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Space–Time Methods

1. Let u be a weak solution of the homogeneous heat equation

$$\partial_t u(x,t) - \Delta_x u(x,t) = 0 \quad \text{for } (x,t) \in Q = \Omega \times (0,T),$$

where we assume some inhomogeneous Dirichlet boundary conditions on Σ. Let $Y = L^2(0,T;H^1_0(\Omega))$. Prove that

$$\|\partial_t u\|_{Y'} \leq \|\nabla_x u\|_{L^2(Q)}\,.$$

2. Let $w \in Y = L^2(0,T;H^1_0(\Omega))$ be the unique solution of the elliptic variational formulation such that

$$\langle \nabla_x w, \nabla_x v\rangle_{L^2(Q)} = \langle \partial_t u, v\rangle_Q \quad \text{for all } v \in Y.$$

Prove that

$$\|\partial_t u\|_{Y'} = \|w\|_Y\,.$$

3. Let $w_h \in Y_h \subset Y$ be the unique solution of the Galerkin variational formulation

$$\langle \nabla_x w_h, \nabla_x v_h\rangle_{L^2(Q)} = \langle \partial_t u, v_h\rangle_Q \quad \text{for all } v_h \in Y_h.$$

Prove that

$$\|w_h\|_Y \leq \|w\|_Y$$

where $w \in Y$ is the unique solution of the variational formulation in **2.**

4. Let $w \in Y = L^2(0,T;H^1_0(\Omega))$ be the unique solution of the elliptic variational formulation such that

$$\langle \nabla_x w, \nabla_x v\rangle_{L^2(Q)} = \langle \partial_t u, v\rangle_Q + \langle \nabla_x u, \nabla_x v\rangle_{L^2(Q)} \quad \text{for all } v \in Y.$$

Prove that

$$\|\partial_t u - \Delta_x u\|^2_{Y'} = \|w\|^2_Y = b(u,w),$$

where

$$b(u,v) = \int_0^T \int_\Omega \Big[\partial_t u(x,t) v(x,t) + \nabla_x u(x,t) \cdot \nabla_x v(x,t)\Big]\, dx\, dt.$$

5. Prove, by using the result of **4.**, the inf–sup condition

$$\|\partial_t u - \Delta_x u\|_{Y'} \leq \sup_{0 \neq v \in Y} \frac{|b(u,v)|}{\|v\|_Y} \quad \text{for all } u \in X,$$

where

$$X := \Big\{ u \in Y : \partial_t u \in Y', u(x,0) = 0 \quad \text{for } x \in \Omega \Big\}.$$

Prove that $\|\partial_t u - \Delta_x u\|_{Y'}$ defines a norm in X which is equivalent to the graph norm

$$\|u\|^2_X = \|u\|^2_Y + \|\partial_t u\|^2_{Y'}.$$