Optimisation of the lowest eigenvalue induced by singular interactions

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TU Graz, Austria, 24.04.2017

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Other boundary conditions

The Neumann Laplacian: similar spectral inequality is trivial: $\lambda_1^{\rm N}(\Omega) = 0$.
Non-trivial for δ-interactions on manifolds and for the Robin Laplacian.

I. Schrödinger operators with δ-interactions on hypersurfaces

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The lowest spectral point for H_α^Σ

$\mu_1^\alpha(\Sigma) := \inf \sigma(\mathsf{H}_\alpha^\Sigma)$.

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(i) 'Leaky' quantum systems: a particle is confined to Σ but the tunneling between different parts of Σ is not neglected.

(ii) Inverse scattering problem for H_α^Σ is linked to the Calderon problem with non-smooth conductivity.

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Spectral geometry

Characterise the spectrum of H_α^Σ in terms of Σ!

- An explicit mapping $\Sigma \mapsto \sigma(\mathsf{H}_\alpha^\Sigma)$ can not be constructed.
- Particular spectral results might be very difficult to obtain.

δ-interactions on loops

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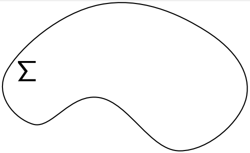

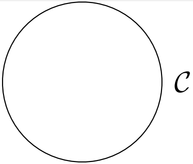

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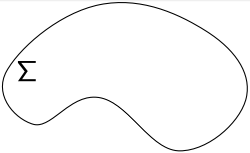

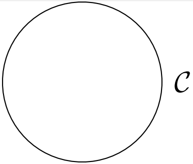

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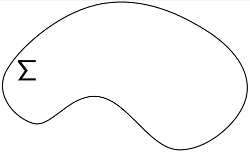

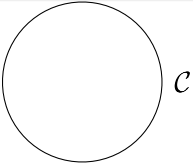

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$$\begin{cases} |\Sigma| = |\mathcal{C}| \\ \Sigma \not\cong \mathcal{C} \end{cases} \quad \Longrightarrow \quad \mu_1^\alpha(\mathcal{C}) > \mu_1^\alpha(\Sigma), \quad \forall\, \alpha > 0.$$

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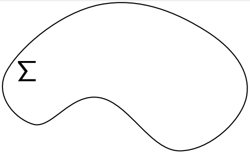

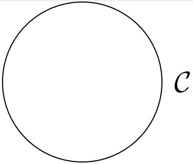

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Operator theory: Birman-Schwinger and min-max principles.

Geometry: mean-chord length inequality (Lükő-66).

Classical analysis: decay and convexity of $K_0(\cdot)$, Jensen's inequality.

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Geometry: line segment – the shortest path between two endpoints.

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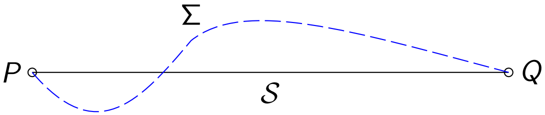

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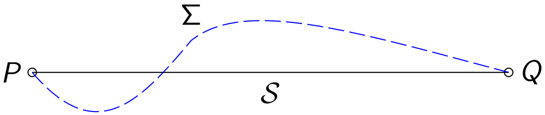

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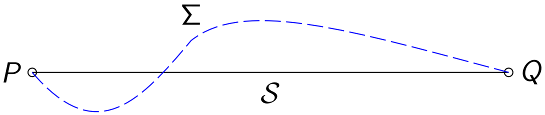

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Open questions

(i) Shape of the optimizer under two constraints: fixed endpoints $P, Q \in \mathbb{R}^2$ and fixed length $L > |P - Q|$?

(ii) A generalization for Laplace-Beltrami operator on a 2-manifold $\mathcal{M}$ with $\mathcal{S}$ being the geodesic connecting $P, Q \in \mathcal{M}$?

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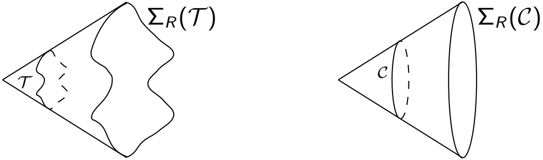

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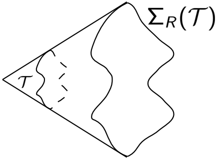

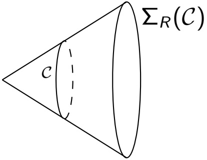

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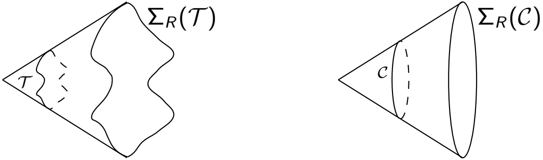

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$$\begin{cases} |\mathcal{C}| = |\mathcal{T}| \\ \mathcal{C} \not\cong \mathcal{T} \end{cases} \quad \Longrightarrow \quad \begin{cases} \mu_1^\alpha(\Sigma_R(\mathcal{C})) > \mu_1^\alpha(\Sigma_R(\mathcal{T})), \quad \forall\, \alpha > \alpha_*(\Sigma_R(\mathcal{C})) \\ \mu_1^\alpha(\Sigma_R(\mathcal{T})) < 0 \ \textit{for}\ \alpha = \alpha_*(\Sigma_R(\mathcal{C})). \end{cases}$$

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Passing in the result for truncated cones to the limit $R \to +\infty$.

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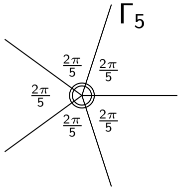

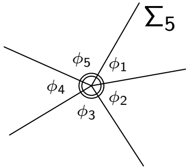

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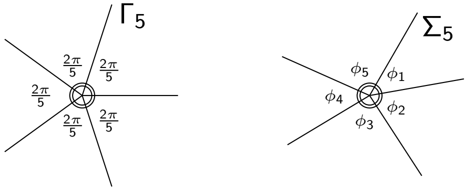

Theorem (Exner-Ichinose-01, Khalile-Pankrashkin-17, Exner-VL-17)

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(ii) $\mu_1^\alpha(\Sigma_N) \leq \mu_1^\alpha(\Gamma_N)$ *for all* $\alpha > 0$ *(*EXNER-VL-17*).*

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Questions

(i) Is the circle a local optimiser under fixed length constraint? Shape derivative of $\mu_1^{\alpha,B}(\Sigma)$ with respect to Σ.

(ii) Is the circle still a global optimiser under fixed length constraint?

(iii) Does the "non-magnetic" strategy of the proof apply?

II. The Robin Laplacian on exterior domains

Definition of the Robin Laplacian

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$G \subset \mathbb{R}^d$ – an unbounded Lipschitz domain with compact boundary ∂G.

- Exterior domain. • Complement of a hypersurface.

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Applications in physics

(i) Oscillating, elastically supported membranes in mechanics.

(ii) Linearized Ginzburg-Landau equation in superconductivity.

(iii) Thin layers with impedance BC condition in electromagnetism.

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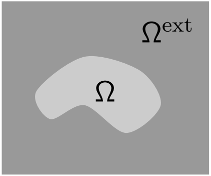

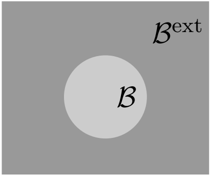

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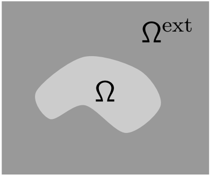

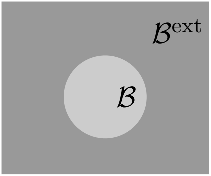

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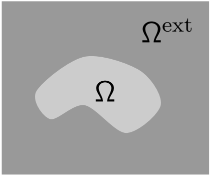

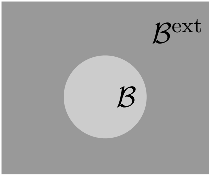

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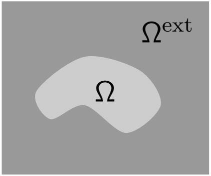

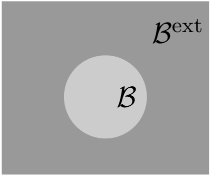

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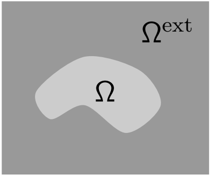

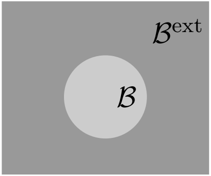

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Non-convex case: joint work in progress with D. Krejčiřík.

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Curvature constraints for $d \geq 3$: joint work in progress with D. Krejčiřík.

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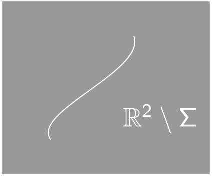

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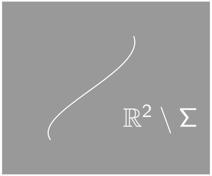

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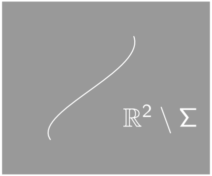

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Robin cones

An analogue of the optimisation result for δ-interactions supported on conical surfaces in the Robin setting.

Thank you

Thank you for your attention!