Chapter 2

Conservation and Balance Equations

In this chapter we consider some applications of Reynold's transport theorem, Theorem 1.1. For a balance equation of the general type

$$\frac{d}{dt}\int_{\omega(t)} u(t,\boldsymbol{y})\,d\boldsymbol{y} \;=\; \int_{\omega(t)} f(t,\boldsymbol{y})\,d\boldsymbol{y} \tag{2.1}$$

we find from (1.12)

$$\int_{\omega(t)} \left[\frac{\partial}{\partial t}u(t,\boldsymbol{y}) + \mathrm{div}_y[u(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})]\right] d\boldsymbol{y} \;=\; \int_{\omega(t)} f(t,\boldsymbol{y})\,d\boldsymbol{y}$$

for all control volumina $\omega(t) \subset \Omega(t)$. Hence, for continuous integrands,

$$\frac{\partial}{\partial t}u(t,\boldsymbol{y}) + \mathrm{div}_y[u(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})] \;=\; f(t,\boldsymbol{y}) \quad \text{for } \boldsymbol{y} \in \Omega(t) \tag{2.2}$$

follows.

2.1 Conservation of Volume

For an arbitrary domain $\omega(t)$ we define the volumen

$$V_{\omega(t)} := \int_{\omega(t)} d\boldsymbol{y},$$

and the conservation of volume states

$$V_{\omega(t)} = V_{\omega(t_0)} \quad \text{for all } t > t_0,$$

i.e.

$$\frac{d}{dt}V_{\omega(t)} = \frac{d}{dt}\int_{\omega(t)} d\boldsymbol{y} \;=\; 0.$$

When comparing this with (2.1), this corresponds to $u(t,\boldsymbol{y}) = 1$ and $f(t,\boldsymbol{y}) = 0$, and therefore we obtain from (2.2) the partial differential equation

$$\mathrm{div}_y \boldsymbol{v}(t,\boldsymbol{y}) \;=\; 0 \quad \text{for } \boldsymbol{y} \in \Omega(t), \tag{2.3}$$

which describes incompressible materials or fluids. The conservation of volume also implies

$$\int_{\omega(t)} d\boldsymbol{y} = \int_{\omega(t_0)} J(t)\, d\boldsymbol{x} = \int_{\omega(t_0)} d\boldsymbol{x}$$

for all $t > t_0$, and for all controll volumina $\omega(t_0)$, and therefore

$$J(t) = 1 \quad \text{for all } t > t_0 \tag{2.4}$$

follows.

2.2 Conservation of Mass

The mass of material with mass density $\varrho(t,\boldsymbol{y})$ in an arbitrary domain $\omega(t)$ is given by

$$M_{\omega(t)} := \int_{\omega(t)} \varrho(t,\boldsymbol{y})\, d\boldsymbol{y}\,.$$

The conservation of mass states

$$M_{\omega(t)} = M_{\omega(t_0)} \quad \text{for all } t > t_0,$$

i.e.

$$\frac{d}{dt} M_{\omega(t)} = \frac{d}{dt} \int_{\omega(t)} \varrho(t,\boldsymbol{y})\, d\boldsymbol{y} \;=\; 0\,.$$

When comparing this with (2.1) this corresponds to $u(t,\boldsymbol{y}) = \varrho(t,\boldsymbol{y})$ and $f(t,\boldsymbol{y}) = 0$, and therefore we obtain from (2.2) the continuity equation

$$\frac{\partial}{\partial t}\varrho(t,\boldsymbol{y}) + \mathrm{div}_y[\varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})] \;=\; 0 \quad \text{for } \boldsymbol{y} \in \Omega(t). \tag{2.5}$$

By using (1.6) we further obtain

$$\begin{aligned} \frac{\partial}{\partial t}\varrho(t,\boldsymbol{y}) + \mathrm{div}_y[\varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})] &= \frac{\partial}{\partial t}\varrho(t,\boldsymbol{y}) + \nabla_y \varrho(t,\boldsymbol{y}) \cdot \boldsymbol{v}(t,\boldsymbol{y}) + \varrho(t,\boldsymbol{y})\, \mathrm{div}_y \boldsymbol{v}(t,\boldsymbol{y}) \\ &= \frac{d}{dt}\varrho(t,\boldsymbol{y}) + \varrho(t,\boldsymbol{y})\, \mathrm{div}_y \boldsymbol{v}(t,\boldsymbol{y}). \end{aligned}$$

Hence we can write the continuity equation (2.5) as

$$\frac{d}{dt}\varrho(t,\boldsymbol{y}) + \varrho(t,\boldsymbol{y})\, \mathrm{div}_y \boldsymbol{v}(t,\boldsymbol{y}) \;=\; 0 \quad \text{for } \boldsymbol{y} \in \Omega(t). \tag{2.6}$$

In particular for incompressible materials we have $\operatorname{div}_y \boldsymbol{v}(t,\boldsymbol{y}) = 0$ and therefore

$$\frac{d}{dt}\varrho(t,\boldsymbol{y}) = 0 \quad \text{for } \boldsymbol{y} = \boldsymbol{\varphi}(t,\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega,$$

follows.

The conservation of mass also implies

$$\int_{\omega(t)} \varrho(t,\boldsymbol{y})\, d\boldsymbol{y} = \int_{\omega(t_0)} \varrho(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\, J(t)\, d\boldsymbol{x} = \int_{\omega(t_0)} \varrho(t_0,\boldsymbol{x})\, d\boldsymbol{x}$$

for all $\omega(t_0) \subset \Omega$, and therefore

$$\varrho_0(\boldsymbol{x}) := \varrho(t_0,\boldsymbol{x}) = \varrho(t,\boldsymbol{\varphi}(t,\boldsymbol{x}))\, J(t) \quad \text{for } \boldsymbol{x} \in \Omega. \tag{2.7}$$

2.3 An Auxiliary Result

Next we consider the application of Reynolds transport theorem, the conservation of mass (2.5) and (1.6) to compute, for a scalar function $f(t,\boldsymbol{y}) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$,

$$\begin{aligned}
\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y}) f(t,\boldsymbol{y})\, d\boldsymbol{y} &= \int_{\omega(t)} \left[\frac{\partial}{\partial t}(\varrho(t,\boldsymbol{y}) f(t,\boldsymbol{y})) + \operatorname{div}_y(\varrho(t,\boldsymbol{y}) f(t,\boldsymbol{y}) \boldsymbol{v}(t,\boldsymbol{y}))\right] d\boldsymbol{y} \\
&= \int_{\omega(t)} \left[f(t,\boldsymbol{y}) \left(\frac{\partial}{\partial t}\varrho(t,\boldsymbol{y}) + \operatorname{div}_y(\varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y}))\right)\right. \\
&\qquad \left. + \varrho(t,\boldsymbol{y}) \left(\frac{\partial}{\partial t} f(t,\boldsymbol{y}) + \boldsymbol{v}(t,\boldsymbol{y}) \cdot \nabla_y f(t,\boldsymbol{y})\right)\right] d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y}) \left(\frac{\partial}{\partial t} f(t,\boldsymbol{y}) + \boldsymbol{v}(t,\boldsymbol{y}) \cdot \nabla_y f(t,\boldsymbol{y})\right) d\boldsymbol{y} \\
&= \int_{\omega(t)} \varrho(t,\boldsymbol{y}) \frac{d}{dt} f(t,\boldsymbol{y}) d\boldsymbol{y}.
\end{aligned}$$

i.e.,

$$\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y}) f(t,\boldsymbol{y})\, d\boldsymbol{y} = \int_{\omega(t)} \varrho(t,\boldsymbol{y}) \frac{d}{dt} f(t,\boldsymbol{y})\, d\boldsymbol{y}. \tag{2.8}$$

2.4 Balance of Linear Momentum

The postulate of balance of linear momentum is the statement that the rate of change of linear momentum of a fixed mass of a body is equal to the sum of the forces acting on the body, i.e. for $i = 1, \ldots, n$ we have

$$\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y}) v_i(t,\boldsymbol{y})\, d\boldsymbol{y} = \int_{\omega(t)} \varrho(t,\boldsymbol{y}) f_i(t,\boldsymbol{y})\, d\boldsymbol{y} + \int_{\partial\omega(t)} t_i(t,\boldsymbol{y},\boldsymbol{n})\, ds_{\boldsymbol{y}},$$

where $\boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})$ is the Cauchy stress vector for $\boldsymbol{y} \in \partial\omega(t)$, and $\boldsymbol{n}$ is the exterior normal vector on the boundary of the test volumen $\omega(t)$. Note that there holds

$$\boldsymbol{t}(t,\boldsymbol{y},-\boldsymbol{n}) = -\boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n}). \tag{2.9}$$

The application of Reynold's transport theorem (Theorem 1.1) gives, by using (2.8),

$$\frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y})v_i(t,\boldsymbol{y})\,d\boldsymbol{y} \;=\; \int_{\omega(t)} \varrho(t,\boldsymbol{y})\frac{d}{dt}v_i(t,\boldsymbol{y})\,d\boldsymbol{y},$$

and we obtain

$$\int_{\omega(t)} \left[\varrho(t,\boldsymbol{y})\frac{d}{dt}v_i(t,\boldsymbol{y}) - \varrho(t,\boldsymbol{y})f_i(t,\boldsymbol{y})\right] d\boldsymbol{y} \;=\; \int_{\partial\omega(t)} t_i(t,\boldsymbol{y},\boldsymbol{n})\,ds_{\boldsymbol{y}}. \tag{2.10}$$

In what follows we aim to rewrite the integral balance (2.10) in form of a partial differential equation. For this we have to transform the surface integral into a domain integral, for which we introduce a reformulation of the Cauchy stress vector $\boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})$ first.

Lemma 2.1 *The Cauchy stress vector $\boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})$ can be written as*

$$\boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n}) = \boldsymbol{T}(t,\boldsymbol{y})\boldsymbol{n} \tag{2.11}$$

where $\boldsymbol{T}(t,\boldsymbol{y})$ is the Cauchy stress tensor.

Proof: We consider the two–dimensional case first. Let $\omega(t)$ be some test volumen with boundary $\partial\omega(t)$. Let $\boldsymbol{y}_0 \in \partial\omega$ be arbitrary but fixed. We assume, without loss of generality, that we can write the exterior normal vector $\boldsymbol{n}_0$ in $\boldsymbol{y}_0$ as

$$\boldsymbol{n}_0 = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2, \quad n_1 > 0,\; n_2 > 0,$$

where the $\boldsymbol{e}_k$, $k = 1,2$, are the Euclidean unit vectors in $\mathbb{R}^2$, see Fig. 2.1. We define a triangle $T(\boldsymbol{y}_0)$ via its nodal points

$$\boldsymbol{P}_0 = \boldsymbol{y}_0, \quad \boldsymbol{P}_1 = \boldsymbol{y}_0 - \alpha\boldsymbol{e}_1, \quad \alpha > 0, \quad \boldsymbol{P}_2 = \boldsymbol{y}_0 - \beta\boldsymbol{e}_2, \quad \beta > 0,$$

such that $-\boldsymbol{n}_0$ is the exterior normal vector of the edge $E_0(\boldsymbol{P}_1,\boldsymbol{P}_2)$, while $\boldsymbol{e}_1$ is the exterior normal vector of the edge $E_1(\boldsymbol{P}_2,\boldsymbol{y}_0)$, and $\boldsymbol{e}_2$ is the exterior normal vector of the edge $E_2(\boldsymbol{y}_0,\boldsymbol{P}_1)$, respectively, see Fig. 2.1.
Note that we have

$$0 = (\boldsymbol{P}_2 - \boldsymbol{P}_1, \boldsymbol{n}_0) = (\alpha\boldsymbol{e}_1 - \beta\boldsymbol{e}_2, n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2) = \alpha n_1 - \beta n_2\,.$$

Due to $n_2 > 0$ we have

$$\beta = \alpha\,\frac{n_1}{n_2}\,. \tag{2.12}$$

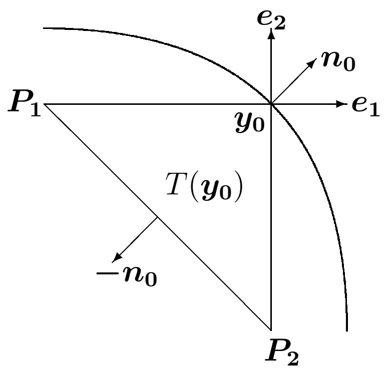

Figure 2.1: Local coordinate system in $\boldsymbol{y}_0 \in \partial\omega(t)$.

For the control volumen $T(\boldsymbol{y}_0)$ the balance of linear momentum (2.10) gives, for $i = 1, 2$,

$$\int_{T(\boldsymbol{y}_0)} \left[\varrho(t,\boldsymbol{y}) \frac{d}{dt} v_i(t,\boldsymbol{y}) - \varrho(t,\boldsymbol{y}) f_i(t,\boldsymbol{y}) \right] d\boldsymbol{y} \;=\; \int_{\partial T(\boldsymbol{y}_0)} t_i(t,\boldsymbol{y},\boldsymbol{n})\, ds_{\boldsymbol{y}}$$

$$= \int_{E_0} t_i(t,\boldsymbol{y},-\boldsymbol{n}_0)\, ds_{\boldsymbol{y}} + \int_{E_1} t_i(t,\boldsymbol{y},\boldsymbol{e}_1)\, ds_{\boldsymbol{y}} + \int_{E_2} t_i(t,\boldsymbol{y},\boldsymbol{e}_2)\, ds_{\boldsymbol{y}}\,.$$

When applying the mean value theorem to all integrals this gives

$$\left[\varrho(t,\widetilde{\boldsymbol{y}}) \frac{d}{dt} v_i(t,\widetilde{\boldsymbol{y}}) - \varrho(t,\widetilde{\boldsymbol{y}}) f_i(t,\widetilde{\boldsymbol{y}}) \right] \operatorname{area}(T(\boldsymbol{y}_0))$$

$$= t_i(t,\widetilde{\boldsymbol{y}}_0,-\boldsymbol{n}_0)\, |E_0| + t_i(t,\widetilde{\boldsymbol{y}}_1,\boldsymbol{e}_1)\, |E_1| + t_i(t,\widetilde{\boldsymbol{y}}_2,\boldsymbol{e}_2)\, |E_2|,$$

where $\widetilde{\boldsymbol{y}} \in T(\boldsymbol{y}_0)$ and $\widetilde{\boldsymbol{y}}_k \in E_k$, $k = 0, 1, 2$, are appropriately chosen. By using

$$|E_0| = \sqrt{\alpha^2+\beta^2}, \quad |E_1| = \beta, \quad |E_2| = \alpha, \quad \operatorname{area}(T(\boldsymbol{y}_0)) = \frac{1}{2}\alpha\beta$$

we further conclude

$$\left[\varrho(t,\widetilde{\boldsymbol{y}}) \frac{d}{dt} v_i(t,\widetilde{\boldsymbol{y}}) - \varrho(t,\widetilde{\boldsymbol{y}}) f_i(t,\widetilde{\boldsymbol{y}}) \right] \frac{1}{2}\alpha\beta$$

$$= t_i(t,\widetilde{\boldsymbol{y}}_0,-\boldsymbol{n}_0)\, \sqrt{\alpha^2+\beta^2} + t_i(t,\widetilde{\boldsymbol{y}}_1,\boldsymbol{e}_1)\, \beta + t_i(t,\widetilde{\boldsymbol{y}}_2,\boldsymbol{e}_2)\, \alpha.$$

By using (2.12) we obtain

$$\left[\varrho(t,\widetilde{\boldsymbol{y}}) \frac{d}{dt} v_i(t,\widetilde{\boldsymbol{y}}) - \varrho(t,\widetilde{\boldsymbol{y}}) f_i(t,\widetilde{\boldsymbol{y}}) \right] \frac{1}{2}\alpha\, n_1$$

$$= t_i(t,\widetilde{\boldsymbol{y}}_0,-\boldsymbol{n}_0)\; + t_i(t,\widetilde{\boldsymbol{y}}_1,\boldsymbol{e}_1)\, n_1 + t_i(t,\widetilde{\boldsymbol{y}}_2,\boldsymbol{e}_2)\, n_2.$$

In the limiting case $\alpha \to 0$ we therefore conclude

$$t_i(t,\boldsymbol{y}_0,-\boldsymbol{n}_0)\; + t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_1)\, n_1 + t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_2)\, n_2 \;=\; 0,$$

from which

$$\begin{aligned} t_i(t,\boldsymbol{y}_0,\boldsymbol{n}_0) &= t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_1)\, n_1 + t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_2)\, n_2 \\ &= T_{i1}(t,\boldsymbol{y}_0)n_1 + T_{i2}(t,\boldsymbol{y}_0)n_2 \end{aligned}$$

with

$$T_{i1}(t,\boldsymbol{y}_0) = t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_1), \quad T_{i2}(t,\boldsymbol{y}_0) = t_i(t,\boldsymbol{y}_0,\boldsymbol{e}_2)$$

follows.

In the three–dimensional case we proceed in the same way. For an arbitrary but fixed $\boldsymbol{y}_0 \in \partial\omega(t)$ we use the Euclidean unit vectors $\boldsymbol{e}_k$, $k = 1,2,3$, to write the exterior normal vector $\boldsymbol{n}_0$ in $\boldsymbol{y}_0$ as

$$\boldsymbol{n}_0 = n_1\boldsymbol{e}_1 + n_2\boldsymbol{e}_2 + n_3\boldsymbol{e}_3,$$

where we assume

$$n_k > 0 \quad \text{for } k = 1,2,3.$$

Note that such a configuration is always possible due to appropriately chosen coordinate transformations to define $\omega(t)$. We define a tetrahedron $T(\boldsymbol{y}_0)$ via its nodal points

$$\boldsymbol{P}_0 = \boldsymbol{y}_0, \quad \boldsymbol{P}_1 = \boldsymbol{y}_0 - \alpha\boldsymbol{e}_1, \quad \alpha > 0, \quad \boldsymbol{P}_2 = \boldsymbol{y}_0 - \beta\boldsymbol{e}_2, \quad \beta > 0, \quad \boldsymbol{P}_3 = \boldsymbol{y}_0 - \gamma\boldsymbol{e}_3, \quad \gamma > 0,$$

such that $-\boldsymbol{n}_0$ is the normal vector of the face $F_0(\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3)$, while $\boldsymbol{e}_k$ are the normal vectors of the faces $F_k(\{\boldsymbol{P}_0,\boldsymbol{P}_1,\boldsymbol{P}_2,\boldsymbol{P}_3\}\backslash\boldsymbol{P}_k)$ for $k = 1,2,3$, see also Fig. 2.1.

For the control volumen $\omega(t) = T(\boldsymbol{y}_0)$ we then have (2.10), i.e. for $i = 1,\ldots,3$

$$\begin{aligned} \int_{T(\boldsymbol{y}_0)} \left[\varrho(t,\boldsymbol{y})\frac{d}{dt}v_i(t,\boldsymbol{y}) - \varrho(t,\boldsymbol{y})f_i(t,\boldsymbol{y})\right] d\boldsymbol{y} &= \int_{\partial T(\boldsymbol{y}_0)} t_i(t,\boldsymbol{y},\boldsymbol{n}_y)\, ds_{\boldsymbol{y}} \\ &= \sum_{k=1}^{3} \int_{F_k} t_i(t,\boldsymbol{y},\boldsymbol{e}_k)\, ds_{\boldsymbol{y}} + \int_{F_0} t_i(t,\boldsymbol{y},-\boldsymbol{n}_0)\, ds_{\boldsymbol{y}}\,. \end{aligned}$$

When applying the mean value theorem to all integrals this gives

$$\begin{aligned} &\left[\varrho(t,\widetilde{\boldsymbol{y}})\frac{d}{dt}v_i(t,\widetilde{\boldsymbol{y}}) - \varrho(t,\widetilde{\boldsymbol{y}})f_i(t,\widetilde{\boldsymbol{y}})\right] \mathrm{vol}(T(\boldsymbol{y}_0)) = \\ &\qquad = \sum_{k=1}^{3} t_i(t,\widetilde{\boldsymbol{y}}_k,\boldsymbol{e}_k)\,\mathrm{area}(F_k) + t_i(t,\widetilde{\boldsymbol{y}}_0,-\boldsymbol{n}_0)\,\mathrm{area}(F_0), \end{aligned} \tag{2.13}$$

where $\widetilde{\boldsymbol{y}}_k \in F_k$ and $\widetilde{\boldsymbol{y}} \in T(\boldsymbol{y}_0)$ are appropriately chosen. The normal vector $-\boldsymbol{n}_0$ of F_0 can be computed from

$$-\boldsymbol{n}_0 = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$$

where

$$\boldsymbol{a} = \boldsymbol{P}_3 - \boldsymbol{P}_1 = \begin{pmatrix} \alpha \\ 0 \\ -\gamma \end{pmatrix}, \quad \boldsymbol{b} = \boldsymbol{P}_2 - \boldsymbol{P}_1 = \begin{pmatrix} \alpha \\ -\beta \\ 0 \end{pmatrix}.$$

Hence we obtain

$$n_k = (\boldsymbol{n}_0, \boldsymbol{e}_k) = -\frac{(\boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{e}_k)}{|\boldsymbol{a} \times \boldsymbol{b}|}$$

i.e.

$$n_k \, |\boldsymbol{a} \times \boldsymbol{b}| \; = \; (\boldsymbol{b} \times \boldsymbol{a}, \boldsymbol{e}_k) \; = \; \left(\begin{pmatrix} \beta\gamma \\ \alpha\gamma \\ \alpha\beta \end{pmatrix}, \boldsymbol{e}_k \right),$$

and therefore

$$n_1 \, |\boldsymbol{a} \times \boldsymbol{b}| \; = \; \beta\gamma, \quad n_2 \, |\boldsymbol{a} \times \boldsymbol{b}| \; = \; \alpha\gamma, \quad n_3 \, |\boldsymbol{a} \times \boldsymbol{b}| \; = \; \alpha\beta$$

follows. Note that

$$\text{area}(F_0) = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\sqrt{[\beta\gamma]^2 + [\alpha\gamma]^2 + [\alpha\beta]^2},$$

and hence we conclude

$$\begin{aligned}
\text{area}(F_1) &= \frac{1}{2}\beta\gamma = \frac{1}{2}n_1 \, |\boldsymbol{a} \times \boldsymbol{b}| = n_1 \, \text{area}(F_0), \\
\text{area}(F_2) &= \frac{1}{2}\alpha\gamma = \frac{1}{2}n_2 \, |\boldsymbol{a} \times \boldsymbol{b}| = n_2 \, \text{area}(F_0), \\
\text{area}(F_3) &= \frac{1}{2}\alpha\beta = \frac{1}{2}n_3 \, |\boldsymbol{a} \times \boldsymbol{b}| = n_3 \, \text{area}(F_0).
\end{aligned}$$

Now we can write (2.13) as

$$\varrho(t, \widetilde{\boldsymbol{y}}) \left[\frac{d}{dt} v_i(t, \widetilde{\boldsymbol{y}}) - f_i(t, \widetilde{\boldsymbol{y}}) \right] \frac{\text{vol}(T(\boldsymbol{y}_0))}{\text{area}(F_0)} = \sum_{k=1}^{3} t_i(t, \widetilde{\boldsymbol{y}}_k, \boldsymbol{e}_k) n_k + t_i(t, \widetilde{\boldsymbol{y}}_0, -\boldsymbol{n}_0) \, .$$

Recall that

$$\text{vol}(T(\boldsymbol{y}_0)) \; = \; \frac{1}{6}\alpha\beta\gamma.$$

Hence, when considering the scaling

$$\alpha = h\widehat{\alpha}, \quad \beta = h\widehat{\beta}, \quad \gamma = h\widehat{\gamma},$$

we find

$$\frac{\text{vol}(T(\boldsymbol{y}_0))}{\text{area}(F_0)} \; = \; \frac{1}{3} h \frac{\widehat{\alpha} \, \widehat{\beta} \, \widehat{\gamma}}{\sqrt{[\widehat{\beta} \, \widehat{\gamma}]^2 + [\widehat{\alpha} \, \widehat{\gamma}]^2 + [\widehat{\alpha} \, \widehat{\beta}]^2}} \; \to \; 0 \quad \text{as } h \to 0 \, .$$

Note that all normal vectors remain the same for $h \to 0$. In the limit we therefore obtain

$$\sum_{k=1}^{3} t_i(t, \boldsymbol{y}, \boldsymbol{e}_k) n_k + t_i(t, \boldsymbol{y}, -\boldsymbol{n}_0) \; = \; 0$$

which is equivalent to, for $i = 1, 2, 3$,

$$t_i(t, \boldsymbol{y}, \boldsymbol{n}_0) = \sum_{k=1}^{3} t_i(t, \boldsymbol{y}, \boldsymbol{e}_k) n_k = \sum_{k=1}^{3} T_{ik}(t, \boldsymbol{y}) n_k$$

with

$$T_{ik}(t, \boldsymbol{y}) = t_i(t, \boldsymbol{y}, \boldsymbol{e}_k) \quad \text{for } i, k = 1, 2, 3.$$

■

Now, using the representation (2.11) we can write the integral balance of linear momentum (2.10) as, for $i = 1, 2, 3$,

$$\begin{aligned} \int_{\omega(t)} \left[\varrho(t, \boldsymbol{y}) \frac{d}{dt} v_i(t, \boldsymbol{y}) - \varrho(t, \boldsymbol{y}) f_i(t, \boldsymbol{y}) \right] d\boldsymbol{y} &= \int_{\partial\omega(t)} t_i(t, \boldsymbol{y}, \boldsymbol{n}) \, ds_{\boldsymbol{y}} \\ &= \int_{\partial\omega(t)} \sum_{j=1}^{n} T_{ij}(t, \boldsymbol{y}) n_j \, ds_{\boldsymbol{y}} \\ &= \int_{\omega(t)} \sum_{j=1}^{n} \frac{\partial}{\partial y_j} T_{ij}(t, \boldsymbol{y}) \, d\boldsymbol{y} \, . \end{aligned}$$

Since this holds for all test volumina $\omega(t)$, we conclude, for continuous functions, the Cauchy equilibrium equations

$$\varrho(t, \boldsymbol{y}) \frac{d}{dt} v_i(t, \boldsymbol{y}) = \varrho(t, \boldsymbol{y}) f_i(t, \boldsymbol{y}) + \sum_{j=1}^{n} \frac{\partial}{\partial y_j} T_{ij}(t, \boldsymbol{y}) \quad \text{for } i = 1, \ldots, n, \tag{2.14}$$

i.e.

$$\varrho(t, \boldsymbol{y}) \frac{d}{dt} \boldsymbol{v}(t, \boldsymbol{y}) = \varrho(t, \boldsymbol{y}) \boldsymbol{f}(t, \boldsymbol{y}) + \operatorname{div} \boldsymbol{T}(t, \boldsymbol{y}). \tag{2.15}$$

2.5 Balance of Angular Momentum

To derive symmetry relations of the Cauchy stress tensor $\boldsymbol{T}(t, \boldsymbol{y})$ as defined in (2.11) we will consider the balance of angular momentum which is the statement that the rate of change of angular momentum of a fixed material region arises from the combined torques on the body. In the absence of body couples, the integral form of the balance of angular momentum can be written as

$$\frac{d}{dt} \int_{\omega(t)} \boldsymbol{y} \times \varrho(t, \boldsymbol{y}) \boldsymbol{v}(t, \boldsymbol{y}) d\boldsymbol{y} = \int_{\omega(t)} \boldsymbol{y} \times \varrho(t, \boldsymbol{y}) \boldsymbol{f}(t, \boldsymbol{y}) d\boldsymbol{y} + \int_{\partial\omega(t)} \boldsymbol{y} \times \boldsymbol{t}(t, \boldsymbol{y}, \boldsymbol{n}) ds_{\boldsymbol{y}}. \tag{2.16}$$

The integral on the left–hand side is the angular momentum of the material body at time t. The integrals on the right–hand side are the resultant torques due to body and surfaces forces, respectively.

Lemma 2.2 *For the Cauchy stress tensor as defined in* (2.11) *there hold the symmetry relations*

$$T_{32}(t,\boldsymbol{y}) = T_{23}(t,\boldsymbol{y}), \quad T_{13}(t,\boldsymbol{y}) = T_{31}(t,\boldsymbol{y}), \quad T_{21}(t,\boldsymbol{y}) = T_{12}(t,\boldsymbol{y}).$$

Proof: We first note that

$$\boldsymbol{y} \times [\varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})] = \varrho(t,\boldsymbol{y})[\boldsymbol{y} \times \boldsymbol{v}(t,\boldsymbol{y})],$$

hence we obtain, by using (2.8),

$$\begin{aligned} \frac{d}{dt}\int_{\omega(t)} [\boldsymbol{y} \times \varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})]\, d\boldsymbol{y} &= \frac{d}{dt}\int_{\omega(t)} \varrho(t,\boldsymbol{y})[\boldsymbol{y} \times \boldsymbol{v}(t,\boldsymbol{y})]\, d\boldsymbol{y} \\ &= \int_{\omega(t)} \varrho(t,\boldsymbol{y})\frac{d}{dt}[\boldsymbol{y} \times \boldsymbol{v}(t,\boldsymbol{y})]\, d\boldsymbol{y}\,. \end{aligned}$$

With the product rule

$$\frac{d}{dt}[\boldsymbol{y} \times \boldsymbol{v}(t,\boldsymbol{y})] = \boldsymbol{v}(t,\boldsymbol{y}) \times \boldsymbol{v}(t,\boldsymbol{y}) + \boldsymbol{y} \times \frac{d}{dt}\boldsymbol{v}(t,\boldsymbol{y}) = \boldsymbol{y} \times \frac{d}{dt}\boldsymbol{v}(t,\boldsymbol{y})$$

we further conclude

$$\frac{d}{dt}\int_{\omega(t)} \boldsymbol{y} \times \varrho(t,\boldsymbol{y})\boldsymbol{v}(t,\boldsymbol{y})\, d\boldsymbol{y} = \int_{\omega(t)} \boldsymbol{y} \times \varrho(t,\boldsymbol{y})\frac{d}{dt}\boldsymbol{v}(t,\boldsymbol{y})\, d\boldsymbol{y}.$$

Then the balance of angular momentum reads

$$\int_{\omega(t)} \boldsymbol{y} \times \varrho(t,\boldsymbol{y}) \left(\frac{d}{dt}\boldsymbol{v}(t,\boldsymbol{y}) - \boldsymbol{f}(t,\boldsymbol{y})\right) d\boldsymbol{y} \;=\; \int_{\partial\omega(t)} \boldsymbol{y} \times \boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n})\, ds_{\boldsymbol{y}}.$$

By using (2.11) we can write the surface integral as

$$\begin{aligned} \int_{\partial\omega(t)} \boldsymbol{y} \times \boldsymbol{t}(t,\boldsymbol{y},\boldsymbol{n}) ds_{\boldsymbol{y}} &= \int_{\partial\omega(t)} \boldsymbol{y} \times [\boldsymbol{T}(t,\boldsymbol{y})\boldsymbol{n}]\, ds_{\boldsymbol{y}} \\ &= \int_{\partial\omega(t)} \begin{pmatrix} \sum_{k=1}^{3} \Big[y_2 T_{3k}(t,\boldsymbol{y}) - y_3 T_{2k}(t,\boldsymbol{y})\Big] n_k \\ \sum_{k=1}^{3} \Big[y_3 T_{1k}(t,\boldsymbol{y}) - y_1 T_{3k}(t,\boldsymbol{y})\Big] n_k \\ \sum_{k=1}^{3} \Big[y_1 T_{2k}(t,\boldsymbol{y}) - y_2 T_{1k}(t,\boldsymbol{y})\Big] n_k \end{pmatrix} ds_{\boldsymbol{y}} \\ &= \int_{\omega(t)} \begin{pmatrix} \displaystyle\sum_{k=1}^{3} \frac{\partial}{\partial y_k}\Big[y_2 T_{3k}(t,\boldsymbol{y}) - y_3 T_{2k}(t,\boldsymbol{y})\Big] \\ \displaystyle\sum_{k=1}^{3} \frac{\partial}{\partial y_k}\Big[y_3 T_{1k}(t,\boldsymbol{y}) - y_1 T_{3k}(t,\boldsymbol{y})\Big] \\ \displaystyle\sum_{k=1}^{3} \frac{\partial}{\partial y_k}\Big[y_1 T_{2k}(t,\boldsymbol{y}) - y_2 T_{1k}(t,\boldsymbol{y})\Big] \end{pmatrix} d\boldsymbol{y} \end{aligned}$$

$$= \int_{\omega(t)} \begin{pmatrix} T_{32}(t,\boldsymbol{y}) - T_{23}(t,\boldsymbol{y}) + y_2 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{3k}(t,\boldsymbol{y}) - y_3 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{2k}(t,\boldsymbol{y}) \\ T_{13}(t,\boldsymbol{y}) - T_{31}(t,\boldsymbol{y}) + y_3 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{1k}(t,\boldsymbol{y}) - y_1 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{3k}(t,\boldsymbol{y}) \\ T_{21}(t,\boldsymbol{y}) - T_{12}(t,\boldsymbol{y}) + y_1 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{2k}(t,\boldsymbol{y}) - y_2 \sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{1k}(t,\boldsymbol{y}) \end{pmatrix} d\boldsymbol{y}$$

$$= \int_{\omega(t)} \begin{pmatrix} T_{32}(t,\boldsymbol{y}) - T_{23}(t,\boldsymbol{y}) \\ T_{13}(t,\boldsymbol{y}) - T_{31}(t,\boldsymbol{y}) \\ T_{21}(t,\boldsymbol{y}) - T_{12}(t,\boldsymbol{y}) \end{pmatrix} d\boldsymbol{y} + \int_{\omega(t)} \boldsymbol{y} \times \left(\sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{ik}(t,\boldsymbol{y}) \right)_{i=1,2,3} d\boldsymbol{y}.$$

Hence we have

$$\int_{\omega(t)} \boldsymbol{y} \times \varrho(t,\boldsymbol{y}) \left(\frac{d}{dt} \boldsymbol{v}(t,\boldsymbol{y}) - \boldsymbol{f}(t,\boldsymbol{y}) \right) d\boldsymbol{y} =$$

$$= \int_{\omega(t)} \begin{pmatrix} T_{32}(t,\boldsymbol{y}) - T_{23}(t,\boldsymbol{y}) \\ T_{13}(t,\boldsymbol{y}) - T_{31}(t,\boldsymbol{y}) \\ T_{21}(t,\boldsymbol{y}) - T_{12}(t,\boldsymbol{y}) \end{pmatrix} d\boldsymbol{y} + \int_{\omega(t)} \boldsymbol{y} \times \left(\sum_{k=1}^{3} \frac{\partial}{\partial y_k} T_{ik}(t,\boldsymbol{y}) \right)_{i=1,2,3} d\boldsymbol{y}$$

from which we conclude, by using (2.14),

$$\int_{\omega(t)} \begin{pmatrix} T_{32}(t,\boldsymbol{y}) - T_{23}(t,\boldsymbol{y}) \\ T_{13}(t,\boldsymbol{y}) - T_{31}(t,\boldsymbol{y}) \\ T_{21}(t,\boldsymbol{y}) - T_{12}(t,\boldsymbol{y}) \end{pmatrix} d\boldsymbol{y} = \boldsymbol{0}$$

for all control volumina $\omega(t)$, i.e. there hold the symmetry relations

$$T_{32}(t,\boldsymbol{y}) = T_{23}(t,\boldsymbol{y}), \quad T_{13}(t,\boldsymbol{y}) = T_{31}(t,\boldsymbol{y}), \quad T_{21}(t,\boldsymbol{y}) = T_{12}(t,\boldsymbol{y}).$$

■

2.6 Equilibrium Equations in Reference Coordinates

Next we will rewrite the Cauchy equilibrium equations (2.14) in terms of the reference coordinates $\boldsymbol{x} \in \Omega$. By introducing vectors $\boldsymbol{p}_i \in \mathbb{R}^n$, $i = 1, \ldots, n$, by

$$p_{ij}(t,\boldsymbol{y}) = T_{ij}(t,\boldsymbol{y}) \quad \text{for } j = 1, \ldots, n,$$

we can rewrite the equilibrium equations (2.14) as

$$\varrho(t,\boldsymbol{y}) \frac{d}{dt} v_i(t,\boldsymbol{y}) = \varrho(t,\boldsymbol{y}) f_i(t,\boldsymbol{y}) + \operatorname{div}_y \boldsymbol{p}_i(t,\boldsymbol{y}) \quad \text{for } i = 1, \ldots, n.$$

Now, by using $\boldsymbol{y} = \boldsymbol{\varphi}(t, \boldsymbol{x})$, $J(t) = \det \boldsymbol{F}$, (1.9) and (1.10) we obtain, for $i = 1, \ldots, n$,

$$\varrho(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) \frac{d}{dt} v_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) = \varrho(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) f_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) + \frac{1}{J(t)} \operatorname{div}_x \widetilde{\boldsymbol{p}}_i(t, \boldsymbol{x}),$$

where

$$\widetilde{\boldsymbol{p}}_i(t, \boldsymbol{x}) = J(t)\, \boldsymbol{F}^{-1} \boldsymbol{p}_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x}))\,.$$

Hence we have

$$J(t) \varrho(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) \frac{d}{dt} v_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) = J(t) \varrho(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) f_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) + \operatorname{div}_x \widetilde{\boldsymbol{p}}_i(t, \boldsymbol{x}),$$

and with (2.7) this gives

$$\varrho_0(\boldsymbol{x}) \frac{d}{dt} v_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) = \varrho_0(\boldsymbol{x}) f_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x})) + \operatorname{div}_x \widetilde{\boldsymbol{p}}_i(t, \boldsymbol{x}), \quad i = 1, \ldots, n.$$

When using the displacement (1.3) we further compute

$$\frac{d}{dt} \boldsymbol{v}(t, \boldsymbol{y}) = \frac{d^2}{dt^2} \boldsymbol{y}(t) = \frac{d^2}{dt^2} \boldsymbol{\varphi}(t, \boldsymbol{x}) = \frac{d^2}{dt^2} [\boldsymbol{x} + \boldsymbol{u}(t, \boldsymbol{x})] = \frac{d^2}{dt^2} \boldsymbol{u}(t, \boldsymbol{x}),$$

and with

$$\widetilde{f}_i(t, x) := f_i(t, \boldsymbol{\varphi}(t, \boldsymbol{x}))$$

we conclude

$$\varrho_0(\boldsymbol{x}) \frac{d^2}{dt^2} u_i(t, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}) \widetilde{f}_i(t, \boldsymbol{x}) + \operatorname{div}_x \widetilde{\boldsymbol{p}}_i(t, \boldsymbol{x}), \quad i = 1, \ldots, n.$$

A simple computation shows, recall the definition of $\boldsymbol{p}_i$,

$$\begin{pmatrix} \widetilde{p}_{11} & \widetilde{p}_{12} & \widetilde{p}_{13} \\ \widetilde{p}_{21} & \widetilde{p}_{22} & \widetilde{p}_{23} \\ \widetilde{p}_{31} & \widetilde{p}_{32} & \widetilde{p}_{33} \end{pmatrix} = J(t) \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \boldsymbol{F}^{-1},$$

i.e.

$$\boldsymbol{P}(t, x) := J(t) \boldsymbol{T}(t, \boldsymbol{\varphi}(t, \boldsymbol{x}))\, \boldsymbol{F}^{-\top} \tag{2.17}$$

defines the first Piola transformation. Therefore we can rewrite the equilibrium equations (2.15) in Lagrange coordinates as

$$\varrho_0(\boldsymbol{x}) \frac{d^2}{dt^2} \boldsymbol{u}(t, \boldsymbol{x}) = \varrho_0(\boldsymbol{x}) \widetilde{\boldsymbol{f}}(t, \boldsymbol{x}) + \operatorname{div}_x \boldsymbol{P}(t, \boldsymbol{x}). \tag{2.18}$$

Although the Cauchy stress tensor $\boldsymbol{T}(t, \boldsymbol{y})$ is symmetric, see Lemma 2.2, the first Piola transformation $\boldsymbol{P}(t, \boldsymbol{x})$ as defined in (2.17) is in general not symmetric. Hence we introduce the second Piola transformation

$$\boldsymbol{\Sigma}(t, \boldsymbol{x}) := \boldsymbol{F}^{-1} \boldsymbol{P} = J(t) \boldsymbol{F}^{-1} \boldsymbol{T}(t, \boldsymbol{\varphi}(t, \boldsymbol{x}))\, \boldsymbol{F}^{-\top}\,. \tag{2.19}$$

It remains to find suitable representations of the Cauchy stress tensor $\boldsymbol{T}$, the first Piola transform $\boldsymbol{P}$, and the second Piola transform $\boldsymbol{\Sigma}$, respectively.