Eigenvalue inequalities for partial differential operators

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joint work with Jussi Behrndt and Jonathan Rohleder

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Preliminaries

Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ be an open set.

$$\mathcal{L}_i := -\sum_{j,k=1}^{d} \partial_j a_{jk,i} \partial_k + a_i,$$

$a_{jk,i} : \Omega \to \mathbb{C}$, $a_i : \Omega \to \mathbb{R}$, such that $\mathcal{L}_i$ is uniformly elliptic, $i = 1, 2$.

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$a_{jk,i} : \Omega \to \mathbb{C}$, $a_i : \Omega \to \mathbb{R}$, such that $\mathcal{L}_i$ is uniformly elliptic, $i = 1, 2$.

$$A_i u := \mathcal{L}_i u, \quad \mathrm{dom} A_i \subseteq L^2(\Omega), \quad i = 1, 2$$

self-adjoint operators in $L^2(\Omega)$ (for suitably chosen coefficient functions and domains)

Question

$\lambda_1(A_i) \leq \lambda_2(A_i) \leq ... < M$ discrete eigenvalues of A_i, counted with multiplicities, below some upper bound $M \in \mathbb{R} \cup \{\infty\}$

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Question:

Under what conditions does it hold

$$\lambda_n(A_2) \leq \lambda_{n+k}(A_1) \quad \text{or even} \quad \lambda_n(A_2) < \lambda_{n+k}(A_1)$$

for some $k \in \mathbb{N}$ and all $n \in \mathbb{N}$?

History

Results for $\mathcal{L}_i = -\Delta$ on bounded domain Ω subject to Dirichlet or Neumann boundary conditions:

- G. Polya (1952), G. Szegő (1954): $\lambda_2^{\mathcal{N}} < \lambda_1^{\mathcal{D}}$
- L. Payne (1955): $\lambda_2^{\mathcal{N}} \leq \lambda_1^{\mathcal{D}}$ for $\Omega \subseteq \mathbb{R}^2$ convex
- H. Levine & H. Weinberger (1986): $\lambda_{n+d}^{\mathcal{N}} < \lambda_n^{\mathcal{D}}$ for $\Omega \subseteq \mathbb{R}^d$ convex with C^∞-boundary
- L. Friedlander (1991): $\lambda_{n+1}^{\mathcal{N}} \leq \lambda_n^{\mathcal{D}}$ for $\Omega \subseteq \mathbb{R}^d$ with C^1-boundary
- N. Filonov (2005): $\lambda_{n+1}^{\mathcal{N}} < \lambda_n^{\mathcal{D}}$ e.g. for $\Omega \subseteq \mathbb{R}^d$ Lipschitz domain

Assumptions

- $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, unbounded domain with compact Lipschitz boundary (i.e. the complement of $\overline{\Omega}$ is a bounded Lipschitz domain)
- $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta + V$, with $V \in L^\infty(\Omega)$ real valued

Schrödinger operator with Dirichlet boundary condition

Definition

$$A_{\mathcal{D}}u = (-\Delta + V)u$$

$$\mathrm{dom}(A_{\mathcal{D}}) = \left\{ u \in H^1(\Omega) \;\middle|\; \Delta u \in L^2(\Omega) \text{ and } u|_{\partial\Omega} = 0 \right\}$$

Schrödinger operator with Robin/mixed boundary condition

Definition

$$
\begin{aligned}
A_{\mathcal{R}} u &= (-\Delta + V)u \\
\operatorname{dom}(A_{\mathcal{R}}) &= \Bigg\{ u \in H^1(\Omega) \,\Bigg|\, \Delta u \in L^2(\Omega),\ u|_{\omega'} = 0 \\
&\qquad \text{and } \alpha u|_\omega + \frac{\partial u}{\partial \nu}\Big|_\omega = 0 \Bigg\}
\end{aligned}
$$

where $\alpha \in \mathbb{R}$, $\varnothing \neq \omega \subseteq \partial\Omega$ open and $\omega' = \partial\Omega \setminus \omega$.

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where $\alpha \in \mathbb{R}$, $\varnothing \neq \omega \subseteq \partial\Omega$ open and $\omega' = \partial\Omega \setminus \omega$.

Special case: Neumann b.c. ($\alpha = 0$ and $\omega = \partial\Omega$)

The spectra of $A_{\mathcal{D}}$ and $A_{\mathcal{R}}$

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- For $V \in L^2(\Omega)$ their essential spectrum is equal to $[0, \infty)$.

For $M := \inf \sigma_{ess}(A_{\mathcal{D}}) = \inf \sigma_{ess}(A_{\mathcal{R}})$ denote by

- $\lambda_1(A_{\mathcal{D}}) \leq \lambda_2(A_{\mathcal{D}}) \leq \cdots < M$ the discrete eigenvalues of $A_{\mathcal{D}}$ in $(-\infty, M)$ counted with multiplicity
- $\lambda_1(A_{\mathcal{R}}) \leq \lambda_2(A_{\mathcal{R}}) \leq \cdots < M$ the discrete eigenvalues of $A_{\mathcal{R}}$ in $(-\infty, M)$ counted with multiplicity

Characterization of the discrete eigenvalues

- Bilinear form corresponding to $A_{\mathcal{D}}$

$$\begin{aligned} a_{\mathcal{D}}(u,v) &= (\nabla u, \nabla v)_{(L^2(\Omega))^d} + (Vu, v)_{L^2(\Omega)} \\ \mathrm{dom}(a_{\mathcal{D}}) &= H_0^1(\Omega) \end{aligned}$$

- Bilinear form corresponding to $A_{\mathcal{R}}$

$$\begin{aligned} a_{\mathcal{R}}(u,v) &= (\nabla u, \nabla v)_{(L^2(\Omega))^d} + (Vu, v)_{L^2(\Omega)} + \alpha(u|_{\partial\Omega}, v|_{\partial\Omega}) \\ \mathrm{dom}(a_{\mathcal{D}}) &= \left\{ u \in H^1(\Omega) \;\middle|\; u|_{\omega'} = 0 \right\} \end{aligned}$$

Characterization of the discrete eigenvalues

- Bilinear form corresponding to $A_{\mathcal{D}}$

$$\begin{aligned} a_{\mathcal{D}}(u,v) &= (\nabla u, \nabla v)_{(L^2(\Omega))^d} + (Vu, v)_{L^2(\Omega)} \\ \mathrm{dom}(a_{\mathcal{D}}) &= H_0^1(\Omega) \end{aligned}$$

- Bilinear form corresponding to $A_{\mathcal{R}}$

$$\begin{aligned} a_{\mathcal{R}}(u,v) &= (\nabla u, \nabla v)_{(L^2(\Omega))^d} + (Vu, v)_{L^2(\Omega)} + \alpha(u|_{\partial\Omega}, v|_{\partial\Omega}) \\ \mathrm{dom}(a_{\mathcal{D}}) &= \left\{ u \in H^1(\Omega) \;\middle|\; u|_{\omega'} = 0 \right\} \end{aligned}$$

Theorem (min-max principle)

$$\lambda_n(A_i) = \min_{\substack{L \text{ subspace of } \mathrm{dom}(a_i) \\ \dim L = n}} \left\{ \max_{u \in L \setminus \{0\}} \frac{a_i(u,u)}{(u,u)} \right\}, \quad i \in \{\mathcal{D}, \mathcal{R}\}$$

An inequality for eigenvalue counting functions

For an interval $\mathcal{I} \subseteq \mathbb{R}$ define the eigenvalue counting functions

- $N_\mathcal{D}(\mathcal{I}) := \dim \operatorname{ran} E_\mathcal{D}(\mathcal{I})$,
- $N_\mathcal{R}(\mathcal{I}) := \dim \operatorname{ran} E_\mathcal{R}(\mathcal{I})$,

$E_\mathcal{D}$, $E_\mathcal{R}$ spectral measures of $A_\mathcal{D}$ and $A_\mathcal{R}$ respectively.

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$E_{\mathcal{D}}$, $E_{\mathcal{R}}$ spectral measures of $A_{\mathcal{D}}$ and $A_{\mathcal{R}}$ respectively.

Theorem

Let $M = \inf \sigma_{ess}(A_{\mathcal{D}}) = \inf \sigma_{ess}(A_{\mathcal{R}})$. Then for each $\mu < M$ the inequality

$$N_{\mathcal{R}}\big((-\infty, \mu)\big) \geq N_{\mathcal{D}}\big((-\infty, \mu]\big)$$

holds.

Idea of the proof 1

Let $\mu < M$. Then we have

$$N_{\mathcal{D}}\big((-\infty,\mu]\big) = \max\big\{\dim L : L \subset \operatorname{dom}(a_{\mathcal{D}}) \text{ subspace}, \\ a_{\mathcal{D}}(u,u) \leq \mu \|u\|^2_{L^2(\Omega)}, u \in L\big\}.$$

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Let $F \subseteq \mathrm{dom}(a_{\mathcal{D}})$ be such that $\dim F = N_{\mathcal{D}}\big((-\infty,\mu]\big)$ and $a_{\mathcal{D}}(u,u) \le \mu \|u\|^2_{L^2(\Omega)}$, for all $u \in F$.

It holds for $u \in F$ and $v \in \ker(A_{\mathcal{R}} - \mu)$:

$$a_{\mathcal{R}}(u+v) \le \mu \|u+v\|^2_{L^2(\Omega)}.$$

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It holds for $u \in F$ and $v \in \ker(A_{\mathcal{R}} - \mu)$:

$$a_{\mathcal{R}}(u+v) \leq \mu\|u+v\|^2_{L^2(\Omega)}.$$

The sum $F + \ker(A_{\mathcal{R}} - \mu)$ is direct and hence

$$\begin{aligned} N_{\mathcal{R}}\big((-\infty,\mu]\big) &\geq \dim(F) + \dim\ker\left(A_{\mathcal{R}} - \mu\right) \\ &= N_{\mathcal{D}}\big((-\infty,\mu]\big) + \dim\ker\left(A_{\mathcal{R}} - \mu\right). \end{aligned}$$

Idea of the proof 2

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\begin{aligned}
N_{\mathcal{R}}\big((-\infty,\mu]\big) &\geq \dim(F) + \dim\ker\left(A_{\mathcal{R}} - \mu\right) \\
&= N_{\mathcal{D}}\big((-\infty,\mu]\big) + \dim\ker\left(A_{\mathcal{R}} - \mu\right)
\end{aligned}
$$

Idea of the proof 2

$$\begin{aligned} N_{\mathcal{R}}\big((-\infty,\mu]\big) &\geq \dim(F) + \dim\ker\left(A_{\mathcal{R}} - \mu\right) \\ &= N_{\mathcal{D}}\big((-\infty,\mu]\big) + \dim\ker\left(A_{\mathcal{R}} - \mu\right) \end{aligned}$$

$$N_{\mathcal{R}}\big((-\infty,\mu]\big) - \dim\ker\left(A_{\mathcal{R}} - \mu\right) \geq N_{\mathcal{D}}\big((-\infty,\mu]\big)$$

Idea of the proof 2

$$\begin{aligned} N_{\mathcal{R}}((-\infty,\mu]) &\geq \dim(F) + \dim\ker(A_{\mathcal{R}} - \mu) \\ &= N_{\mathcal{D}}((-\infty,\mu]) + \dim\ker(A_{\mathcal{R}} - \mu) \end{aligned}$$

$$N_{\mathcal{R}}((-\infty,\mu]) - \dim\ker(A_{\mathcal{R}} - \mu) \geq N_{\mathcal{D}}((-\infty,\mu])$$

$$N_{\mathcal{R}}((-\infty,\mu)) \geq N_{\mathcal{D}}((-\infty,\mu])$$

□

The corresponding eigenvalue inequality

Inserting $\mu = \lambda_k(A_{\mathcal{D}})$ one obtains the following result

Theorem

If there exist m eigenvalues of $A_{\mathcal{D}}$ in $(-\infty, M)$ then there exist at least m eigenvalues of $A_{\mathcal{R}}$ in $(-\infty, M)$ and the strict inequality

$$\lambda_k(A_{\mathcal{R}}) < \lambda_k(A_{\mathcal{D}})$$

holds for all $k \in \{1, \dots, m\}$.

Assumptions

- $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, open, nonempty and (for simplicity) connected
- $\mathcal{L}_i := -\sum_{j,k=1}^{d} \partial_j a_{jk,i} \partial_k + a_i$,
  - $a_{jk,i} : \overline{\Omega} \to \mathbb{C}$ bounded Lipschitz functions such that $a_{jk,i}(x) = \overline{a_{kj,i}(x)}$ for all $x \in \overline{\Omega}$
  - $a_i : \Omega \to \mathbb{R}$ bounded and measurable

such that $\mathcal{L}_i$ is uniformly elliptic, $i = 1, 2$.

Associated linear operators and bilinear forms

Definition

$$A_i u = -\sum_{j,k=1}^{d} \partial_j a_{jk,i} \partial_k u + a_i u$$

$$\mathrm{dom}(A_i) = \left\{ u \in H_0^1(\Omega) \;\middle|\; \mathcal{L}_i u \in L^2(\Omega) \right\}$$

Bilinear form corresponding to A_i

$$\mathfrak{a}_i(u,v) = \sum_{j,k=1}^{d} \int_\Omega a_{jk,i} \partial_k u \overline{\partial_j v} \,\mathrm{d}x + \int_\Omega a_i u \overline{v} \,\mathrm{d}x$$

$$\mathrm{dom}(\mathfrak{a}_i) = H_0^1(\Omega)$$

Additional assumptions

- The essential spectra of A_1 and A_2 coincide
- $\sum_{j,k=1}^{d} a_{jk,1}(x)\xi_j\overline{\xi_k} \leq \sum_{j,k=1}^{d} a_{jk,2}(x)\xi_j\overline{\xi_k}$,
  for any $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^d$
  (i.e. $(a_{jk,2}(x) - a_{jk,1}(x))_{j,k=1}^{d} \geq 0, \forall x \in \overline{\Omega}$)
- $a_1(x) \leq a_2(x)$ for all $x \in \Omega$

(in particular $\mathfrak{a}_1(u,u) \leq \mathfrak{a}_2(u,u)$ for any $u \in H_0^1(\Omega)$)

The spectra of A_1 and A_2

A_1 and A_2 are selfadjoint in $L^2(\Omega)$.

Define:

- $M := \inf \sigma_{ess}(A_1) = \inf \sigma_{ess}(A_2)$
- $\lambda_1(A_i) \leq \lambda_2(A_i) \leq \cdots < M$ discrete eigenvalues of A_i in $(-\infty, M)$ counted with multiplicity
- $N_i(\mathcal{I}) := \dim \operatorname{ran} E_i(\mathcal{I})$, $\mathcal{I} \subseteq \mathbb{R}$ interval, where E_i is the spectral measure of A_i.

Eigenvalue inequality for different coefficients

Theorem

Assume there exists an open ball $\mathcal{O} \subseteq \Omega$ such that at least one of the following conditions is satisfied:

- $a_1(x) < a_2(x)$ *for all* $x \in \mathcal{O}$,
- *the matrix* $(a_{jk,2}(x) - a_{jk,1}(x))_{j,k}$ *is invertible for all* $x \in \mathcal{O}$.

Then for all $\mu < M$ *the inequality*

$$N_1((-\infty, \mu)) \geq N_2((-\infty, \mu])$$

holds. In particular, if there exist l *eigenvalues of* A_2 *in* $(-\infty, M)$ *then*

$$\lambda_k(A_1) < \lambda_k(A_2)$$

holds for all $k \in \{1, \dots, l\}$.

Idea of the proof

Let $\mu < M$. Then we have

$$N_2\big((-\infty,\mu]\big) = \max\big\{\dim L : L \subset \operatorname{dom}(\mathfrak{a}_2) \text{ subspace}, \\ \mathfrak{a}_2(u,u) \leq \mu \|u\|^2_{L^2(\Omega)}, u \in L\big\}.$$

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Let $F \subseteq \mathrm{dom}(\mathfrak{a}_2)$ be such that $\dim F = N_2\big((-\infty,\mu]\big)$ and $\mathfrak{a}_2(u,u) \leq \mu\|u\|^2_{L^2(\Omega)}$, $\forall u \in F$.

It holds for $u \in F$ and $v \in \ker(A_1 - \mu)$:

$$\mathfrak{a}_1(u+v) \leq \mu\|u+v\|^2_{L^2(\Omega)}.$$

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$$\begin{aligned} N_1\big((-\infty,\mu]\big) &\geq \dim(F) + \dim\ker(A_1 - \mu) \\ &= N_2\big((-\infty,\mu]\big) + \dim\ker(A_1-\mu). \end{aligned}$$

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\end{aligned}
$$

$$
N_1\big((-\infty,\mu]\big) - \dim\ker\,(A_1-\mu) \geq N_2\big((-\infty,\mu]\big)
$$

Idea of the proof

$$\begin{aligned} N_1\big((-\infty,\mu]\big) &\geq \dim(F) + \dim\ker\left(A_1 - \mu\right) \\ &= N_2\big((-\infty,\mu]\big) + \dim\ker\left(A_1 - \mu\right) \end{aligned}$$

$$N_1\big((-\infty,\mu]\big) - \dim\ker\left(A_1 - \mu\right) \geq N_2\big((-\infty,\mu]\big)$$

$$N_1\big((-\infty,\mu)\big) \geq N_2\big((-\infty,\mu]\big)$$

□

Thank you for your attention.