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Space–Time Methods

Let
$$Y := L^2(0,T;H_0^1(\Omega)), \quad X := \Big\{ u \in Y : \partial_t u \in Y',\ u(0) = 0 \text{ in } \Omega \Big\},$$
and
$$b(u,v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} \quad \text{for } u \in X, v \in Y.$$
Define
$$\mathcal{B}(u,p;q,v) := \langle \nabla_x p, \nabla_x v \rangle_{L^2(Q)} + b(u,v) + b(q,p) \quad \text{for } u,q \in X, p,v \in Y,$$
and
$$\|(u,p)\|_{X\times Y} := \sqrt{\|u\|_X^2 + \|p\|_Y^2}.$$

6. Prove the boundedness
$$|\mathcal{B}(u,p;q,v)| \leq c_2^B \, \|(u,p)\|_{X\times Y} \|(q,v)\|_{X\times Y}$$
with a suitable positive constant c_2^B.

7. Prove the inf–sup stability condition
$$c_1^B \, \|(u,p)\|_{X\times Y} \leq \sup_{(0,0)\neq(q,v)\in X\times Y} \frac{\mathcal{B}(u,p;q,v)}{\|(q,v)\|_{X\times Y}} \quad \text{for all } (u,p) \in X \times Y.$$

Hints: For $u \in X$ define $w \in Y$ such that $\|\partial_t u\|_{Y'} = \|w\|_Y$. For $p \in Y$ and for a suitable chosen $\alpha \in \mathbb{R}_+$ define $\overline{v} := u + w + \alpha p \in Y$ and $\overline{q} := -\alpha u \in X$. Use
$$ab \leq \frac{1}{2\gamma}\, a^2 + \frac{1}{2}\,\gamma\, b^2, \quad \gamma > 0$$
and define γ and α appropriately.

Instead of the heat equation we now consider the diffusion–convection equation
$$-\Delta u(x) + \underline{b}(x) \cdot \nabla u(x) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = 0 \quad \text{for } x \in \Gamma := \partial\Omega,$$
where $\underline{b}$ is a given bounded velocity field satisfying $\operatorname{div} \underline{b} = 0$. Define
$$Y := H_0^1(\Omega), \quad X := \Big\{ u \in Y : \underline{b} \cdot \nabla u \in Y' \Big\}.$$

8. Derive a variational formulation to determine a solution $u \in X$ and prove a related inf–sup stability condition.

9. Derive a related least–squares formulation and prove ellipticity of the associated Schur complement operator which is of the form $S = B'A^{-1}B$.

10. Introduce conforming finite element spaces $X_H \subset X$ and $Y_h \subset Y$ of piecewise linear basis functions, and define an appropriate approximation $\widetilde{S}u = B'p_h$ by an approximate solution of the operator equation $Ap = Bu$. Prove discrete ellipticity of $\widetilde{S}$ when considering an appropriate choice of the mesh sizes h and H.

Hint: Recall the regularity of piecewise linear finite element functions.