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Michael Reichelt and Olaf Steinbach

Abstract In this note we formulate and analyze a hybrid space-time finite element method for the numerical solution of parabolic evolution equation. We combine the more standard variational formulation in Bochner spaces, and a more recent formulation in anisotropic Sobolev spaces using a modified Hilbert transformation. The Galerkin discretization then results in symmetric and positive definite stiffness matrices for both the temporal and spatial derivatives, and a remainder which is in general non-symmetric, but non-negative. We present related error estimates a series of different numerical examples which confirm the theoretical findings.

1 Introduction

Space-time finite element methods, e.g., [1, 6, 8, 11], are well established for the numerical solution of time-dependent partial differential equations. In the case of parabolic evolution equations, the space-time variational formulation is usually considered in Bochner spaces, and the stability and error analysis is based on a discrete inf-sup stability condition. As an alternative, one may consider the variational formulation in anisotropic Sobolev spaces, see, e.g., [5, 9], and using either the standard or a modified Hilbert transformation, the related bilinear form turns out to be elliptic in suitable spaces. While in the Bochner space setting the space-time finite element discretization of the spatial part results in a symmetric and positive definite stiffness matrix, the discretization of the first order time derivative becomes symmetric and positive definite when using the modified Hilbert transformation [9]. In this note we propose and analyze a hybrid formulation of both approaches which results in symmetric and positive definite stiffness matrices for both the temporal and spatial

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differential operators, and a non-negative remainder. Numerical results for all three formulations are given to confirm the theoretical findings.

2 Space-time variational formulations

As a model problem for a parabolic evolution equation, we consider the Dirichlet boundary value problem for the heat equation,

$$\partial_t u - \Delta_x u = f \text{ in } Q := \Omega \times (0,T), \; u = 0 \text{ on } \Sigma := \partial\Omega \times (0,T), \; u(0) = 0 \text{ in } \Omega, \quad (1)$$

where $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$, is a bounded Lipschitz domain. The abstract space-time variational formulation of (1) reads to find $u \in X$ such that

$$b(u,v) := \int_0^T \int_\Omega \Big[\partial_t u \, v + \nabla_x u \cdot \nabla_x v \Big] \, dx \, dt = \int_0^T \int_\Omega f \, v \, dx \, dt \qquad (2)$$

is satisfied for all $v \in Y$, where X and Y are appropriate Hilbert spaces to be fixed. As in [8] we can consider the Bochner spaces $X = L^2(0,T; H^1_0(\Omega)) \cap H^1_{0,}(0,T; H^{-1}(\Omega))$ and $Y = L^2(0,T; H^1_0(\Omega))$ with the associated norms

$$\|v\|_Y = \|\nabla_x v\|_{L^2(Q)}, \quad \|u\|_X = \sqrt{\|u\|_Y^2 + \|\partial_t u\|^2_{Y^*}}, \quad \|\partial_t u\|_{Y^*} = \|w_u\|_Y,$$

where $w_u \in Y$ is the unique solution of the variational formulation

$$\int_0^T \int_\Omega \nabla_x w_u \cdot \nabla_x v \, dx \, dt = \langle \partial_t u, v \rangle_Q \quad \text{for all } v \in Y. \qquad (3)$$

In this case, unique solvability of (2) is based on the inf-sup stability condition, see [8, Theorem 2.1], and [3] for an improved estimate,

$$\|u\|_X \leq \sup_{0 \neq v \in Y} \frac{b(u,v)}{\|v\|_Y} \quad \text{for all } u \in X. \qquad (4)$$

For the Galerkin discretization of the variational formulation (2) we introduce a conforming space-time finite element space $X_h = \text{span}\{\varphi_k\}_{k=1}^M \subset X$ of piecewise (multi-)linear and continuous basis functions φ_k which are defined with respect to some admissible decomposition of Q into shape-regular simplicial or tensor-product finite elements q_ℓ of local mesh size h_ℓ. As in [8], chosing $Y_h = X_h$, we then consider the space-time Galerkin variational formulation to find $u_h^1 \in X_h$ such that

$$\int_0^T \int_\Omega \Big[\partial_t u_h^1 \, v_h + \nabla_x u_h^1 \cdot \nabla_x v_h \Big] \, dx \, dt = \int_0^T \int_\Omega f \, v_h \, dx \, dt \quad \text{for all } v_h \in X_h. \quad (5)$$

The stability and error analysis of (5) is based on a discrete inf-sup stability condition with respect to a discrete norm,

$$\|u\|_{X,h} = \sqrt{\|u\|_Y^2 + \|w_{u,h}\|_Y^2} \leq \sqrt{\|u\|_Y^2 + \|w_u\|_Y^2} = \|u\|_X,$$

where $w_{u,h} \in Y_h$ is the unique solution of the Galerkin variational formulation

$$\int_0^T \int_\Omega \nabla_x w_{u,h} \cdot \nabla_x v_h \, dx\, dt = \langle \partial_t u, v_h \rangle_Q \quad \text{for all } v_h \in Y_h.$$

With this, and assuming $u \in H^s(Q)$ for some $s \in [1,2]$, we are able to derive an error estimate for the spatial part $\|\nabla_x (u - u_h^1)\|_{L^2(Q)}$, see [8, Corollary 3.4], but the error for the temporal part is only measured in the discrete norm, which in general is not equivalent to the full norm. Due to the anisotropy in the temporal and spatial derivatives in the heat equation, and due to the maximal parabolic regularity $u \in H^{2,1}(Q)$ for the solution of the heat equation with $f \in L^2(Q)$, one can also derive linear convergence in the energy norm in this case [2], when assuming a space-time tensor product discretization, see also [7].

As an alternative to the Bochner space setting as described above, and as in [9], we may also consider the variational formulation (2) in the anisotropic Sobolev spaces $X = H^{1,1/2}_{0;0,}(Q) := L^2(0,T; H^1_0(\Omega)) \cap H^{1/2}_{0,}(0,T; L^2(\Omega))$, and $Y = H^{1,1/2}_{0;,0}(Q)$. Unique solvability of (2) is now based on the inf-sup stability condition [9, Corollary 3.3]

$$\frac{1}{2}\, \|u\|_{H^{1,1/2}_{0;0,}(Q)} \leq \sup_{0 \neq v \in H^{1,1/2}_{0;,0}(Q)} \frac{\langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)}}{\|v\|_{H^{1,1/2}_{0;,0}(Q)}}, \; u \in H^{1,1/2}_{0;0,}(Q).$$

When using the finite element space $X_h = \operatorname{span}\{\varphi_k\}_{k=1}^M \subset X$ as in the first approach, and using the modified Hilbert transformation $\mathcal{H}_T : H^{1,1/2}_{0;0,}(Q) \to H^{1,1/2}_{0;,0}(Q)$, see [9], this results in a space-time Galerkin–Bubnov variational formulation to find $u_h^2 \in X_h$ such that

$$\langle \partial_t u_h^2, \mathcal{H}_T v_h \rangle_{L^2(Q)} + \langle \nabla_x u_h^2, \nabla_x \mathcal{H}_T v_h \rangle_{L^2(Q)} = \langle f, \mathcal{H}_T v_h \rangle_Q \quad \text{for all } v_h \in X_h. \tag{6}$$

While stability of the space-time finite element scheme (6) follows for any choice of the conforming finite element space $X_h \subset X$, related error estimates are given in [9] only in the case of a space-time tensor product discretization. This is due to the fact that $\langle \partial_t u, \mathcal{H}_T u \rangle_Q$ defines a norm in $H^{1/2}_{0,}(Q)$, but the non-negative spatial part $\langle \nabla_x u, \mathcal{H}_T \nabla_x u \rangle_Q \geq 0$ does not define a norm in $L^2(0,T; H^1_0(\Omega))$. However, when assuming sufficient regularity for the solution, e.g., $u \in H^2(Q)$, we can conclude optimal convergence on tensor product meshes, see [9, Theorem 3.4].

3 A hybrid space-time finite element method

The aim of this section is the formulation and numerical analysis of a space-time finite element method which ensures the control of both temporal and spatial derivatives, which is not restricted to space-time tensor product meshes, but also allows the use of adaptive decompositions into simplicial elements. In some sense, this is the counter part of the approach in [5] where the classical Hilbert transformation was used with respect to the infinite time interval $[0,\infty)$. Here we assume $u \in H^s(Q)$ for some $s \in [1,2]$ only, and we will not consider the regularity in anisotropic Sobolev spaces. We start to consider the variational formulation (2) in the Bochner spaces $X = L^2(0,T;H^1_0(\Omega)) \cap H^1_{0,}(0,T;H^{-1}(\Omega))$ and $Y = L^2(0,T;H^1_0(\Omega))$. As before we use $X_h = \text{span}\{\varphi_k\}_{k=1}^M \subset X$, but we now define $Y_h = \text{span}\{\varphi_k + \mathcal{H}_T\varphi_k\}_{k=1}^M \subset Y$ which covers neither zero initial nor terminal conditions. Hence, $\varphi_k + \mathcal{H}_T\varphi_k \in H^{1,1/2}_{0;,}(Q)$ only. In particular for $s \in [0,\frac{1}{2})$, the spaces $H^{1,s}_{0;0,}(Q)$, $H^{1,s}_{0;,0}(Q)$, and $H^{1,s}_{0;,}(Q)$ coincide. Now we consider the variational formulation to find $u_h \in X_h$ such that

$$\langle \partial_t u_h, v_h + \mathcal{H}_T v_h\rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x(v_h + \mathcal{H}_T v_h)\rangle_{L^2(Q)} = \langle f, v_h + \mathcal{H}_T v_h\rangle_Q \tag{7}$$

is satisfied for all $v_h \in X_h$. In fact, the variational formulation (7) is the sum of the variational formulations (5) and (6). Since each of these variational formulations is uniquely solvable, unique solvability of (7) follows.

Theorem 1 *The bilinear form*

$$b_{hybrid}(u_h, v_h) := \langle \partial_t u_h, v_h + \mathcal{H}_T v_h\rangle_{L^2(Q)} + \langle \nabla_x u_h, \nabla_x(v_h + \mathcal{H}_T v_h)\rangle_{L^2(Q)}$$

is elliptic, satisfying

$$b_{hybrid}(v_h, v_h) \geq \|v_h\|^2_{H^{1,1/2}_{0;0,}(Q)} \quad \text{for all } v_h \in X_h \subset H^{1,1/2}_{0;0,}(Q). \tag{8}$$

Proof For $u_h = v_h \in X_h$ we have

$$\begin{aligned} b_{\text{hybrid}}(v_h, v_h) &= \langle \partial_t v_h, v_h + \mathcal{H}_T v_h\rangle_{L^2(Q)} + \langle \nabla_x v_h, \nabla_x(v_h + \mathcal{H}_T v_h)\rangle_{L^2(Q)} \\ &= \frac{1}{2}\int_0^T \frac{d}{dt}\int_\Omega [v_h(x,t)]^2\, dx\, dt + \langle \partial_t v_h, \mathcal{H}_T v_h\rangle_{L^2(Q)} \\ &\qquad + \langle \nabla_x v_h, \nabla_x v_h\rangle_{L^2(Q)} + \langle \nabla_x v_h, \mathcal{H}_T \nabla_x v_h\rangle_{L^2(Q)} \\ &\geq \|v_h\|^2_{H^{1/2}_{0;0,}(Q)} + \|\nabla_x v_h\|^2_{L^2(Q)} = \|v_h\|^2_{H^{1,1/2}_{0;0,}(Q)}, \end{aligned}$$

i.e., the assertion. □

For any $\phi \in H^{1,1/2}_{0;0,}(Q)$ we define the Galerkin projection $\phi_h = G_h\phi \in X_h$ as unique solution of the variational formulation

$$b_{\text{hybrid}}(\phi_h, v_h) = b_{\text{hybrid}}(\phi, v_h) \quad \text{for all } v_h \in X_h. \tag{9}$$

Lemma 1 *The Galerkin projection* $G_h : H^{1,1/2+\varepsilon}_{0;0,}(Q) \to X_h \subset H^{1,1/2}_{0;0,}(Q)$ *is bounded for all* $\varepsilon \in (0, \frac{1}{2}]$, *satisfying*

$$\|G_h \phi\|_{H^{1,1/2}_{0;0,}(Q)} \leq 2 \ \|\phi\|_{H^{1,1/2+\varepsilon}_{0;0,}(Q)} \quad \textit{for all } \phi \in H^{1,1/2+\varepsilon}_{0;0,}(Q). \tag{10}$$

Proof With the ellipticity estimate (8), using duality for $\phi_h + \mathcal{H}_T \phi_h \notin H^{1,1/2}_{0;,0}(Q)$, Hölders inequality, and

$$H^{1/2}_{0,}(0,T; L^2(\Omega)) \subset H^{1/2-\varepsilon}_{0,}(0,T; L^2(\Omega)) = H^{1/2-\varepsilon}_{,0}(0,T; L^2(\Omega)),$$

we conclude, for $\phi_h = G_h \phi$,

$$\begin{aligned}
&\|\phi_h\|^2_{H^{1,1/2}_{0;0,}(Q)} \leq b_{\text{hybrid}}(\phi_h, \phi_h) \ = \ b_{\text{hybrid}}(\phi, \phi_h) \\
&\quad = \ \langle \partial_t \phi, \phi_h + \mathcal{H}_T \phi_h \rangle_Q + \langle \nabla_x \phi, \nabla_x (\phi_h + \mathcal{H}_T \phi_h) \rangle_{L^2(Q)} \\
&\quad \leq \ \|\partial_t \phi\|_{[H^{1/2-\varepsilon}_{,0}(0,T;L^2(\Omega))]^*} \|\phi_h + \mathcal{H}_T \phi_h\|_{H^{1/2-\varepsilon}_{,0}(0,T;L^2(\Omega))} \\
&\qquad\qquad + \|\nabla_x \phi\|_{L^2(Q)} \|\nabla_x (\phi_h + \mathcal{H}_T \phi_h)\|_{L^2(Q)} \\
&\quad \leq \ 2 \left[\|\phi\|_{H^{1/2+\varepsilon}_{0,}(0,T;L^2(\Omega))} \|\phi_h\|_{H^{1/2-\varepsilon}_{,0}(0,T;L^2(\Omega))} + \|\nabla_x \phi\|_{L^2(Q)} \|\nabla_x \phi_h\|_{L^2(Q)} \right] \\
&\quad \leq \ 2 \sqrt{\|\phi\|^2_{H^{1/2+\varepsilon}_{0,}(0,T;L^2(\Omega))} + \|\nabla_x \phi\|^2_{L^2(Q)}} \sqrt{\|\phi_h\|^2_{H^{1/2-\varepsilon}_{,0}(0,T;L^2(\Omega))} + \|\nabla_x \phi_h\|^2_{L^2(Q)}} \\
&\quad \leq \ 2 \, \|\phi\|_{H^{1,1/2+\varepsilon}_{0;0,}(Q)} \|\phi_h\|_{H^{1,1/2}_{0;0,}(Q)},
\end{aligned}$$

i.e., the assertion. □

With the projection property $v_h = G_h v_h$ for all $v_h \in X_h$ and (10) we further conclude Cea's lemma,

$$\begin{aligned}
\|u - u_h\|_{H^{1,1/2}_{0;0,}(Q)} &= \|u - G_h u\|_{H^{1,1/2}_{0;0,}(Q)} \ = \ \|u - v_h + G_h v_h - G_h u\|_{H^{1,1/2}_{0;0,}(Q)} \\
&\leq \|u - v_h\|_{H^{1,1/2}_{0;0,}(Q)} + \|G_h (v_h - u)\|_{H^{1,1/2}_{0;0,}(Q)} \\
&\leq \|u - v_h\|_{H^{1,1/2}_{0;0,}(Q)} + 2 \, \|u - v_h\|_{H^{1,1/2+\varepsilon}(Q)} \\
&\leq 3 \, \|u - v_h\|_{H^{1,1/2+\varepsilon}(Q)} \quad \text{for all } v_h \in X_h.
\end{aligned} \tag{11}$$

When combining (11) with the approximation property of piecewise (multi-)linear basis functions in $H^1(Q)$ we can formulate the main result of this paper.

Theorem 2 *Let* $u \in H^{1,1/2}_{0;0,}(Q) \cap H^s(Q)$ *for some* $s \in [1, 2]$ *be the unique solution of* (1)*, and let* $u_h \in X_h$ *be the unique solution of the hybrid variational formulation* (7)*. Then there holds the error estimate*

$$\|u - u_h\|_{H^{1,1/2}_{0;0,}(Q)} \leq c \, h^{s-1} \, |u|_{H^s(Q)}. \tag{12}$$

Proof The assertion follows from (11),

$$\|u-u_h\|_{H^{1,1/2}_{0;0,}(Q)} \le 3 \inf_{v_h\in X_h} \|u-v_h\|_{H^{1,1/2+\varepsilon}_{0;0,}(Q)} \le 3 \inf_{v_h\in X_h} \|u-v_h\|_{H^{1,1}_{0;0,}(Q)},$$

and the approximation property of X_h in $H^1(Q)$. □

In the case of tensor-product finite element spaces it is possible to improve the error estimate (12) when using suitable projection operators with respect to spatial and temporal components separately. This also allows to consider less regular solutions $u \in H^{2,1}(Q)$ when assuming $f \in L^2(Q)$ and convex spatial domains Ω, which then requires the parabolic scaling $h_t = h_x^2$ for the temporal and spatial mesh sizes h_t and h_x, respectively. For a more detailed discussion, we refer to [2], and [7].

4 Numerical results

The variational formulation (5) in the Bochner space setting is equivalent to a linear system of algebraic equations,

$$[\widetilde{A}_h + K_h]\underline{u}^1 = \underline{f}^1, \tag{13}$$

where

$$K_h[j,k] = \int_Q \nabla_x\varphi_k \cdot \nabla_x\varphi_j \, dx\, dt, \quad \widetilde{A}_h[j,k] = \int_Q \partial_t\varphi_k \, \varphi_j \, dx\, dt,$$

for $k, j = 1, \dots, M$, and $f_j^1 = \langle f, \varphi_j\rangle_Q$. Note that the stiffness matrix K_h is symmetric and positive definite, and $\widetilde{A}$ is non-symmetric, and non-negative. In contrast, the variational formulation (6) in the anisotropic Sobolev space results in the linear system

$$[A_h + \widetilde{K}_h]\underline{u}^2 = \underline{f}^2, \tag{14}$$

where

$$A_h[j,k] = \int_Q \partial_t\varphi_k \, \mathcal{H}_T\varphi_j \, dx\, dt, \quad \widetilde{K}_h[j,k] = \int_Q \nabla_x\varphi_k \cdot \nabla_x\mathcal{H}_T\varphi_j \, dx\, dt,$$

for $k, j = 1, \dots, M$, and $f_j^2 = \langle f, \mathcal{H}_T\varphi_j\rangle_Q$. Now the stiffness matrix A_h is symmetric and positive definite, and $\widetilde{K}_h$ is non-sysmmetric, and non-negative. The hybrid space-time variational formulation (7) then results in the linear system

$$[A_h + K_h + \widetilde{A}_h + \widetilde{K}_h]\underline{u} = \underline{f}^1 + \underline{f}^2 =: \underline{f}, \tag{15}$$

where $A_h + K_h$ is symmetric and positive definite, and $\widetilde{A}_h + \widetilde{K}_h$ is non-symmetric, but non-negative. Note that the representation (15) holds true for discretizations using either simplicial or tensor-product meshes. To simplify the implementation [10] of the modified Hilbert transformation $\mathcal{H}_T$, at this time we only consider the discretization using space-time tensor-product meshes with n_x quadrilaterals in space, and n_t elements in time. Recall that X_h is the space-time finite element space of multilinear and continuous basis functions. In the case of space-time tensor product meshes it is also possible to construct efficient direct solvers for the solution of the linear system (15), see [4].

In what follows we will consider numerical examples for a regular solution $u_1 \in H^2(Q)$ as used in [1], a solution u_2 with a singularity in space, and u_3 with a singularity in time,

$$u_1(x,t) = \sin(10\pi t)\sin(\pi x_1)\sin(\pi x_2),$$
$$u_2(x,t) = t^{3/2}\,(x_1(1-x_1))^{3/2}\sin(\pi x_2),$$
$$u_3(x,t) = t^{2/3}\sin(\pi x_1)\sin(\pi x_2).$$

In all cases we consider $\Omega = (0,1)^2$ and $T = 1$, i.e., $Q = (0,1)^3$. The observed errors $\|\nabla_x(u_i - u_{i,h})\|_{L^2(Q)}$ as well as the experimental order of convergence (eoc) are listed in Tables 1, 2, and 3, respectively. The respective L^2 errors $\|u_i - u_{i,h}\|_{L^2(Q)}$ only slightly differ for the different formulations, and hence are only presented for the hybrid formulation in Table 4, where we see a breakdown of the convergence order for the two singular solutions. In Table 5 it is visible that in the case of the temporal singularity the optimal order of convergence can be retrieved by using parabolic scaling, i.e. $n_t = n_x^2$.

Table 1 Convergence of the finite element method for the regular solution $u_1(x,t)$ in the modified Hilbert transform, the Bochner, and the hybrid setting.

DoF	$\|\nabla_x(u_1 - u_{1,h}^{\mathcal{H}})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_1 - u_{1,h}^{B})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_1 - u_{1,h}^{hyb})\|_{L^2(Q)}$	eoc
44	$1.57 \cdot 10^{0}$		$1.57 \cdot 10^{0}$		$1.57 \cdot 10^{0}$	
189	$7.85 \cdot 10^{-1}$	1.00	$7.40 \cdot 10^{-1}$	1.09	$7.62 \cdot 10^{-1}$	1.04
1025	$3.63 \cdot 10^{-1}$	1.11	$3.59 \cdot 10^{-1}$	1.04	$3.62 \cdot 10^{-1}$	1.07
6561	$1.79 \cdot 10^{-1}$	1.02	$1.78 \cdot 10^{-1}$	1.01	$1.79 \cdot 10^{-1}$	1.02
46529	$8.91 \cdot 10^{-2}$	1.00	$8.91 \cdot 10^{-2}$	1.00	$8.91 \cdot 10^{-2}$	1.00
349569	$4.45 \cdot 10^{-2}$	1.00	$4.45 \cdot 10^{-2}$	1.00	$4.45 \cdot 10^{-2}$	1.00

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Table 2 Convergence of the finite element method for the singular solution $u_2(x,t)$ in the modified Hilbert transform, the Bochner and the hybrid setting.

DoF	$\|\nabla_x(u_2 - u_{2,h}^{\mathcal{H}})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_2 - u_{2,h}^{B})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_2 - u_{2,h}^{hyb})\|_{L^2(Q)}$	eoc
44	$1.36 \cdot 10^{-1}$		$1.36 \cdot 10^{-1}$		$1.36 \cdot 10^{-1}$	
189	$5.68 \cdot 10^{-2}$	1.27	$5.68 \cdot 10^{-2}$	1.27	$5.68 \cdot 10^{-2}$	1.27
1025	$3.55 \cdot 10^{-2}$	0.68	$3.55 \cdot 10^{-2}$	0.68	$3.55 \cdot 10^{-2}$	0.68
6561	$2.12 \cdot 10^{-2}$	0.75	$2.12 \cdot 10^{-2}$	0.75	$2.12 \cdot 10^{-2}$	0.75
46529	$1.20 \cdot 10^{-2}$	0.81	$1.20 \cdot 10^{-2}$	0.81	$1.20 \cdot 10^{-2}$	0.81
349569	$6.66 \cdot 10^{-3}$	0.85	$6.66 \cdot 10^{-3}$	0.85	$6.66 \cdot 10^{-3}$	0.85

Table 3 Convergence of the finite element method for the singular solution $u_3(x,t)$ in the modified Hilbert transform, the Bochner and the hybrid setting.

DoF	$\|\nabla_x(u_3 - u_{3,h}^{\mathcal{H}})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_3 - u_{3,h}^{B})\|_{L^2(Q)}$	eoc	$\|\nabla_x(u_3 - u_{3,h}^{hyb})\|_{L^2(Q)}$	eoc
44	$1.45 \cdot 10^{0}$		$1.45 \cdot 10^{0}$		$1.45 \cdot 10^{0}$	
189	$6.53 \cdot 10^{-1}$	1.15	$6.53 \cdot 10^{-1}$	1.15	$6.53 \cdot 10^{-1}$	1.15
1025	$3.28 \cdot 10^{-1}$	0.99	$3.28 \cdot 10^{-1}$	0.99	$3.28 \cdot 10^{-1}$	0.99
6561	$1.65 \cdot 10^{-1}$	1.00	$1.65 \cdot 10^{-1}$	1.00	$1.65 \cdot 10^{-1}$	1.00
46529	$8.24 \cdot 10^{-2}$	1.00	$8.24 \cdot 10^{-2}$	1.00	$8.24 \cdot 10^{-2}$	1.00
349569	$4.12 \cdot 10^{-2}$	1.00	$4.12 \cdot 10^{-2}$	1.00	$4.12 \cdot 10^{-2}$	1.00

Table 4 L^2 convergence for the different solutions in the hybrid setting.

DoF	$\|u_1 - u_{1,h}^{hyb}\|_{L^2(Q)}$	eoc	$\|u_2 - u_{2,h}^{hyb}\|_{L^2(Q)}$	eoc	$\|u_3 - u_{3,h}^{hyb}\|_{L^2(Q)}$	eoc
44	$3.53 \cdot 10^{-1}$		$3.57 \cdot 10^{-3}$		$3.27 \cdot 10^{-1}$	
189	$9.10 \cdot 10^{-2}$	1.96	$6.48 \cdot 10^{-4}$	2.46	$7.59 \cdot 10^{-2}$	2.11
1025	$2.15 \cdot 10^{-2}$	2.08	$3.22 \cdot 10^{-4}$	1.01	$1.91 \cdot 10^{-2}$	1.99
6561	$5.29 \cdot 10^{-3}$	2.02	$1.11 \cdot 10^{-4}$	1.54	$4.83 \cdot 10^{-3}$	1.98
46529	$1.32 \cdot 10^{-3}$	2.00	$3.36 \cdot 10^{-5}$	1.72	$1.28 \cdot 10^{-3}$	1.91
349569	$3.31 \cdot 10^{-4}$	2.00	$9.64 \cdot 10^{-6}$	1.80	$4.20 \cdot 10^{-4}$	1.61

Table 5 L^2 convergence for the time singular solution in the hybrid setting using parabolic scaling.

DoF	n_x	n_t	$\|u_3 - u_{3,h}^{hyb}\|_{L^2(Q)}$	eoc
8	1	1	$3.27 \cdot 10^{-1}$	
45	2	4	$7.56 \cdot 10^{-2}$	2.11
425	4	16	$1.91 \cdot 10^{-2}$	1.98
5265	8	64	$4.85 \cdot 10^{-3}$	1.98
74273	16	256	$1.24 \cdot 10^{-3}$	1.96

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