Technische Universität Graz
Institut für Angewandte Mathematik
Assoc. Prof. Dr. Günther Of

Numerics and Simulation
Elective subject mathematics
Exercise sheet 1, April 10, 2024

Exercise 1: Start from the provided initial Python script, which solves the Dirichlet boundary value problem of the Poisson equation, and enhance it as follows:

a) Compute the $H^1(\Omega)$ error of the approximate solution.

b) Refine the mesh several times and create a related table providing the refinement level, the number of vertices (dofs), the errors and the related experimental orders of convergence (eoc).

Exercise 2:

a) Enhance your Python script of Exercise 1 by a log-log plot (check `matplotlib.plot`) of the $H^1(\Omega)$ error versus the number of degrees of freedom. What is the slop of the curve?

b) Compare the $H^1(\Omega)$ error curves for finite elements of polynomial degree (order) 1, 2, and 3 in a single log-log plot. How are your observation related to the theoretical results?

Exercise 3:

a) Set up a variational formulation for the Neumann boundary value problem of the Yukawa operator

$$-\Delta u(x) + u(x) = f(x) \quad \text{for } x \in \Omega,$$
$$\frac{\partial u}{\partial n}(x) = g(x) \quad \text{for } x \in \Gamma.$$

Why is the bilinear form $H^1(\Omega)$ elliptic?

b) Modify your Python script to solve this boundary value problem and your variational problem, respectively, by lowest order FEM. Check for convergence for the exact solution $u(x) = \sin(\pi x)\sin(\pi y)$ for several refinement steps.

Exercise 4:

a) Set up a variational formulation for the mixed boundary value problem of the Poisson equation

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega,$$
$$u(x) + \frac{\partial u}{\partial n}(x) = g_R(x) \quad \text{for } x \in \Gamma_{\text{top}},$$
$$\frac{\partial u}{\partial n}(x) = g_N(x) \quad \text{for } x \in \Gamma \setminus \Gamma_{\text{top}}.$$

Why is the bilinear form $H^1(\Omega)$ elliptic?

b) Modify your Python script to solve this boundary value problem and your variational problem, respectively, by lowest order FEM. Check for convergence for the exact solution $u(x) = \sin(\pi x)\sin(\pi y)$ for several refinement steps.

Exercise 5:

a) Set up a variational formulation for the following mixed boundary value problem of the Poisson equation with non-zero Dirichlet data g_D:

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega,$$
$$\frac{\partial u}{\partial n}(x) = g_N(x) \quad \text{for } x \in \Gamma_{\text{left}} \cup \Gamma_{\text{right}},$$
$$u(x) = g_D(x) \quad \text{for } x \in \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}}.$$

b) Study the instructions from on how to implement non-zero Dirichlet data and describe it: `https://docu.ngsolve.org/latest/i-tutorials/unit-1.3-dirichlet/dirichlet.html`

c) Modify your python script in the suggested way to solve the above boundary value problem and your variational problem, respectively, by lowest order FEM. Check for convergence for the exact solution $u(x) = x_1^2 + 1$ for several refinement steps.