gamma \geq c_1^V \|w\|^2_{H^{-1/2}(\Gamma)} \quad \text{for all } w \in H^{-1/2}(\Gamma) \tag{2.10}$$

where $C_V := V - V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, and $V = V_0$ is the Laplace single layer boundary integral operator which is $H^{-1/2}(\Gamma)$–elliptic. Moreover, for $t, \tau \in H^{-1/2}(\Gamma)$ we obtain

$$\begin{aligned} \langle V_\kappa t, \tau\rangle_\Gamma &= \int_\Gamma (V_\kappa t)(x)\overline{\tau(x)}\, ds_x = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|} t(y) ds_y\, \overline{\tau(x)}\, ds_x \\ &= \frac{1}{4\pi}\int_\Gamma t(y)\overline{\int_\Gamma \frac{e^{-i\kappa|x-y|}}{|x-y|}\tau(y) ds_x}\, ds_y = \int_\Gamma t(y)\overline{(V_{-\kappa}\tau)(y)}\, ds_y = \langle t, V_{-\kappa}\tau\rangle_\Gamma. \end{aligned}$$

Finally, for $\kappa \in \mathbb{R}$ we find that, see, e.g., [10, Lemma 3.1],

$$\Im(\langle V_\kappa \tau, \tau\rangle_\Gamma) \geq 0 \quad \text{for all } \tau \in H^{-1/2}(\Gamma).$$

Analoguesly, the hypersingular boundary integral operator $D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is coercive satisfying

$$\langle (D_\kappa + C_D)v, v\rangle_\Gamma = \langle \widetilde{D}v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma)$$

where $C_D := \widetilde{D} - D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, and $\widetilde{D}$ is the stabilised Laplace hypersingular boundary integral operator defined as [21]

$$\langle \widetilde{D}u, v\rangle_\Gamma := \langle D_0 u, v\rangle_\Gamma + \langle u, V^{-1}1\rangle_\Gamma \overline{\langle v, V^{-1}1\rangle_\Gamma} \quad \text{for } u, v \in H^{1/2}(\Gamma). \tag{2.11}$$

Moreover, $\frac{1}{2}I - K_\kappa : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive satisfying

$$\langle [(\frac{1}{2}I - K_\kappa) + C_K]v, v\rangle_{V^{-1}} = \langle (\frac{1}{2}I - K)v, v\rangle_{V^{-1}} \geq (1 - c_K)\, \|v\|^2_{V^{-1}} \tag{2.12}$$

for all $v \in H^{1/2}(\Gamma)$ where $C_K := K_\kappa - K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact and $K = K_0$ is the Laplace double layer boundary integral operator. Note that in (2.12) we have used the contraction property of the Laplace double layer boundary integral operator $\frac{1}{2}I + K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, see [28],

$$\|(\frac{1}{2}I + K)v\|_{V^{-1}} \leq c_K\, \|v\|_{V^{-1}} \quad \text{for all } v \in H^{1/2}(\Gamma), \quad c_K < 1.$$

In a similar way we conclude that $\frac{1}{2}I - K'_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is coercive, i.e.

$$\langle [(\frac{1}{2}I - K'_\kappa) + C_{K'}]\tau, \tau\rangle_V = \langle (\frac{1}{2}I - K')\tau, \tau\rangle_V \geq (1 - c_K)\, \|\tau\|^2_V \tag{2.13}$$

for all $\tau \in H^{-1/2}(\Gamma)$ where $C_{K'} := K'_\kappa - K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact. Moreover, as for the single layer boundary integral operator V_κ we conclude

$$\langle K_\kappa v, \tau \rangle = \langle v, K'_{-\kappa} \tau \rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma), \tau \in H^{-1/2}(\Gamma).$$

Since all boundary integral operators are coercive, we can apply Fredholm's alternative to investigate the unique solvability of boundary integral equations which are related to certain transmission and boundary value problems for the Helmholtz equation. It remains to consider the injectivity of the involved boundary integral operators. This is related to interior eigenvalue problems of the Laplace operator.

The next two results are direct consequences of the boundary integral equations (2.8) which hold for any solution of the interior Helmholtz equation.

Proposition 2.1 *Let* $(\lambda, u_\lambda) \in \mathbb{R} \times H^1(\Omega)$, $\lambda > 0$, *be a solution of the interior Dirichlet eigenvalue problem*

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \quad u_\lambda(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.14}$$

For $\kappa := \pm\sqrt{\lambda} \in \mathbb{R}$ *and* $t_\lambda := n_x \cdot \nabla u_\lambda \in H^{-1/2}(\Gamma)$ *we then conclude for* $x \in \Gamma$

$$(V_\kappa t_\lambda)(x) = 0, \quad (\frac{1}{2} I - K'_\kappa) t_\lambda(x) = 0. \tag{2.15}$$

In particular, the single layer boundary integral operator V_κ and the adjoint double layer boundary integral operator $\frac{1}{2}I - K'_\kappa$ are not injective if κ^2 corresponds to an eigenvalue λ of the interior Dirichlet eigenvalue problem (2.14).

Proposition 2.2 *Let* $(\mu, u_\mu) \in \mathbb{R} \times H^1(\Omega)$, $\mu \geq 0$, *be a solution of the interior Neumann eigenvalue problem*

$$-\Delta u_\mu(x) = \mu u_\mu(x) \quad \text{for } x \in \Omega, \quad t_\mu(x) := \frac{\partial}{\partial n_x} u_\mu(x) = 0 \quad \text{for } x \in \Gamma. \tag{2.16}$$

For $\kappa := \pm\sqrt{\mu} \in \mathbb{R}$ *we then conclude for* $x \in \Gamma$

$$(\frac{1}{2} I + K_\kappa) u_\mu(x) = 0, \quad (D_\kappa u_\mu)(x) = 0. \tag{2.17}$$

Hence, the hypersingular boundary integral operator D_κ and the double layer boundary integral operator $\frac{1}{2}I + K_\kappa$ are not injective if κ^2 corresponds to an eigenvalue μ of the interior Neumann eigenvalue problem (2.16).

To characterise the null spaces of all boundary integral operators as discussed above we need to analyse the equivalence of the boundary integral operator eigenvalue problems (2.15) and (2.17) with the eigenvalue problems (2.14) and (2.16). This has to be done in several steps, first we consider the single layer boundary integral operator V_κ.

Proposition 2.3 *Let* $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ *be an eigensolution of the single layer boundary integral operator eigenvalue problem* $V_\kappa t = 0$. *Then,*

$$\lambda := \kappa^2, \quad u_\lambda(x) := \int_\Gamma U^*_\kappa(x,y)t(y)ds_y \quad \textit{for } x \in \Omega$$

defines an eigensolution of the interior Dirichlet eigenvalue problem (2.14).

Proof. Since u_λ is defined as a single layer potential, by definition, u_λ is a solution of the interior Helmholtz equation, i.e.

$$-\Delta u_\lambda(x) - \kappa^2 u_\lambda(x) = 0 \quad \text{for } x \in \Omega.$$

Moreover,

$$u_\lambda(x) = \int_\Gamma U^*_\kappa(x,y)t(y)ds_y = (V_\kappa t)(x) = 0 \quad \text{for } x \in \Gamma.$$

Hence, u_λ is a solution of the interior Dirichlet eigenvalue problem (2.14). ■

Remark 2.1 *In addition to real eigenvalues* $\kappa \in \mathbb{R}$ *there also exist complex solutions* $\kappa \in \mathbb{C}$ *of the single layer boundary integral operator eigenvalue problem* $V_\kappa t = 0$ *which correspond to the scattering poles, and where the associated single layer potential is zero inside* Ω. *For an approximate solution of the single layer boundary integral operator eigenvalue problem* $V_\kappa t = 0$ *by using boundary element methods, see, e.g., [27, 32].*

Corollary 2.4 *Since any eigensolution* $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ *of the single layer boundary integral operator eigenvalue problem* $V_\kappa t = 0$ *induces an eigensolution* $(\lambda, u_\lambda) \in \mathbb{R} \times H^1(\Omega)$ *of the interior Dirichlet eigenvalue problem (2.14), Proposition 2.1 implies* $(\frac{1}{2}I - K'_\kappa)t = 0$.

A similar result as given in Proposition 2.3 for the single layer boundary integral operator V_κ also holds for the hypersingular boundary integral operator D_κ.

Proposition 2.5 *Let* $(\kappa, u) \in \mathbb{R} \times H^{1/2}(\Gamma)$ *be an eigensolution of the hypersingular boundary integral operator eigenvalue problem* $D_\kappa u = 0$. *Then,*

$$\mu := \kappa^2, \quad u_\mu(x) := -\int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)u(y)ds_y \quad \textit{for } x \in \Omega$$

defines an eigensolution of the interior Neumann eigenvalue problem (2.16). Moreover, we also have $(\frac{1}{2}I + K_\kappa)u = 0$.

More involved is the consideration of the adjoint double layer boundary integral operator eigenvalue problem $(\frac{1}{2}I - K'_\kappa)t = 0$.

Lemma 2.6 *Let* $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ *be an eigensolution of the adjoint boundary integral operator eigenvalue problem* $(\frac{1}{2}I - K'_\kappa)t = 0$. *Then,*

$$\lambda := \kappa^2, \quad u_\lambda(x) := \int_\Gamma U^*_\kappa(x,y)t(y)ds_y \quad \text{for } x \in \Omega$$

defines an eigensolution of the interior Dirichlet eigenvalue problem (2.14), and we also have $V_\kappa t = 0$.

Proof. By definition, u_λ is a solution of the interior Helmholtz equation. For the Dirichlet trace of u_λ we obtain

$$u_\lambda(x) = (V_\kappa t)(x) \quad \text{for } x \in \Gamma,$$

and therefore, by using (2.9),

$$(\frac{1}{2}I - K_\kappa)u_\lambda(x) = (\frac{1}{2}I - K_\kappa)(V_\kappa t)(x) = V_\kappa(\frac{1}{2}I - K'_\kappa)t(x) = 0 \quad \text{for } x \in \Gamma$$

follows. On the other hand, by considering the Neumann trace of u_λ we obtain

$$\frac{\partial}{\partial n_x}u_\lambda(x) = (\frac{1}{2}I + K'_\kappa)t(x) = t(x) - (\frac{1}{2}I - K'_\kappa)t(x) = t(x) \quad \text{for } x \in \Gamma,$$

and by using the second boundary integral equation of (2.8)

$$(D_\kappa u_\lambda)(x) = (\frac{1}{2}I - K'_\kappa)t(x) = 0 \quad \text{for } x \in \Gamma$$

follows. Since u_λ implies a solution of the Neumann eigenvalue problem (2.16), see Proposition 2.5, we further conclude, by using Proposition 2.2, that

$$(\frac{1}{2}I + K_\kappa)u_\lambda(x) = 0 \quad \text{for } x \in \Gamma,$$

and therefore

$$u_\lambda(x) = (V_\kappa t)(x) = 0 \quad \text{for } x \in \Gamma$$

follows. ■

By using Proposition 2.1, Corollary 2.4, and Lemma 2.6, we finally conclude

$$\begin{aligned} \ker V_\kappa &= \ker(\frac{1}{2}I - K'_\kappa) \\ &= \left\{ \frac{\partial}{\partial n}u_\lambda \in H^{-1/2}(\Gamma) : -\Delta u_\lambda = \kappa^2 u_\lambda \,\text{in}\, \Omega, u_\lambda = 0 \,\text{on}\, \Gamma \right\}. \end{aligned} \tag{2.18}$$

A similar result as given for the adjoint double layer integral operator $\frac{1}{2}I - K'_\kappa$ in Lemma 2.6 also holds for the double layer boundary integral operator $\frac{1}{2}I + K_\kappa$.

Lemma 2.7 *Let* $(\kappa, u) \in \mathbb{R} \times H^{1/2}(\Gamma)$ *be an eigensolution of the double layer boundary integral operator eigenvalue problem* $(\frac{1}{2}I + K_\kappa)u = 0$. *Then,*

$$\mu := \kappa^2, \quad u_\mu(x) := -\int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x, y)u(y)ds_y \quad \textit{for } x \in \Omega$$

defines an eigensolution of the interior Neumann eigenvalue problem (2.16), and we have $D_\kappa u = 0$. *Moreover,*

$$ker D_\kappa = ker(\frac{1}{2}I + K_\kappa) = \left\{ u_{\mu|\Gamma} \in H^{1/2}(\Gamma) : -\Delta u_\mu = \kappa^2 u_\mu \textit{ in } \Omega, \frac{\partial}{\partial n}u_\mu = 0 \textit{ on } \Gamma \right\}.$$

From Propositions 2.1 and 2.2 we conclude that the solution of the interior Helmholtz equation (2.1) with either Dirichlet or Neumann boundary conditions is not unique when κ^2 corresponds to either a Dirichlet or a Neumann eigenvalue, respectively. Note that in these cases one has to assume certain solvability conditions to ensure existence of a solution. But when considering the interior Helmholtz equation (2.1) with a Robin type boundary condition this admits a unique solution for all wave numbers.

Theorem 2.8 *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded Lipschitz domain, and let* $\kappa \in \mathbb{R}$. *Let* $f \in \widetilde{H}^{-1}(\Omega)$ *and* $g \in H^{-1/2}(\Gamma)$ *be given. Then there exists a unique solution of the interior Robin boundary value problem,* $\eta \in \mathbb{R}$, $\eta \neq 0$,

$$-\Delta u(x) - \kappa^2 u(x) = f(x) \quad \textit{for } x \in \Omega, \quad \frac{\partial}{\partial n_x}u(x) - i\eta u(x) = g(x) \quad \textit{for } x \in \Gamma. \tag{2.19}$$

Proof. The variational formulation of the Robin boundary value problem (2.19) is to find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)dx - \kappa^2 \int_\Omega u(x)v(x)dx - i\eta \int_\Gamma u(x)v(x)ds_x = \int_\Omega f(x)v(x)dx + \int_\Gamma g(x)v(x)ds_x$$

is satisfied for all $v \in H^1(\Omega)$. Since the bilinear form

$$a(u, v) := \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}dx - \kappa^2 \int_\Omega u(x)\overline{v(x)}dx - i\eta \int_\Gamma u(x)\overline{v(x)}ds_x$$

is bounded for all $u, v \in H^1(\Omega)$, and since it satisfies a Gårding inequality, i.e.

$$\begin{aligned}\Re(a(v, v)) &= \int_\Omega |\nabla v(x)|^2 dx + \int_\Omega |v(x)|^2 dx - (1 + \kappa^2) \int_\Omega |v(x)|^2 dx \\ &= \|v\|^2_{H^1(\Omega)} - c(v, v) \quad \text{for all } v \in H^1(\Omega)\end{aligned}$$

with a compact bilinear form $c(\cdot, \cdot)$, it remains to prove injectivity.

Let $w \in H^1(\Omega)$ be a solution of the homogeneous Robin boundary value problem

$$-\Delta w(x) - \kappa^2 w(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} w(x) - i\eta w(x) = 0 \quad \text{for } x \in \Gamma.$$

Then we have

$$\int_\Omega \nabla w(x) \cdot \nabla v(x) dx - \kappa^2 \int_\Omega w(x) v(x) dx - i\eta \int_\Gamma w(x) v(x) ds_x = 0$$

for all $v \in H^1(\Omega)$. In particular for $v = \overline{w}$ this gives

$$\int_\Omega |\nabla w(x)|^2 dx - \kappa^2 \int_\Omega |w(x)|^2 dx - i\eta \int_\Gamma |w(x)|^2 ds_x = 0,$$

and by considering the imaginary part

$$\int_\Gamma |w(x)|^2 ds_x = 0$$

follows due to $\kappa \in \mathbb{R}$. Hence we conclude $w = 0$ and $\frac{\partial}{\partial n} w = 0$ on Γ and therefore $w = 0$ in Ω follows when considering the representation formula (2.4). ■

Remark 2.2 *The unique solvability of the interior Robin boundary value problem (2.19) and the proof as presented are the essential tools to ensure unique solvability of combined boundary integral equations as to be discussed in the subsequent sections. Obviously, the proof of Theorem 2.8 remains true when considering complex wave numbers $\kappa \in \mathbb{C}$ when assuming $\eta > 0$ for $Re(\kappa) \cdot \Im(\kappa) > 0$ and $\eta < 0$ for $\Re(\kappa) \cdot \Im(\kappa) < 0$. Moreover, we may replace the Robin boundary condition in (2.19) by the regularised boundary condition*

$$\frac{\partial}{\partial n_x} u(x) - i\eta (Bu)(x) = g(x) \quad \textit{for } x \in \Gamma$$

where $B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is some self–adjoint and $H^{1/2}(\Gamma)$–elliptic operator such that

$$\langle Bw, w \rangle_\Gamma > 0 \quad \textit{for all } w \in H^{1/2}(\Gamma).$$

As for the interior Helmholtz equation (2.1) we can describe the solution of the exterior Helmholtz problem (2.2), and satisfying the radiation condition (2.3), by the representation formula

$$u(x) = -\int_\Gamma U_\kappa^*(x,y) t(y) ds_y + \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y) u(y) ds_y \quad \text{for } x \in \Omega^c. \tag{2.20}$$

By considering the exterior Dirichlet and Neumann traces of the representation formula (2.20) we obtain the related system of boundary integral equations

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I + K_\kappa & -V_\kappa \\ -D_\kappa & \frac{1}{2} I - K_\kappa' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix}. \tag{2.21}$$

In addition, by applying Green's formula with respect to the interior domain Ω, this gives

$$-\int_\Gamma U^*_\kappa(x,y)t(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)u(y)ds_y = 0 \quad \text{for } x \in \Omega \tag{2.22}$$

for the solution of the exterior Helmholtz equation.

Recall that the exterior Helmholtz problem (2.2) with either Dirichlet or Neumann boundary conditions admits in all cases, due to the Sommerfeld radiation condition (2.3), a unique solution.

3 Exterior Dirichlet boundary value problem

As a model problem we consider the exterior Dirichlet boundary value problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c, \quad u(x) = g(x) \quad \text{for } x \in \Gamma, \tag{3.1}$$

and satisfying the Sommerfeld radiation condition (2.3).

In what follows we will review and discuss different direct and indirect formulations of boundary integral equations to describe solutions of the exterior Dirichlet boundary value problem (3.1).

Note that the exterior Dirichlet boundary value problem (3.1) serves as a model problem, all further considerations can be done analoguesly when considering either an exterior Neumann or Robin boundary value problem.

3.1 Direct boundary integral equations

In the direct approach, the solution of the exterior Dirichlet boundary value problem (3.1) is given by the representation formula (2.20), i.e.

$$u(x) = -\int_\Gamma U^*_\kappa(x,y)t(y)ds_y + \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)g(y)ds_y \quad \text{for } x \in \Omega^c.$$

Hence it remains to find the yet unknown Neumann datum $t \in H^{-1/2}(\Gamma)$ as the solution of one of the boundary integral equations as given in (2.21). In fact, we can use either the first kind boundary integral equation

$$(V_\kappa t)(x) = (-\frac{1}{2}I + K_\kappa)g(x) \quad \text{for } x \in \Gamma, \tag{3.2}$$

or the second kind boundary integral equation

$$(\frac{1}{2}I + K'_\kappa)t(x) = -(D_\kappa g)(x) \quad \text{for } x \in \Gamma. \tag{3.3}$$

To investigate the unique solvability of the boundary integral equation (3.2) we first combine the results on coercivity and injectivity of the single layer boundary integral operator V_κ.

Corollary 3.1 *The single layer boundary integral operator $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive and satisfies the Gårding inequality (2.10). Moreover, if κ^2 does not coincide with an eigenvalue λ of the interior Dirichlet eigenvalue problem (2.14), then V_κ is also injective, and the boundary integral equation (3.2) admits, for any $g \in H^{1/2}(\Gamma)$, a unique solution $t \in H^{-1/2}(\Gamma)$ satisfying the variational problem*

$$\langle V_\kappa t, \tau\rangle_\Gamma = \langle(-\frac{1}{2}I + K_\kappa)g, \tau\rangle_\Gamma \quad \textit{for all}\ \tau \in H^{-1/2}(\Gamma).$$

Although the single layer boundary integral operator V_κ is not injective when $\lambda = \kappa^2$ is an eigenvalue of the interior Dirichlet eigenvalue problem (2.14), the boundary integral equation (3.2) of the direct approach is solvable. By the closed range theorem we have $\operatorname{Im} V_\kappa = (\ker V_{-\kappa})^0$ where the polar space is given as

$$(\ker V_{-\kappa})^0 := \left\{v \in H^{1/2}(\Gamma) : \langle v, t_\lambda\rangle_\Gamma = 0 \text{ for all } t_\lambda \in \ker V_{-\kappa}\right\}.$$

By using $\ker V_\kappa = \ker(\frac{1}{2}I - K'_\kappa)$ for all $\kappa \in \mathbb{R}$, see (2.18), we obtain for the right hand side of the boundary integral equation (3.2)

$$\langle(-\frac{1}{2}I + K_\kappa)g, t_\lambda\rangle_\Gamma = -\langle g, (\frac{1}{2}I - K'_{-\kappa})t_\lambda\rangle_\Gamma = 0,$$

and therefore $(-\frac{1}{2}I + K_\kappa)g \in \operatorname{Im} V_\kappa$ follows. In particular, the boundary integral equation (3.2) of the direct approach is solvable, but the solution is not unique. Hence one needs to introduce some suitable constraint for scaling.

A natural choice for doing so would be to require

$$t \in H_\lambda^{-1/2}(\Gamma) := \left\{w \in H^{-1/2}(\Gamma) : \langle Vw, t_\lambda\rangle_\Gamma = 0\right\},$$

i.e. to select a solution which is orthogonal to the eigensolution t_λ, where the orthogonality in $H^{-1/2}(\Gamma)$ is realized by using an inner product which is induced by the Laplace single layer boundary integral operator V. Then we can consider an extended variational problem to find $t \in H^{-1/2}(\Gamma)$ such that

$$\langle V_\kappa t, \tau\rangle_\Gamma + \langle Vt, t_\lambda\rangle_\Gamma \overline{\langle V\tau, t_\lambda\rangle_\Gamma} = \langle(-\frac{1}{2}I + K_\kappa)g, \tau\rangle_\Gamma \tag{3.4}$$

is satisfied for all $\tau \in H^{-1/2}(\Gamma)$. Since the bilinear form of the extended variational problem (3.4) is coercive and injective, unique solvability follows.

Obviously, the extended variational problem (3.4) is only of practical interest, when the eigensolution t_λ is known, which restricts the applicability of this approach.

Instead one may combine the boundary integral equation (3.2) with the interior representation formula (2.22), see [25], i.e. we consider the problem to find $t \in H^{-1/2}(\Gamma)$ such that

$$\langle V_\kappa t, \tau\rangle_\Gamma = \langle(-\frac{1}{2}I + K_\kappa)g, \tau\rangle_\Gamma \quad \text{for all } \tau \in H^{-1/2}(\Gamma), \tag{3.5}$$

and

$$\int_\Gamma U_\kappa^*(x,y)t(y)ds_y = \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y)g(y)ds_y \quad \text{for all } x \in \Omega. \tag{3.6}$$

Lemma 3.2 *For all wave numbers $\kappa \in \mathbb{R}$ there exists a unique solution $t \in H^{-1/2}(\Gamma)$ of the boundary integral formulation (3.5) which also satisfies (3.6).*

Proof. Since the variational formulation (3.5) admits a unique solution when κ^2 is not an eigenvalue of the interior Dirichlet eigenvalue problem (2.14), it is sufficient to consider the remaining case only. If $\lambda = \kappa^2$ is an eigenvalue of the interior Dirichlet problem (2.14), the variational problem (3.5) is solvable, but the solution is not unique.

Let $t_i \in H^{-1/2}(\Gamma)$, $i = 1, 2$, denote two solutions of the system (3.5) and (3.6), i.e.

$$V_\kappa t_i = (-\frac{1}{2}I + K_\kappa)g \text{ on } \Gamma, \int_\Gamma U^*_\kappa(x,y)t_i(y)ds_y = \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)g(y)ds_y \text{ in } \Omega.$$

For the difference $t_0 := t_1 - t_2$ we therefore conclude

$$V_{\kappa_i} t_0 = 0 \quad \text{on } \Gamma, \quad \int_\Gamma U^*_\kappa(x,y)t_0(y)ds_y = 0 \quad \text{in } \Omega.$$

From the first equation we find $t_0 = \alpha t_\lambda$, $\alpha \in \mathbb{R}$, while the second one gives

$$0 = \alpha \int_\Gamma U^*_\kappa(x,y)t_\lambda(y)ds_y = \alpha u_\lambda(x) \quad \text{for all } x \in \Omega,$$

and hence, $\alpha = 0$ follows, i.e. $t_1 = t_2$. ■

The discretisation of the coupled formulation (3.5) and (3.6) by using either a collocation or Galerkin scheme for (3.5), and choosing a finite set of interior nodes for (3.6), results in an overdetermined system of linear equations to be solved. This approach is known as *Combined Helmholtz Integral Equation Formulation* (CHIEF) [25], see also [1] for a further discussion.

In any case, the boundary integral equation (3.2) admits a unique solution $t \in H^{-1/2}(\Gamma)$ when κ^2 is not an eigenvalue of the Dirichlet eigenvalue problem (2.14), or a unique solution $t \in H^{-1/2}_\lambda(\Gamma)$ when $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (2.14). In the latter case, the general solution of the boundary integral equation (3.2) is given by

$$t = -V^{-1}_\kappa(\frac{1}{2}I - K_\kappa)g + \alpha t_\lambda, \quad \alpha \in \mathbb{R},$$

where the application of V^{-1}_κ acts between appropriate factor spaces. When using, in addition, the second boundary integral equation as given by the exterior Calderon projection (2.21), this gives

$$\begin{aligned} t &= -D_\kappa g + (\frac{1}{2}I - K'_\kappa)t \\ &= -D_\kappa g + (\frac{1}{2}I - K'_\kappa)\Big[-V^{-1}_\kappa(\frac{1}{2}I - K_\kappa)g + \alpha t_\lambda\Big] \\ &= -\Big[D_\kappa + (\frac{1}{2}I - K'_\kappa)V^{-1}_\kappa(\frac{1}{2}I - K_\kappa)\Big]g + \alpha(\frac{1}{2}I - K'_\kappa)t_\lambda = -S^{\text{ext}}_\kappa g, \end{aligned}$$

i.e. the symmetric representation of the exterior Steklov–Poincaré operator

$$S_\kappa^{\text{ext}} := D_\kappa + (\frac{1}{2}I - K'_\kappa)V_\kappa^{-1}(\frac{1}{2}I - K_\kappa), \tag{3.7}$$

which is related to the exterior Dirichlet boundary value problem, and which is well defined for all wave numbers $\kappa \in \mathbb{R}$. Note that the non–symmetric representation

$$S_\kappa^{\text{ext}} = V_\kappa^{-1}(\frac{1}{2}I - K_\kappa) \tag{3.8}$$

is not well defined when $\lambda = \kappa^2$ is an eigenvalue of the Dirichlet eigenvalue problem (2.14). However, superpositions $V_\kappa S_\kappa^{\text{ext}}$ and $(\frac{1}{2}I - K'_\kappa)S_\kappa^{\text{ext}}$ are well defined for all wave numbers $\kappa \in \mathbb{R}$ also for the non–symmetric representation (3.8).

Instead of the first kind boundary integral equation (3.2) we may also use the second kind boundary integral equation (3.3) to describe the solution of the exterior Dirichlet boundary value problem (3.1). To investigate the unique solvability of the boundary integral equation (3.3) we first recall that the boundary integral operator $\frac{1}{2}I + K'_\kappa$ is not injective when $\frac{1}{2}I + K_\kappa$ is not injective, and vice versa. Moreover, it is possible to characterise the null space of $\frac{1}{2}I + K'_\kappa$. As a motivation we first recall the situation when considering the Laplace equation.

Remark 3.1 *In the case of the Laplace equation we have $(\frac{1}{2}I + K)u_0 = 0$ for $u_0 \equiv 1$. Since the Laplace single layer boundary integral operator V is bijective, this implies*

$$V^{-1}(\frac{1}{2}I + K)u_0 = (\frac{1}{2}I + K')V^{-1}u_0 = (\frac{1}{2}I + K')t_{eq} = 0,$$

where $t_{eq} = V^{-1}u_0$ is the natural density.

The relation as just described for the Laplace equation can be generalised to the case when considering the Helmholtz equation.

Lemma 3.3 *Let $\mu = \kappa^2$ be an eigenvalue of the Neumann eigenvalue problem (2.16). Any eigensolution $u_\mu \in ker(\frac{1}{2}I + K_\kappa)$ implies an eigensolution $t_\mu \in ker(\frac{1}{2}I + K'_\kappa)$, and vice versa. In particular, there holds the relation $u_\mu = V_\kappa t_\mu$.*

Proof. Let $t \in \ker(\frac{1}{2}I + K'_\kappa)$, i.e. $(\frac{1}{2}I + K'_\kappa)t = 0$. Define $u = V_\kappa t$ which turns out to be non–trivial. Indeed, in the case $u \equiv 0$, $V_\kappa t = 0$ implies, by using Corollary 2.4, $(\frac{1}{2}I - K'_\kappa)t = 0$, and therefore $t \equiv 0$ follows. Hence we have that $u = V_\kappa t$ is non–trivial satisfying, by using (2.9),

$$(\frac{1}{2}I + K_\kappa)u = (\frac{1}{2}I + K_\kappa)V_\kappa t = V_\kappa(\frac{1}{2}I + K'_\kappa)t = 0.$$

Vice versa, let $u \in \ker(\frac{1}{2}I + K_\kappa)$, i.e. $(\frac{1}{2}I + K_\kappa)u = 0$. It remains to prove that there exists a $t \in H^{-1/2}(\Gamma)$ satisfying $V_\kappa t = u$. This is trivial when V_κ is injective. In this case we further conclude

$$(\frac{1}{2}I + K_\kappa)u = (\frac{1}{2}I + K_\kappa)V_\kappa t = V_\kappa(\frac{1}{2}I + K'_\kappa)t = 0,$$

and therefore

$$(\frac{1}{2}I + K'_\kappa)t = 0, \quad t = V_\kappa^{-1}u.$$

In the case that V_κ is not injective, let $t_\lambda \in \ker V_\kappa = \ker(\frac{1}{2}I - K'_{-\kappa})$. From

$$\langle u, t_\lambda\rangle_\Gamma = \langle(\frac{1}{2}I + K_\kappa)u, t_\lambda\rangle_\Gamma + \langle u, (\frac{1}{2}I - K'_{-\kappa})t_\lambda\rangle_\Gamma = 0$$

we conclude, by the closed range theorem, $u \in \operatorname{Im} V_\kappa$. Hence there exists a unique solution $t \in H^{-1/2}(\Gamma)$ satisfying

$$V_\kappa t = u, \quad \langle Vt, t_\lambda\rangle_\Gamma = 0 \quad \text{for all } t_\lambda \in \ker V_\kappa.$$

Moreover, again by using (2.9), we obtain

$$0 = (\frac{1}{2}I + K_\kappa)u = (\frac{1}{2}I + K_\kappa)V_\kappa t = V_\kappa(\frac{1}{2}I + K'_\kappa)t.$$

In the case $(\frac{1}{2}I + K'_\kappa)t = \alpha t_\lambda \in \ker V_\kappa$ for some $\alpha \in \mathbb{R}$ we further conclude

$$t = \alpha t_\lambda + (\frac{1}{2}I - K'_\kappa)t,$$

which obviously is satisfied for $t = \alpha t_\lambda$. The orthogonality

$$\langle Vt, t_\lambda\rangle_\Gamma = \alpha\langle Vt_\lambda, t_\lambda\rangle_\Gamma = 0$$

gives $\alpha = 0$, and finally we obtain

$$(\frac{1}{2}I + K'_\kappa)t = 0.$$

■

Now we are in a position to investigate the unique solvability of the second kind boundary integral equation (3.3). Since the boundary integral operator $\frac{1}{2}I + K'_\kappa$ may not be injective, in particular when $\mu = \kappa^2$ is an eigenvalue of the interior Neumann eigenvalue problem (2.16), we introduce the polar space

$$\left(\ker(\frac{1}{2}I + K_{-\kappa})\right)^0 := \left\{w \in H^{-1/2}(\Gamma) : \langle w, u_\mu\rangle_\Gamma \text{ for all } u_\mu \in \ker(\frac{1}{2}I + K_{-\kappa})\right\},$$

and by the closed range theorem we have

$$\operatorname{Im}(\frac{1}{2}I + K'_\kappa) = \left(\ker(\frac{1}{2}I + K_{-\kappa})\right)^0.$$

Indeed, for $u_\mu \in \ker(\frac{1}{2}I + K_{-\kappa}) = \ker D_{-\kappa}$ we have

$$\langle D_\kappa g, u_\mu\rangle_\Gamma = \langle g, D_{-\kappa}u_\mu\rangle_\Gamma = 0,$$

and therefore, $D_\kappa g \in \operatorname{Im}(\frac{1}{2}I + K'_\kappa)$. In fact, the boundary integral equation (3.3) is solvable, but the solution may not be unique. Since $\frac{1}{2}I + K'_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is coercive and satisfies a Gårding inequality similar to (2.13), we can formulate the following result.

Corollary 3.4 *If κ^2 does not coincide with an eigenvalue μ of the interior Neumann eigenvalue problem (2.16), then $\frac{1}{2}I + K'_\kappa$ is injective, and the boundary integral equation (3.3) admits, for any $g \in H^{1/2}(\Gamma)$, a unique solution $t \in H^{-1/2}(\Gamma)$. Moreover, if $\mu = \kappa^2$ is an eigenvalue of the interior Neumann eigenvalue problem (2.16), the boundary integral equation (3.3) is solvable, but the solution is only unique up to eigensolutions $t_\lambda \in ker(\frac{1}{2}I + K'_\kappa)$.*

When the solution of the boundary integral equation (3.3) is not unique, we may proceed as in the previous case of the boundary integral equation (3.2) to introduce an appropriate scaling condition. In the case that the eigensolutions of the Neumann eigenvalue problem (2.16) are known, we can define a suitable factor space. In the other case, as in the CHIEF method, we may use the interior representation formula (2.22) for scaling.

Although both boundary integral equations (3.2) and (3.3) are solvable for all wave numbers $\kappa \in \mathbb{R}$, the solutions are not unique when κ^2 corresponds either to a Dirichlet or to a Neumann eigenvalue of the interior Laplace equation. Even when the Dirichlet and Neumann eigenvalues coincide, the underlying eigensolutions of the single layer boundary integral operator V_κ and of the adjoint double layer boundary integral operator $\frac{1}{2}I + K'_\kappa$ are different in general. This motivates to use suitable linear combinations of the boundary integral equations (3.2) and (3.3) to derive combined boundary integral equations which admit unique solutions for all wave numbers.

Following the approach of Burton and Miller [5] in the case of an exterior Neumann boundary value problem, in the case of the exterior Dirichlet boundary value problem (3.1) a complex linear combination of the boundary integral equations (3.2) and (3.3) results in the combined boundary integral equation, $\eta \in \mathbb{R}$, $\eta \neq 0$,

$$\Big(\frac{1}{2}I + K'_\kappa + i\eta V_\kappa\Big)t(x) = \Big(-D_\kappa + i\eta(-\frac{1}{2}I + K_\kappa)\Big)g(x) \quad \text{for } x \in \Gamma. \tag{3.9}$$

Lemma 3.5 *The boundary integral operator $\frac{1}{2}I + K'_\kappa + i\eta V_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, $\eta \in \mathbb{R}$, $\eta \neq 0$, is bounded, coercive and injective. Hence, for any $g \in H^{1/2}(\Gamma)$, there exists a unique solution $t \in H^{-1/2}(\Gamma)$ of the boundary integral equation (3.9).*

Proof. The boundedness of $\frac{1}{2}I + K'_\kappa + i\eta V_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ follows from $\frac{1}{2}I + K_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma)$. Since $K'_\kappa - K' + i\eta V_\kappa : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, Gårdings inequality follows.

To prove injectivity, let $w \in H^{-1/2}(\Gamma)$ be a solution of the homogeneous equation

$$\Big(\frac{1}{2}I + K'_\kappa + i\eta V_\kappa\Big)w(x) = 0 \quad \text{for } x \in \Gamma.$$

Define

$$u(x) = \int_\Gamma U^*_\kappa(x,y)w(y)ds_y \quad \text{for } x \in \Omega,$$

which is a solution of the interior Helmholtz equation. Moreover,

$$\frac{\partial}{\partial n_x}u(x) + i\eta u(x) = (\frac{1}{2}I + K'_\kappa)w(x) + i\eta(V_\kappa w)(x) = 0 \quad \text{for } x \in \Gamma.$$

In particular we conclude that u is a solution of the homogeneous interior Robin boundary value problem

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u(x) + i\eta u(x) = 0 \quad \text{for } x \in \Gamma.$$

By using Theorem 2.8 we conclude $u \equiv 0$ in Ω. Then,

$$(V_\kappa w)(x) = 0, \quad (\frac{1}{2} I + K'_\kappa) w(x) = 0 \quad \text{for } x \in \Gamma.$$

Since w is an eigensolution of the single layer boundary integral operator V_κ, we conclude, by using Corollary 2.4,

$$(\frac{1}{2} I - K'_\kappa) w(x) = 0 \quad \text{for } x \in \Gamma,$$

and therefore $w \equiv 0$ follows. ■

Remark 3.2 *Instead of the exterior Dirichlet boundary value problem (3.1), Burton and Miller [5] have considered the exterior Neumann boundary value problem*

$$\Delta u(x) + \kappa^2 u(x) = 0 \quad \textit{for } x \in \Omega^c, \quad \frac{\partial}{\partial n_x} u(x) = 0 \quad \textit{for } x \in \Gamma$$

and the radiation condition (2.3), where the total field $u = u_i + u_s$ *is decomposed into a given incoming field* u_i*, and an unknown scattered field* u_s*. The total field satisfies the modified representation formula for* $x \in \Omega^c$

$$u(x) = u_i(x) + \int_\Gamma \left\{ u(y) \frac{\partial}{\partial n_y} U^*_\kappa(x, y) - U^*_\kappa(x, y) \frac{\partial}{\partial n_y} u(y) \right\} ds_y. \tag{3.10}$$

From (3.10) one may conclude either the second kind boundary integral equation

$$(\frac{1}{2} I - K_\kappa) u(x) = u_i(x) \quad \textit{for } x \in \Gamma, \tag{3.11}$$

or the hypersingular boundary integral equation

$$(D_\kappa u)(x) = \frac{\partial}{\partial n_x} u_i(x) ds_x \quad \textit{for } x \in \Gamma \tag{3.12}$$

which are not uniquely solvable when κ^2 *corresponds either to a Dirichlet eigenvalue* λ *or to a Neumann eigenvalue* μ*, respectively. Hence, Burton and Miller proposed to consider a complex linear combination of the boundary integral equations (3.11) and (3.12), for* $\alpha \in \mathbb{C}$ *with* $\Im(\alpha) \neq 0$*,*

$$\left(\frac{1}{2} I - K_\kappa + \alpha D_\kappa \right) u(x) = u_i(x) + \alpha \frac{\partial}{\partial n_x} u_i(x) \quad \textit{for } x \in \Gamma. \tag{3.13}$$

While in [5] Burton and Miller discussed the uniqueness of the solution of the combined boundary integral equation (3.13) only, they did not comment on an appropriate functional analytic setting. In the case of a smooth surface, T.–C. Lin [15] gives a rigorous proof on the existence and uniqueness of the solution of (3.13) in Hölder spaces.

Although the alternative representation (2.7) of the hypersingular integral operator D_κ was already given in [5], this was not used for discretisation. Instead, using (2.9), a regularised version of (3.13) was derived, i.e.

$$V\left(\frac{1}{2}I - K_\kappa + \alpha[D_\kappa - D]\right)u + \alpha\left(\frac{1}{4}I - K^2\right)u = V\left(u_i + \alpha\frac{\partial}{\partial n_x}u_i\right).$$

Instead of a complex linear combination of the boundary integral equations (3.2) and (3.3) we may also consider a system of both equations to derive a stable boundary integral formulation of the exterior Dirichlet boundary value problem (3.1). The boundary integral equations of the exterior Calderon projection (2.21) can be written as

$$u = (\frac{1}{2}I + K_\kappa)u - V_\kappa t = g, \quad D_\kappa u + (\frac{1}{2}I + K'_\kappa)t = 0.$$

Hence we need to find $(u, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\begin{pmatrix} D_\kappa & \frac{1}{2}I + K'_\kappa \\ -(\frac{1}{2}I + K_\kappa) & V_\kappa \end{pmatrix}\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}. \tag{3.14}$$

Note that our main interest is in the determination of the unknown Neumann datum t, while $u = g$ is given by the Dirichlet boundary condition.

Lemma 3.6 *The boundary integral operator as considered in (3.14) is injective in t. In particular, the homogeneous system*

$$V_\kappa t - (\frac{1}{2}I + K_\kappa)u = 0, \quad D_\kappa u + (\frac{1}{2}I + K'_\kappa)t = 0$$

implies $t = 0$ for all wave numbers κ.

Proof. Let $(u, t) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be a solution of the homogeneous system

$$V_\kappa t - (\frac{1}{2}I + K_\kappa)u = 0, \quad D_\kappa u + (\frac{1}{2}I + K'_\kappa)t = 0.$$

When applying the hypersingular boundary integral operator D_κ to the weakly singular boundary integral equation, and using (2.9), this gives

$$\begin{aligned} 0 &= D_\kappa V_\kappa t - D_\kappa(\frac{1}{2}I + K_\kappa)u \\ &= (\frac{1}{2}I + K'_\kappa)(\frac{1}{2}I - K'_\kappa)t - (\frac{1}{2}I + K'_\kappa)D_\kappa u \\ &= (\frac{1}{2}I + K'_\kappa)(\frac{1}{2}I - K'_\kappa)t + (\frac{1}{2}I + K'_\kappa)(\frac{1}{2}I + K'_\kappa)t = (\frac{1}{2}I + K'_\kappa)t, \end{aligned}$$

and therefore,

$$(\frac{1}{2}I + K'_\kappa)t = 0, \quad D_\kappa u = 0.$$

In the same way, when applying the single layer boundary integral operator V_κ to the hypersingular boundary integral operator, we have

$$\begin{aligned} 0 &= V_\kappa D_\kappa u + V_\kappa(\frac{1}{2}I + K'_\kappa)t \\ &= (\frac{1}{2}I + K_\kappa)(\frac{1}{2}I - K_\kappa)u + (\frac{1}{2}I + K_\kappa)V_\kappa t \\ &= (\frac{1}{2}I + K_\kappa)(\frac{1}{2}I - K_\kappa)u + (\frac{1}{2}I + K_\kappa)(\frac{1}{2}I + K_\kappa)u = (\frac{1}{2}I + K_\kappa)u, \end{aligned}$$

and therefore,

$$(\frac{1}{2}I + K_\kappa)u = 0, \quad V_\kappa t = 0.$$

The latter also implies

$$(\frac{1}{2}I - K'_\kappa)t = 0,$$

and therefore, $t = 0$ follows. ∎

Note that Lemma 3.6 allows no statement concerning u. In particular, the remaining equations

$$D_\kappa u = 0, \quad (\frac{1}{2}I + K_\kappa)u = 0$$

are both satisfied when $\mu = \kappa^2$ is a Neumann eigenvalue.

Indeed, the first boundary integral equation in (3.14) is always solvable. When κ^2 is not a Neumann eigenvalue, the hypersingular boundary integral operator D_κ is bijective and we obtain

$$u = -D_\kappa^{-1}(\frac{1}{2}I + K'_\kappa)t.$$

If $\mu = \kappa^2$ is a Neumann eigenvalue, the general solution is given by

$$u = -D_\kappa^{-1}(\frac{1}{2}I + K'_\kappa)t + \alpha u_\mu,$$

where the application of D_κ^{-1} has to be considered between appropriate factor spaces, see also the related discussion for the definition (3.8) of the exterior Steklov–Poincaré operator. However, in both cases we conclude the Schur complement system

$$T_\kappa t := \Big[V_\kappa + (\frac{1}{2}I + K_\kappa)D_\kappa^{-1}(\frac{1}{2}I + K'_\kappa)\Big]t = -g, \tag{3.15}$$

where $T_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is also coercive, and bijective.

To avoid the use of the inverse hypersingular boundary integral operator D_κ between appropriate factor spaces, in particular when $\mu = \kappa^2$ is an eigenvalue of the interior Neumann eigenvalue problem (2.16), we can modify the system (3.14) as follows. Since $u = g$

is given due to the Dirichlet boundary condition, instead of (3.14) we now consider the modified system, $\eta \in \mathbb{R}$,

$$\begin{pmatrix} D_\kappa + i\eta\widetilde{D} & \frac{1}{2}I + K'_\kappa \\ -(\frac{1}{2}I + K_\kappa) & V_\kappa \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} i\eta\widetilde{D}g \\ -g \end{pmatrix} \tag{3.16}$$

where $\widetilde{D}$ is the stabilised Laplace hypersingular boundary integral operator as given in (2.11). It turns out that $D_\kappa + i\eta\widetilde{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is invertible for all wave numbers $\kappa \in \mathbb{R}$, and hence, instead of (3.15) we may consider the modified Schur complement system

$$\begin{aligned}\Big[V_\kappa + (\frac{1}{2}I + K_\kappa)(D_\kappa + i\eta\widetilde{D})^{-1}(\frac{1}{2}I + K'_\kappa)\Big]t \\ = i\eta(\frac{1}{2}I + K_\kappa)(D_\kappa + i\eta\widetilde{D})^{-1}\widetilde{D}g - g.\end{aligned} \tag{3.17}$$

Although both Schur complement systems (3.15) and (3.17) are unique solvable, the modified version allows a direct application of standard arguments to derive a stability and error analysis of related boundary element methods.

3.2 Indirect boundary integral equations

Instead of the representation formula (2.20) of the direct approach one may also consider an indirect approach to use single and double layer potentials to describe solutions of the exterior Helmholtz equation (2.2), and satisfying the radiation condition (2.3). In particular, one may use either the single layer potential ansatz

$$u(x) = \int_\Gamma U^*_\kappa(x,y)w(y)ds_y \quad \text{for } x \in \Omega^c$$

or the double layer layer potential ansatz

$$u(x) = \int_\Gamma \frac{\partial}{\partial n_y} U^*_\kappa(x,y)v(y)ds_y \quad \text{for } x \in \Omega^c.$$

By considering the Dirichlet boundary condition $u = g$ on Γ we conclude the boundary integral equations to find $w \in H^{-1/2}(\Gamma)$ such that

$$(V_\kappa w)(x) = g(x) \quad \text{for } x \in \Gamma, \tag{3.18}$$

or to find $v \in H^{1/2}(\Gamma)$ such that

$$(\frac{1}{2}I + K_\kappa)v(x) = g(x) \quad \text{for } x \in \Gamma. \tag{3.19}$$

Although the boundary integral equation (3.18) of the indirect approach is similar to the boundary integral equation (3.2) of the direct approach, statements on existence and uniqueness of solutions are rather different, in particular when $\lambda = \kappa^2$ corresponds to an eigenvalue of the interior Dirichlet eigenvalue problem (2.14). In this case, V_κ is not injective, and $g \notin \operatorname{Im} V_\kappa$ in general, i.e. the boundary integral equation (3.18) is not solvable in this case. In all other cases, V_κ is injective, and therefore (3.18) admits a unique solution $w \in H^{-1/2}(\Gamma)$.

Analoguesly, the boundary integral equation (3.19) is in general not solvable when $\mu = \kappa^2$ corresponds to an eigenvalue of the Neumann eigenvalue problem (2.16), and hence the boundary integral operator $\frac{1}{2}I + K_\kappa$ is not injective. In all other cases, the boundary integral operator $\frac{1}{2}I + K_\kappa$ is injective, and since it is also coercive, there exists a unique solution $v \in H^{1/2}(\Gamma)$ of the boundary integral equation (3.19).

To obtain formulations of boundary integral equations which are uniquely solvable for all wave numbers one may consider complex linear combinations of the boundary integral equations (3.18) and (3.19).

As in [2], see also [22], we may consider a complex linear combination of the indirect single and double layer potentials to describe a solution of the exterior Helmholtz equation as

$$u(x) = \int_\Gamma \nu(y)\Big(\frac{\partial}{\partial n_y} - i\eta\Big)U_\kappa^*(x,y)ds_y \quad \text{for } x \in \Omega^c, \tag{3.20}$$

and from the Dirichlet boundary condition $u = g$ on Γ we conclude the boundary integral equation

$$\Big(\frac{1}{2}I + K_\kappa - i\eta V_\kappa\Big)\nu(x) = g(x) \quad \text{for } x \in \Gamma \tag{3.21}$$

to be solved.

Lemma 3.7 *The boundary integral operator* $\frac{1}{2}I + K_\kappa - i\eta V_\kappa : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is bounded, coercive and injective. Hence, for any* $g \in H^{1/2}(\Gamma)$*, there exists a unique solution* $\nu \in H^{1/2}(\Gamma)$ *of the boundary integral equation (3.21).*

Proof. From $\frac{1}{2}I + K_\kappa : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, $V_\kappa : H^{1/2}(\Gamma) \subset H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ we first conclude boundedness and coercivity.

To prove injectivity, let $\nu \in H^{1/2}(\Gamma)$ be a solution of the homogeneous boundary integral equation

$$\Big(\frac{1}{2}I + K_\kappa - i\eta V_\kappa\Big)\nu(x) = 0 \quad \text{for } x \in \Gamma,$$

and we define

$$u(x) = \int_\Gamma \frac{\partial}{\partial n_y} U_\kappa^*(x,y)\nu(y)ds_y - i\eta \int_\Gamma U_\kappa^*(x,y)\nu(y)ds_y \quad \text{for } x \in \mathbb{R}^3\backslash\Gamma.$$

By construction, u is a solution of the exterior Dirichlet boundary value problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c, \quad u(x) = 0 \quad \text{for } x \in \Gamma.$$

Since u satisfies the Sommerfeld radiation condition (2.3), $u \equiv 0$ in Ω^c follows. On the other hand, u is also a solution of the interior Helmholtz equation

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega.$$

For the interior Dirichlet trace of u we further obtain

$$u(x) = (-\frac{1}{2}I + K_\kappa)\nu(x) - i\eta(V_\kappa\nu)(x) = -\nu(x) \quad \text{for } x \in \Gamma,$$

while for the interior Neumann trace of u we conclude

$$\frac{\partial}{\partial n_x}u(x) = -i\eta\nu(x) \quad \text{for } x \in \Gamma.$$

Note that we have used the relations of all boundary integral operators involved, and $u \equiv 0$ in Ω^c. Hence we have

$$\frac{\partial}{\partial n_x}u(x) = i\eta u(x) \quad \text{for } x \in \Gamma.$$

Hence, u is a solution of the homogeneous interior Robin boundary value problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x}u(x) - i\eta u(x) = 0 \quad \text{for } x \in \Gamma.$$

By using Theorem 2.8, $u \equiv 0$ in Ω follows, and by the jump relation of the double layer potential we finally conclude

$$\nu(x) = u_{|\Omega^c}(x) - u_{|(\Omega)}(x) = 0 \quad \text{for } x \in \Gamma.$$

■

Remark 3.3 *In [2], the authors consider the case of a twice differentiable smooth surface* Γ*, and the space of continuous functions, i.e.* $\nu \in C(\Gamma)$*. In this setting, the operator* $K_\kappa - i\eta V_\kappa : C(\Gamma) \to C(\Gamma)$ *is compact, and unique solvability of the boundary integral equation (3.21) is a consequence of Fredholm's alternative. Instead, we may also consider the boundary integral equation (3.21) in* $L_2(\Gamma)$*, but we still need to assume a smooth surface to ensure compactness of* $K_\kappa - i\eta V_\kappa : L_2(\Gamma) \to L_2(\Gamma)$*.*

Instead of the indirect ansatz (3.20) we may also consider the alternative ansatz

$$u(x) = \int_\Gamma \nu(y) \left(i\eta \frac{\partial}{\partial n_y} + 1 \right) U_\kappa^*(x,y) ds_y \quad \text{for } x \in \Omega^c,$$

which results in the boundary integral equation

$$\Big(V_\kappa + i\eta(\frac{1}{2}I + K_\kappa) \Big) \nu(x) = g(x) \quad \text{for } x \in \Gamma. \tag{3.22}$$

Obviously, unique solvability of the boundary integral equation (3.22) follows as for (3.21). In particular, the equivalence with the boundary integral equation (3.21) ensures unique solvability in $H^{1/2}(\Gamma)$.

When we consider second kind boundary integral equations in the natural Sobolev spaces, i.e. in $H^{\pm 1/2}(\Gamma)$, we can ensure unique solvability even in case of a general Lipschitz boundary. These results are based on ellipticity estimates of the Laplace double layer boundary integral operator $\frac{1}{2}I - K$ in $H^{1/2}(\Gamma)$, see [28]. Since a variational formulation in $H^{\pm 1/2}(\Gamma)$ is in general not applicable for a stable and efficient discretisation scheme, alternative formulations are of interest. In most cases we may consider second kind boundary integral equations in $L_2(\Gamma)$, but in this case we need to assume smooth boundaries to ensure compactness of the Laplace double layer boundary integral operator. When considering, e.g. the boundary integral equation (3.21) in $L_2(\Gamma)$, we observe a mismatch in the mapping properties of the single layer boundary integral operator V_κ, and the double layer boundary integral operator $\frac{1}{2}I + K_\kappa$. This motivates to introduce regularised combined boundary integral equations.

3.3 Regularised combined boundary integral equations

Instead of the combined boundary integral equations (3.21) and (3.22) we now consider regularised boundary integral equations to find either $w \in H^{-1/2}(\Gamma)$ such that

$$\Big[(\frac{1}{2}I + K_\kappa)B - i\eta V_\kappa\Big]w(x) = g(x) \quad \text{for } x \in \Gamma, \tag{3.23}$$

or to find $v \in H^{1/2}(\Gamma)$ such that

$$\Big[(\frac{1}{2}I + K_\kappa) - i\eta V_\kappa B^{-1}\Big]v(x) = g(x) \quad \text{for } x \in \Gamma.$$

In both cases, $B : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a suitable given operator. Instead of (3.23) we may also consider the equivalent formulation to find $w \in H^{-1/2}(\Gamma)$ such that

$$\Big[V_\kappa + i\eta(\frac{1}{2}I + K_\kappa)B\Big]w(x) = g(x) \quad \text{for } x \in \Gamma.$$

In particular, for the Laplace–Beltrami operator $B := V^2 : H^{-1}(\Gamma) \to H^1(\Gamma)$ we can use the compact imbedding $H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ to prove unique solvability of the regularised boundary integral equation (3.23), see [4, 8]. Another choice is, see [10, 11], $B := \widetilde{D}^{-1}(\frac{1}{2}I + K'_{-\kappa})$, or, as in (3.17), $B = \frac{1}{i\eta}(D_\kappa + i\eta\widetilde{D})^{-1}(\frac{1}{2}I + K'_\kappa)$.

4 Transmission problems

As a model problem we consider the scattering at an interface between two media of different density [14, 18],

$$\Delta u_i(x) + \kappa_i^2 u_i(x) = 0 \quad \text{for } x \in \Omega, \quad \Delta u_e(x) + \kappa_e^2 u_e(x) = 0 \quad \text{for } x \in \Omega^c, \tag{4.1}$$

where the exterior field u_e also satisfies the Sommerfeld radiation condition (2.3). In addition we consider the inhomogeneous transmission conditions for $x \in \Gamma = \partial\Omega$,

$$\varrho_i u_i(x) - \varrho_e u_e(x) = f(x), \quad t_i(x) - t_e(x) := \frac{\partial}{\partial n_i} u_i(x) - \frac{\partial}{\partial n_i} u_e(x) = g(x), \tag{4.2}$$

where the discontinuity in the normal velocity g and in the pressure discontinuity f represent the effect of boundary layer sources. Unique solvability of the transmission problem (4.1), (4.2) and (2.3) is based on the following result.

Theorem 4.1 [14, Theorem 3.1] *Assume $\kappa_i, \kappa_e, \varrho_i, \varrho_e \in \mathbb{C}$, $0 \leq arg\,\kappa_i, arg\,\kappa_e < \pi$, and*

$$\varrho := \frac{\varrho_i}{\varrho_e} \frac{\overline{\kappa}_i^2}{\overline{\kappa}_e^2} \in \mathbb{R},$$

where $\varrho \geq 0$ (< 0) if $\Re(\kappa_i), \Re(\kappa_e) \geq 0$ (< 0). Then the only solution of the homogeneous transmission problem is the trivial solution.

In what follows we discuss equivalent reformulations of the transmission boundary value problem (4.1), (4.2), and (2.3) by means of boundary integral equations, in particular for wave numbers $\kappa_i, \kappa_e \in \mathbb{R}$.

Related to the interior Helmholtz equation in (4.1) we obtain the boundary integral equations

$$(V_{\kappa_i} t_i)(x) - (\frac{1}{2} I + K_{\kappa_i}) u_i(x) = 0 \quad \text{for } x \in \Gamma \tag{4.3}$$

and

$$(D_{\kappa_i} u_i)(x) - (\frac{1}{2} I - K'_{\kappa_i}) t_i(x) = 0 \quad \text{for } x \in \Gamma, \tag{4.4}$$

while for the exterior Helmholtz problem in (4.1) we conclude the boundary integral equations

$$(V_{\kappa_e} t_e)(x) - (-\frac{1}{2} I + K_{\kappa_e}) u_e(x) = 0 \quad \text{for } x \in \Gamma \tag{4.5}$$

and

$$(D_{\kappa_e} u_e)(x) + (\frac{1}{2} I + K'_{\kappa_e}) t_e(x) = 0 \quad \text{for } x \in \Gamma. \tag{4.6}$$

Together with the transmission conditions (4.2) we therefore have six equations to find the four unknowns $(u_i, t_i; u_e, t_e)$. Recall that the boundary integral equations (4.3) and (4.4) as well as (4.5) and (4.6) are not independent of each other. In what follows we may use different combinations of the above boundary integral equations to describe the unique solution of the transmission problem (4.1), (4.2), and (2.3).

4.1 Steklov–Poincaré operator equations

As in the case of the Laplace equation we may use boundary integral operators to define Steklov–Poincaré operators which realize the Dirichlet to Neumann maps for both the interior and exterior boundary value problems. But special care is required when the local wave numbers, κ_i and κ_e, coincide with eigenvalues of either the interior Dirichlet or Neumann eigenvalue problems (2.14) and (2.16), respectively. Since the transmission problem (4.1) and (4.2) admits, by Theorem 4.1, a unique solution, we aim to derive an equivalent boundary integral equation formulation which is stable for all wave numbers.

From the boundary integral equations (4.3) and (4.4) we find the characterisation

$$t_i(x) = (D_{\kappa_i} u_i)(x) + (\frac{1}{2}I + K'_{\kappa_i})t_i(x), \quad (V_{\kappa_i} t_i)(x) = (\frac{1}{2}I + K_{\kappa_i})u_i(x),$$

while from (4.5) and (4.6) we obtain

$$t_e(x) = -\Big[D_{\kappa_e} + (\frac{1}{2}I - K'_{\kappa_e})V_{\kappa_e}^{-1}(\frac{1}{2}I - K_{\kappa_e})\Big]u_e(x) = -(S^{\text{ext}}_{\kappa_e} u_e)(x),$$

where $S^{\text{ext}}_{\kappa_e}$ is the Steklov–Poincaré operator (3.7) of the exterior Dirichlet boundary value problem, and which is well defined for all wave numbers $\kappa_e \in \mathbb{R}$.

From the Neumann transmission condition in (4.2) we therefore conclude

$$\begin{aligned} g(x) &= t_i(x) - t_e(x) \\ &= (D_{\kappa_i} u_i)(x) + (\frac{1}{2}I + K'_{\kappa_i})t_i(x) + (S^{\text{ext}}_{\kappa_e} u_e)(x), \end{aligned}$$

where, in addition, t_i is a solution of the local boundary integral equation

$$(V_{\kappa_i} t_i)(x) = (\frac{1}{2}I + K_{\kappa_i})u_i(x).$$

When inserting the Dirichlet transmission condition, i.e.

$$u_e(x) = \frac{1}{\varrho_e}\Big[\varrho_i u_i(x) - f(x)\Big] \quad \text{for } x \in \Gamma,$$

we end up with the following system of boundary integral equations

$$\begin{pmatrix} V_{\kappa_i} & -(\frac{1}{2}I + K_{\kappa_i}) \\ \frac{1}{2}I + K'_{\kappa_i} & D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} S^{\text{ext}}_{\kappa_e} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} = \begin{pmatrix} 0 \\ g + \frac{1}{\varrho_e} S^{\text{ext}}_{\kappa_e} f \end{pmatrix}. \tag{4.7}$$

Lemma 4.2 *Let the assumptions of Theorem 4.1 to be satisfied. Then the system (4.7) is injective.*

Proof. Let $(t_i, u_i) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ be a solution of the homogeneous system of boundary integral equations, i.e.

$$\begin{pmatrix} V_{\kappa_i} & -(\frac{1}{2}I + K_{\kappa_i}) \\ \frac{1}{2}I + K'_{\kappa_i} & D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} S^{\text{ext}}_{\kappa_e} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We then define

$$U_i(x) = \int_\Gamma U^*_{\kappa_i}(x,y)t_i(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y)u_i(y)ds_y \quad \text{for } x \in \Omega,$$

which is a solution of the interior Helmholtz equation

$$-\Delta U_i(x) - \kappa_i^2 U_i(x) = 0 \quad \text{for } x \in \Omega,$$

and with the Cauchy data

$$\begin{aligned} U_i(x) &= (V_{\kappa_i} t_i)(x) + (\frac{1}{2}I - K_{\kappa_i})u_i(x) = u_i(x), \\ \frac{\partial}{\partial n_i} U_i(x) &= (\frac{1}{2}I + K'_{\kappa_i})t_i(x) + (D_{\kappa_i} u_i)(x). \end{aligned}$$

Since the Steklov–Poincaré operator $S^{\text{ext}}_{\kappa_e}$ of the exterior Dirichlet boundary value problem is well defined when using the symmetric representation (3.7), we introduce $t_e := -\frac{\varrho_i}{\varrho_e} S^{\text{ext}}_{\kappa_e} u_i$ and we define

$$U_e(x) = -\int_\Gamma U^*_{\kappa_e}(x,y)t_e(y)ds_y + \frac{\varrho_i}{\varrho_e}\int_\Gamma \frac{\partial}{\partial n_i} U^*_{\kappa_e}(x,y)u_i(y)ds_y \quad \text{for } x \in \Omega^c,$$

which is a solution of the exterior Helmholtz equation

$$-\Delta U_e(x) - \kappa_e^2 U_e(x) = 0 \quad \text{for } x \in \Omega^c,$$

and with the Cauchy data

$$\begin{aligned} U_e(x) &= -V_{\kappa_e} t_e(x) + \frac{\varrho_i}{\varrho_e}(\frac{1}{2}I + K_{\kappa_e})u_i(x) \\ &= \frac{\varrho_i}{\varrho_e}\left[V_{\kappa_e} S^{\text{ext}}_{\kappa_e} + (\frac{1}{2}I + K_{\kappa_e})\right] u_i(x) = \frac{\varrho_i}{\varrho_e} u_i(x), \\ \frac{\partial}{\partial n_i} U_e(x) &= (\frac{1}{2}I - K'_{\kappa_e})t_e(x) - \frac{\varrho_i}{\varrho_e}(D_{\kappa_e} u_i)(x) \\ &= -\frac{\varrho_i}{\varrho_e}\left[(\frac{1}{2}I - K'_{\kappa_e})S^{\text{ext}}_{\kappa_e} + D_{\kappa_e}\right] u_i(x) = -\frac{\varrho_i}{\varrho_e}(S^{\text{ext}}_{\kappa_e} u_i)(x). \end{aligned}$$

Note that in the above computations we can use the non–symmetric representation (3.8) of the exterior Steklov–Poincaré operator $S^{\text{ext}}_{\kappa_e}$, since in both cases the boundary integral operators V_{κ_e} and $(\frac{1}{2}I - K'_{\kappa_e})$ eliminate possible eigenfunctions in the case when $\lambda = \kappa_e^2$ is an eigenvalue of the interior Dirichlet eigenvalue problem (2.14).

Hence we conclude that (U_i, U_e) is a solution of the transmission problem

$$-\Delta U_i - \kappa_i^2 U_i = 0 \quad \text{in } \Omega, \quad -\Delta U_e - \kappa_e^2 U_e = 0 \quad \text{in } \Omega^c$$

with the transmission conditions

$$\varrho_i U_i = \varrho_e U_e, \quad \frac{\partial}{\partial n_i} U_i = \frac{\partial}{\partial n_i} U_e \quad \text{on } \Gamma.$$

Since U_e also satisfies the radiation condition (2.3), we can apply Theorem 4.1 to conclude $U_i \equiv 0$ in Ω and $U_e \equiv 0$ in Ω^c. From this,

$$V_{\kappa_i} t_i + (\frac{1}{2} I - K_{\kappa_i}) u_i = 0, \quad (\frac{1}{2} I + K'_{\kappa_i}) t_i + D_{\kappa_i} u_i = 0$$

follows. Together with the first equation of the homogeneous system this results in $u_i = 0$, and therefore

$$V_{\kappa_i} t_i = 0, \quad (\frac{1}{2} I + K'_{\kappa_i}) t_i = 0$$

follows. Since the first relation implies

$$(\frac{1}{2} I - K'_{\kappa_i}) t_i = 0,$$

also $t_i = 0$ follows. ■

Now we are in a position to state the unique solvability of the system (4.7) of boundary integral equations.

Theorem 4.3 *Let the assumptions of Theorem 4.1 to be satisfied. Then there exists a unique solution* $(t_i, u_i) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *of the system (4.7).*

Proof. The system (4.7) of boundary integral equations induces an operator

$$\begin{aligned} \mathcal{A}_\kappa &:= \begin{pmatrix} V_{\kappa_i} & -(\frac{1}{2} I + K_{\kappa_i}) \\ \frac{1}{2} I + K'_{\kappa_i} & D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} S^{\text{ext}}_{\kappa_e} \end{pmatrix} \\ &= \begin{pmatrix} V & -(\frac{1}{2} I + K) \\ \frac{1}{2} I + K' & \widetilde{D} + \frac{\varrho_i}{\varrho_e} S^{\text{ext}} \end{pmatrix} + \begin{pmatrix} V_{\kappa_i} - V & -(K_{\kappa_i} - K) \\ K'_{\kappa_i} - K' & D_{\kappa_i} - \widetilde{D} + \frac{\varrho_i}{\varrho_e} (S^{\text{ext}}_{\kappa_e} - S^{\text{ext}}) \end{pmatrix} \\ &= \mathcal{A} + (\mathcal{A}_\kappa - \mathcal{A}), \end{aligned}$$

where $V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the Laplace single layer boundary integral operator, $K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is the Laplace double layer boundary integral operator, and $K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ its adjoint, and $\widetilde{D} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the stabilised Laplace hypersingular boundary integral operator. Moreover, S^{ext} is the related Steklov–Poincaré operator of the exterior Dirichlet boundary value problem of the Laplacian. The boundary integral operator $\mathcal{A} : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is elliptic, while the operator $\mathcal{A}_\kappa - \mathcal{A}$ is compact. In fact, the boundary integral operator $\mathcal{A}_\kappa$ is coercive, and due to Lemma 4.2, injective. Hence we conclude unique solvability of (4.7) by Fredholm's alternative. ■

If the wave number κ_i is not related to an eigenvalue of the interior Dirichlet eigenvalue problem (2.14), then the single layer boundary integral operator V_{κ_i} is invertible, and we can determine

$$t_i = V_{\kappa_i}^{-1}(\frac{1}{2}I + K_{\kappa_i})u_i \in H^{-1/2}(\Gamma).$$

Hence we conclude the Schur complement boundary integral equation

$$\left[S_{\kappa_i}^{\text{int}} + \frac{\varrho_i}{\varrho_e} S_{\kappa_e}^{\text{ext}}\right] u_i(x) = g(x) + \frac{1}{\varrho_e}(S_{\kappa_e}^{\text{ext}} f)(x) \quad \text{for } x \in \Gamma$$

with the Steklov–Poincaré operator of the interior Dirichlet problem

$$S_{\kappa_i}^{\text{int}} := D_{\kappa_i} + (\frac{1}{2}I + K'_{\kappa_i})V_{\kappa_i}^{-1}(\frac{1}{2}I + K_{\kappa_i}).$$

Although the exterior Steklov–Poincaré operator $S_{\kappa_e}^{\text{ext}}$ is well defined for all wave numbers $\kappa_e \in \mathbb{R}$, alternatively, we may proceed as in (3.16) to end up with a modified system of boundary integral equations. For this, we first rewrite (4.7) as

$$\begin{pmatrix} V_{\kappa_i} & & -(\frac{1}{2}I + K_{\kappa_i}) \\ & \varrho_e V_{\kappa_e} & -\varrho_i(\frac{1}{2}I - K_{\kappa_e}) \\ \frac{1}{2}I + K'_{\kappa_i} & & D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} D_{\kappa_e} \end{pmatrix} \begin{pmatrix} t_i \\ t_e \\ u_i \end{pmatrix} = \begin{pmatrix} 0 \\ -(\frac{1}{2}I - K_{\kappa_e})f \\ g \end{pmatrix}.$$

Moreover, due to the Neumann transmission condition $t_i - t_e = g$ we may add the equality

$$-i\eta\varrho_e V t_i + i\eta\varrho_e V t_e = -i\eta\varrho_e V g, \quad \eta \in \mathbb{R},$$

to the second equation, to obtain the modified system

$$\begin{pmatrix} V_{\kappa_i} & & -(\frac{1}{2}I + K_{\kappa_i}) \\ -i\eta\varrho_e V & \varrho_e(V_{\kappa_e} + i\eta V) & -\varrho_i(\frac{1}{2}I - K_{\kappa_e}) \\ \frac{1}{2}I + K'_{\kappa_i} & & D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} D_{\kappa_e} \end{pmatrix} \begin{pmatrix} t_i \\ t_e \\ u_i \end{pmatrix} = \begin{pmatrix} 0 \\ -(\frac{1}{2}I - K_{\kappa_e})f - i\eta\varrho_e V g \\ g \end{pmatrix}.$$

The operator $V_\kappa + i\eta V$ is invertible for all wave numbers, without any additional restriction. Hence we can eliminate t_e, and we can proceed as before to establish unique solvability of the resulting boundary integral equation system.

4.2 Combined boundary integral equations

The boundary integral equation system (4.7) was based on the use of both the interior and exterior Calderon projections (2.8) and (2.21), and on the elimination of the exterior Cauchy data t_e and u_e. Hence, (4.7) can be seen as a *single trace formulation.* Although the system (4.7) is equivalent to the original transmission problem (4.1) and (4.2), and stable for all wave numbers κ_i and κ_e, one may ask for alternative formulations, in particular from a numerical point of view.

By using parameters $\alpha_i, \alpha_e \in \mathbb{C}$ we can write the general linear combination of the boundary integral equations (4.3) and (4.5) as

$$\alpha_i \left[V_{\kappa_i} t_i - (\frac{1}{2} I + K_{\kappa_i}) u_i \right] + \alpha_e \left[V_{\kappa_e} t_e - (-\frac{1}{2} I + K_{\kappa_e}) u_e \right] = 0 \quad \text{on } \Gamma,$$

while by using parameters $\beta_i, \beta_e \in \mathbb{C}$ we obtain from (4.4) and (4.6)

$$\beta_i \left[D_{\kappa_i} u_i - (\frac{1}{2} I - K'_{\kappa_i}) t_i \right] + \beta_e \left[D_{\kappa_e} u_e - (-\frac{1}{2} I - K'_{\kappa_e}) t_e \right] = 0 \quad \text{on } \Gamma.$$

By using the transmission conditions (4.2) we may insert

$$u_e(x) = \frac{\varrho_i}{\varrho_e} u_i(x) - \frac{1}{\varrho_e} f(x), \quad t_e(x) = t_i(x) - g(x) \quad \text{for } x \in \Gamma$$

to obtain the coupled system, in the most general case,

$$\begin{aligned} \Big[\alpha_i V_{\kappa_i} + \alpha_e V_{\kappa_e} \Big] t_i + \frac{1}{2} \left(\alpha_e \frac{\varrho_i}{\varrho_e} - \alpha_i \right) u_i - \left[\alpha_i K_{\kappa_i} + \alpha_e \frac{\varrho_i}{\varrho_e} K_{\kappa_e} \right] u_i & \\ = \alpha_e \left[V_{\kappa_e} g + \frac{1}{\varrho_e} (\frac{1}{2} I - K_{\kappa_e}) f \right] & \end{aligned}$$

and

$$\begin{aligned} \left[\beta_i D_{\kappa_i} + \beta_e \frac{\varrho_i}{\varrho_e} D_{\kappa_e} \right] u_i + \frac{1}{2} \Big(\beta_e - \beta_i \Big) t_i + \left[\beta_i K'_{\kappa_i} + \beta_e K'_{\kappa_e} \right] t_i & \\ = \beta_e \left[(\frac{1}{2} I + K'_{\kappa_e}) g + \frac{1}{\varrho_e} D_{\kappa_e} f \right]. & \end{aligned}$$

For the particular choice

$$\alpha_i = 1, \quad \alpha_e = \frac{\varrho_e}{\varrho_i}, \quad \beta_i = 1, \quad \beta_e = 1$$

we then conclude the system of boundary integral equations

$$\Big[V_{\kappa_i} + \frac{\varrho_e}{\varrho_i} V_{\kappa_e} \Big] t_i - \Big[K_{\kappa_i} + K_{\kappa_e} \Big] u_i = \frac{\varrho_e}{\varrho_i} V_{\kappa_e} g + \frac{1}{\varrho_i} (\frac{1}{2} I - K_{\kappa_e}) f, \tag{4.8}$$

$$\left[K'_{\kappa_i} + K'_{\kappa_e} \right] t_i + \left[D_{\kappa_i} + \frac{\varrho_i}{\varrho_e} D_{\kappa_e} \right] u_i = (\frac{1}{2} I + K'_{\kappa_e}) g + \frac{1}{\varrho_e} D_{\kappa_e} f. \tag{4.9}$$

Note that the system of the boundary integral equations (4.8) and (4.9) is known as *single trace formulation*, see, e.g., [12, 23]. Since the underlying bilinear form can be shown to be coercive in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, see, e.g., the proof of Theorem 4.3, it remains to establish injectivity.

Lemma 4.4 *The boundary integral operator*

$$\left(\begin{array}{cc} V_{\kappa_i}+\frac{\varrho_e}{\varrho_i}V_{\kappa_e} & -K_{\kappa_i}-K_{\kappa_e} \\ K'_{\kappa_i}+K'_{\kappa_e} & D_{\kappa_i}+\frac{\varrho_i}{\varrho_e}D_{\kappa_e}\end{array}\right) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is injective.

Proof. Let $(t_i, u_i) \in H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ be a solution of the homogeneous system

$$\left(\begin{array}{cc} V_{\kappa_i}+\frac{\varrho_e}{\varrho_i}V_{\kappa_e} & -K_{\kappa_i}-K_{\kappa_e} \\ K'_{\kappa_i}+K'_{\kappa_e} & D_{\kappa_i}+\frac{\varrho_i}{\varrho_e}D_{\kappa_e}\end{array}\right)\left(\begin{array}{c} t_i \\ u_i\end{array}\right) = \left(\begin{array}{c} 0 \\ 0\end{array}\right).$$

We then define

$$U_i(x) = \int_\Gamma U^*_{\kappa_i}(x,y)t_i(y)ds_y - \int_\Gamma \frac{\partial}{\partial n_y}U^*_{\kappa_i}(x,y)u_i(y)ds_y \quad \text{for } x\in\Omega$$

and

$$U_e(x) = -\int_\Gamma U^*_{\kappa_e}(x,y)t_i(y)ds_y + \frac{\varrho_i}{\varrho_e}\int_\Gamma \frac{\partial}{\partial n_y}U^*_{\kappa_e}(x,y)u_i(y)ds_y \quad \text{for } x\in\Omega^c,$$

which are solutions of the interior and exterior Helmholtz equation, respectively. For the Cauchy data we obtain for $x\in\Gamma$

$$\begin{aligned} U_i(x) &= (V_{\kappa_i}t_i)(x) + (\frac{1}{2}I - K_{\kappa_i})u_i(x), \\ \frac{\partial}{\partial n_i}U_i(x) &= (\frac{1}{2}I + K'_{\kappa_i})t_i(x) + (D_{\kappa_i}u_i)(x) \end{aligned}$$

and

$$\begin{aligned} U_e(x) &= -(V_{\kappa_e}t_i)(x) + \frac{\varrho_i}{\varrho_e}(\frac{1}{2}I + K_{\kappa_e})u_i(x), \\ \frac{\partial}{\partial n_i}U_e(x) &= (\frac{1}{2}I - K'_{\kappa_e})t_i(x) - \frac{\varrho_i}{\varrho_e}(D_{\kappa_e}u_i)(x). \end{aligned}$$

Hence we conclude

$$\begin{aligned} U_i - \frac{\varrho_e}{\varrho_i}U_e &= V_{\kappa_i}t_i + (\frac{1}{2}I - K_{\kappa_i})u_i - \frac{\varrho_e}{\varrho_i}\left[-V_{\kappa_e}t_i + \frac{\varrho_i}{\varrho_e}(\frac{1}{2}I + K_{\kappa_e})u_i\right] \\ &= \left[V_{\kappa_i} + \frac{\varrho_e}{\varrho_i}V_{\kappa_e}\right]t_i - \left[K_{\kappa_i} + K_{\kappa_e}\right]u_i = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial n_i}U_i - \frac{\partial}{\partial n_i}U_e &= (\frac{1}{2}I + K'_{\kappa_i})t_i + D_{\kappa_i}u_i - \left[(\frac{1}{2}I - K'_{\kappa_e})t_i - \frac{\varrho_i}{\varrho_e}D_{\kappa_e}u_i\right] \\ &= \left[K'_{\kappa_i} + K'_{\kappa_e}\right]t_i + \left[D_{\kappa_i} + \frac{\varrho_i}{\varrho_e}D_{\kappa_e}\right]u_i = 0, \end{aligned}$$

i.e., (U_i, U_e) is a solution of the homogeneous transmission problem

$$-\Delta U_i - \kappa_i^2 U_i = 0 \quad \text{in } \Omega, \quad -\Delta U_e - \kappa_e^2 U_e = 0 \quad \text{in } \Omega^c$$

with the transmission conditions

$$\varrho_i U_i = \varrho_e U_e, \quad \frac{\partial}{\partial n_i} U_i = \frac{\partial}{\partial n_i} U_e \quad \text{on } \Gamma.$$

Since U_e also satisfies the radiation condition (2.3), we can apply Theorem 4.1 to conclude $U_i \equiv 0$ in Ω and $U_e \equiv 0$ in Ω^c. From this,

$$V_{\kappa_i} t_i + (\frac{1}{2} I - K_{\kappa_i}) u_i = 0, \quad (\frac{1}{2} I + K'_{\kappa_i}) t_i + D_{\kappa_i} u_i = 0 \quad \text{on } \Gamma \tag{4.10}$$

follows. Next we define

$$\widetilde{U}_e(x) = -\int_\Gamma U^*_{\kappa_i}(x,y) t_i(y) ds_y + \int_\Gamma \frac{\partial}{\partial n_y} U^*_{\kappa_i}(x,y) u_i(y) ds_y \quad \text{for } x \in \Omega^c$$

which is a solution of the exterior Helmholtz equation

$$-\Delta \widetilde{U}_e(x) - \kappa_i^2 \widetilde{U}_e(x) = 0 \quad \text{for } x \in \Omega^c,$$

and with the Cauchy data for $x \in \Gamma$, by using (4.10),

$$\widetilde{U}_e(x) = -(V_{\kappa_i} t_i)(x) + (\frac{1}{2} I + K_{\kappa_i}) u_i(x) = u_i(x),$$

$$\frac{\partial}{\partial n_i} \widetilde{U}_e(x) = (\frac{1}{2} I - K'_{\kappa_i}) t_i(x) - (D_{\kappa_i} u_i)(x) = t_i(x).$$

Hence, Green's formula for the exterior Helmholtz equation reads

$$\int_{\Omega^c} |\nabla \widetilde{U}_e(x)|^2 dx - \kappa_i^2 \int_{\Omega^c} |\widetilde{U}_e(x)|^2 dx = -\int_\Gamma t_i(x) \overline{u_i(x)} ds_x,$$

From the radiation condition (2.3) we conclude

$$\begin{aligned} 0 &= \lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial}{\partial n_x} \widetilde{U}_e(x) - i\kappa_i \widetilde{U}_e(x) \right|^2 ds_x \\ &= \lim_{r\to\infty} \left\{ \int_{|x|=r} \left| \frac{\partial}{\partial n_x} \widetilde{U}_e(x) \right|^2 ds_x + \kappa_i^2 \int_{|x|=r} \left| \widetilde{U}_e(x) \right|^2 ds_x \right. \\ &\qquad \left. -2\kappa_i \Im \left(\int_{|x|=r} \frac{\partial}{\partial n_x} \widetilde{U}_e(x) \overline{\widetilde{U}_e(x)} ds_x \right) \right\}, \end{aligned}$$

and using Green's formula for the bounded exterior domain $\Omega^c \cap B_r$ we further obtain

$$\Im \left(\int_\Gamma t_i(x) \overline{u_i(x)} ds_x \right) = \Im \left(\int_{|x|=r} \frac{\partial}{\partial n_x} \widetilde{U}_e(x) \overline{\widetilde{U}_e(x)} ds_x \right) = 0.$$

Hence we have

$$\lim_{r\to\infty}\int_{|x|=r}|\widetilde{U}_e(x)|^2 ds_x = 0,$$

and by the Rellich lemma, e.g., [8, Lemma 3.11], $\widetilde{U}_e \equiv 0$ in Ω^c follows. From this we further conclude $u_i = \gamma_0^{\text{ext}}\widetilde{U}_e = 0$, and therefore, by using (4.10),

$$V_{\kappa_i}t_i = 0, \quad (\frac{1}{2}I + K'_{\kappa_i})t_i = 0.$$

Since the first relation implies $(\frac{1}{2}I - K'_{\kappa_i})t_i = 0$, also $t_i = 0$ follows. ∎

Recall that coercivity and injectivity of the boundary integral operator system as considered in Lemma 4.4 ensures unique solvability of the boundary integral equation system (4.8) and (4.9), in particular we find $(t_i, u_i) \in H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ as the unique solution of the variational problem

$$\langle (V_{\kappa_i} + \frac{\varrho_e}{\varrho_i}V_{\kappa_e})t_i, \tau\rangle_\Gamma - \langle (K_{\kappa_i} + K_{\kappa_e})u_i, \tau\rangle_\Gamma = \langle \frac{\varrho_e}{\varrho_i}V_{\kappa_e}g + \frac{1}{\varrho_i}(\frac{1}{2}I - K_{\kappa_e})f, \tau\rangle_\Gamma$$

$$\langle (K'_{\kappa_i} + K'_{\kappa_e})t_i, v\rangle_\Gamma + \langle (D_{\kappa_i} + \frac{\varrho_i}{\varrho_e}D_{\kappa_e})u_i, v\rangle_\Gamma = \langle (\frac{1}{2}I + K'_{\kappa_e})g + \frac{1}{\varrho_e}D_{\kappa_e}f, v\rangle_\Gamma$$

for all $(\tau, v) \in H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$. In fact, (4.8) and (4.9) are considered as boundary integral equations of the first kind.

Alternatively, one may consider the formulation of second kind boundary integral equations, where the single layer boundary integral operators V_{κ_i} and V_{κ_e} as well as the hypersingular boundary integral operators D_{κ_i} and D_{κ_e} appear in the coupling terms. For the particular choice

$$\alpha_e = 1, \quad \alpha_i = -1, \quad \beta_i = -1, \quad \beta_e = \frac{\varrho_e}{\varrho_i}$$

we conclude the system of boundary integral equations

$$\Big[V_{\kappa_e} - V_{\kappa_i}\Big]t_i + \frac{1}{2}\left(\frac{\varrho_i}{\varrho_e} + 1\right)u_i - \left[\frac{\varrho_i}{\varrho_e}K_{\kappa_e} - K_{\kappa_i}\right]u_i = \frac{1}{\varrho_e}(-\frac{1}{2}I + K_{\kappa_e})f - V_{\kappa_e}g, \tag{4.11}$$

$$[D_{\kappa_e} - D_{\kappa_i}]\,u_i + \frac{1}{2}\Big(\frac{\varrho_e}{\varrho_i} + 1\Big)t_i + \left[\frac{\varrho_e}{\varrho_i}K'_{\kappa_e} - K'_{\kappa_i}\right]t_i = \frac{\varrho_e}{\varrho_i}\left[(-\frac{1}{2}I - K'_{\kappa_e})g - \frac{1}{\varrho_e}D_{\kappa_e}f\right]. \tag{4.12}$$

A closer look on the difference $V_{\kappa_e} - V_{\kappa_i}$ of the single layer boundary integral operators indicates a reduced order of the singularity involved, i.e.

$$\begin{aligned}(V_{\kappa_e}t_i)(x) - (V_{\kappa_i}t_i)(x) &= \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa_e|x-y|} - e^{i\kappa_i|x-y|}}{|x-y|}t_i(y)ds_y \\ &= \frac{i}{2\pi}\int_\Gamma \frac{e^{i\frac{\kappa_e+\kappa_i}{2}|x-y|}\sin\frac{\kappa_e-\kappa_i}{2}|x-y|}{|x-y|}t_i(y)ds_y.\end{aligned}$$

Since also the difference $D_{\kappa_e} - D_{\kappa_i}$ has a reduced order in its singularity, the system of boundary integral equations (4.11) and (4.12) is known as *minimal coupling formulation*, see [18], i.e., *the minimal coupling formulation involves lower–order singularities in the kernels of the coupling terms than does any other formulation generated from* the combined boundary integral equations.

5 Conclusions

In this work, several boundary integral formulations for the solution of the exterior Dirichlet boundary value problem, and for transmission problems of the Helmholtz equation with piecewise constant wave numbers were presented. Although the focus of this contribution was on the existence and uniqueness of solutions, and the equivalence with the solution of the underlying boundary value and transmission problems, almost all formulations allow for the use of standard arguments, see, e.g. [24, 26], within a stability and error analysis of related Galerkin boundary element methods. Note that it is almost impossible to compare and to rate different boundary integral formulations and different boundary element implementations. The method of choice always depends on several aspects, e.g. the application in mind, the mathematical foundation, the required accuracy, the availability of fast and accurate boundary element implementations including parallel and preconditioned iterative solution strategies, the use of a posteriori error estimators and adaptivity, just to name a few. Although there are already some numerical studies around for a comparison of different boundary element approaches, in particular when considering exterior boundary value problems, it seems that more work is required to state a fair comparison of different boundary integral formulations and related boundary element implementations.

The focus of future work will be in the numerical analysis of stable, robust, and efficient boundary element domain decomposition methods for the solution of transmission problems, including multiple trace formulations [7] and tearing and interconnecting methods [30, 31]. In addition to local and global preconditioning strategies, which are required to be robust with respect to local wave numbers, the handling of adaptive and non–matching discretisations becomes more challenging. The latter also includes the coupling with conforming and non–standard finite element methods, including mixed finite elements, discontinuous Galerkin, and Mortar finite elements.

While the current considerations of this contribution were restricted to the case of acoustic scattering problems, almost all approaches and methodologies as presented here can be carried over to the case of electromagnetic scattering problems, see, e.g. the discussion in [7], in [29, 33], and the references given therein.

References

[1] W. Benthien and A. Schenck, Nonexistence and nonuniqueness problems associated with integral equation methods in acoustics. *Comput. & Structure* **65** (1997), 295–305.

[2] H. Brakhage and P. Werner, Über das Dirichletsche Aussenraumproblem für die Hemholtzsche Schwingungsgleichung. *Arch. Math.* **16**, (1965) 325–329.

[3] A. Buffa and R. Hiptmair, Coercive combined field integral equations. *J. Numer. Math.* **11** (2003), 115–133.

[4] A. Buffa and R. Hiptmair, Regularized combined field integral equations. *Numer. Math.* **100** (2005), 1–19.

[5] A. J. Burton and G. F. Miller, The application of integral equation methods to the numerical solution of some exterior boundary value problems. *Proc. R. Soc. Lond. Ser. A* **323** (1971), 201–210.

[6] S. N. Chandler–Wilde, I. G. Graham, S. Langdon and E. A. Spence, Numerical–asymptotic boundary integral methods in high–frequency acoustic scattering. *Acta Numer.* **21** (2012), 89–305.

[7] X. Claeys, R. Hiptmair and C. Jerez–Hanckes, Multi–trace boundary integral equations. Research Report 2012–20, Seminar für Angewandte Mathematik, ETH Zürich.

[8] D. Colton and R. Kress, Integral Equation Methods in Scattering Theory. John Wiley & Sons, Chichester, 1983.

[9] M. Costabel, Boundary integral operators on Lipschitz domains: Elementary results. *SIAM J. Math. Anal.* **19** (1988), 613–626.

[10] S. Engleder and O. Steinbach, Modified boundary integral formulations for the Helmholtz equation. *J. Math. Anal. Appl.* **331** (2007), 396–407.

[11] S. Engleder and O. Steinbach, Stabilized boundary element methods for exterior Helmholtz problems. *Numer. Math.* **110** (2008), 145–160.

[12] R. Hiptmair, C. Jerez–Hanckes, Multiple traces boundary integral formulation for Helmholtz transmission problems. *Adv. Comput. Math.* **37** (2012), 39–91.

[13] G. C. Hsiao and W. L. Wendland, Boundary Integral Equations. Applied Mathematical Sciences, vol. 164. Springer, Berlin, 2008.

[14] R. Kress, G. F. Roach, Transmission problems for the Helmholtz equation. *J. Math. Phys.* **19** (1978), 1433–1437.

[15] T.–C. Lin, A proof for the Burton and Miller integral equation approach for the Helmholtz equation. *J. Math. Anal. Appl.* **103** (1984), 565–574.

[16] A. W. Maue, Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung. *Z. Phys.* **126** (1949), 601–618.

[17] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[18] K. M. Mitzner, Acoustic scattering from an interface between media of greatly different density. *J. Math. Phys.* **7** (1966), 2053–2060.

[19] J.–C. Nédélec, Integral equations with nonintegrable kernels. *Integral Equations Operator Theory* **5** (1982), 562–572.

[20] J.–C. Nédélec, Acoustic and electromagnetic equations. Applied Mathematical Sciences, vol. 144, Springer, New York, 2001.

[21] G. Of and O. Steinbach, A fast multipole boundary element method for a modified hypersingular boundary integral equation. In: Analysis and Simulation of Multifield Problems (W. L. Wendland, M. Efendiev eds.). Lecture Notes in Applied and Computational Mechanics, vol. 12, Springer, Heidelberg, pp. 163–169, 2003.

[22] O. I. Panič, On the solvability of exterior boundary-value problems for the wave equation and for a system of Maxwell's equations. *Uspehi Mat. Nauk* **20** (1965), 221–226.

[23] T. von Petersdorff, Boundary integral equations for mixed Dirichlet, Neumann and transmission problems. *Math. Meth. Appl. Sci.* **11** (1989), 185–213.

[24] S. A. Sauter and C. Schwab, Boundary Element Methods. Springer Series in Computational Mathematics, vol. 39, Springer, Berlin, 2011.

[25] H. A. Schenck, Improved integral formulation for acoustic radiation problems. *J. Acoust. Soc. Am.* **44** (1968), 41–58.

[26] O. Steinbach, Numerical Approximation Methods for Elliptic Boundary Value Problems. Finite and Boundary Elements. Springer, New York, 2008.

[27] O. Steinbach and G. Unger, Convergence analysis of a Galerkin boundary element method for the Dirichlet Laplacian eigenvalue problem. *S*IAM J. Numer. Anal. **50** (2012), 710–728.

[28] O. Steinbach and W. L. Wendland, On C. Neumann's method for second–order elliptic systems in domains with non–smooth boundaries. *J. Math. Anal. Appl.* **262** (2001), 733–748.

[29] O. Steinbach and M. Windisch, Modified combined field integral equations for electromagnetic scattering. *SIAM J. Numer. Anal.* **47** (2009), 1149–1167.

[30] O. Steinbach and M. Windisch, Robust boundary element domain decomposition solvers in acoustics. In: Domain Decomposition Methods in Science and Engineering XIX (Y. Huang, R. Kornhuber, O. B. Widlund, J. Xu eds.). Lecture Notes in

Computational Science and Engineering, vol. 78, Springer, Heidelberg, pp. 277–284, 2011.

[31] O. Steinbach and M. Windisch, Stable boundary element domain decomposition methods for the Helmholtz equation. *Numer. Math.* **118** (2011), 171–195.

[32] G. Unger, Analysis of boundary element methods for Laplacian eigenvalue problems. PhD thesis, Monographic Series TU Graz, Computation in Engineering and Science, vol. 6, 2009.

[33] M. Windisch, Boundary element tearing and interconnecting methods for acoustic and electromagnetic scattering. PhD thesis, Monographic Series TU Graz, Computation in Engineering and Science, vol. 11, 2011.

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