EJIIM for Boundary Value Problems in 3D Elastic Microstructures

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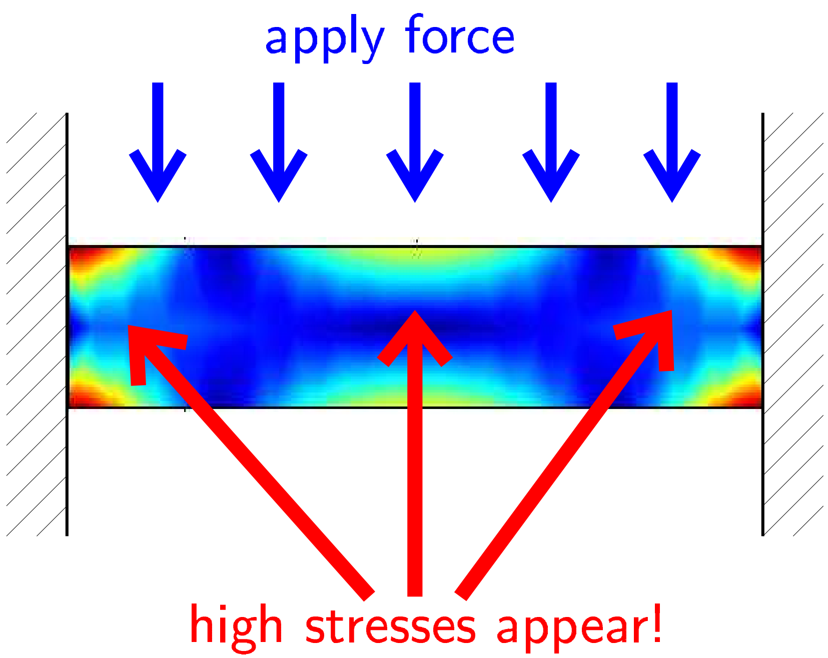
Graduiertenkolleg Mathematik & Praxis

Motivation: Shape Optimization

Optimization tasks:

- minimize the maximal stress
  (usually the von Mises stress used)
- minimize the displacements
- minimize the total volume
- . . .



(FEMLAB simulation)

Design based on **positions** of high/low stresses and/or strains and/or displacements e.g.,
higher stresses ⇒ add material
lower stresses ⇒ remove material

a method to compute the **local** stresses, strains and displacements **fast** and **accurately** is needed!

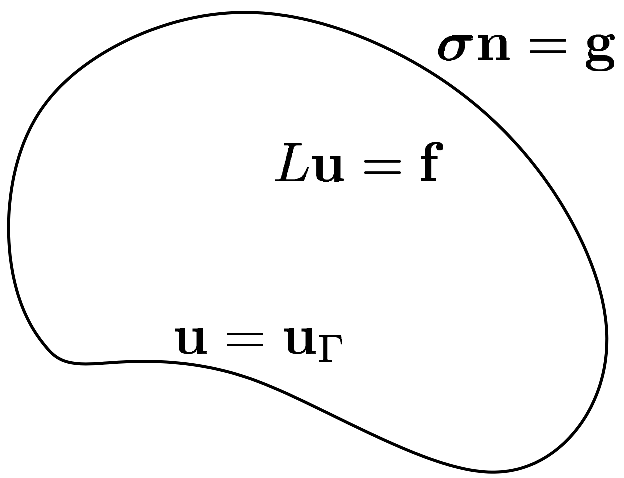
(Thanks to I. Matei for consultation)

Mathematical Model

Linear elasticity:

(“small” deformations, stresses and strains)

$$\nabla \cdot \left(C \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \right) = \mathbf{f}$$



C: stiffness tensor

$\mathbf{u}$: displacement vector

strain tensor: $\boldsymbol{\varepsilon} = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right)$

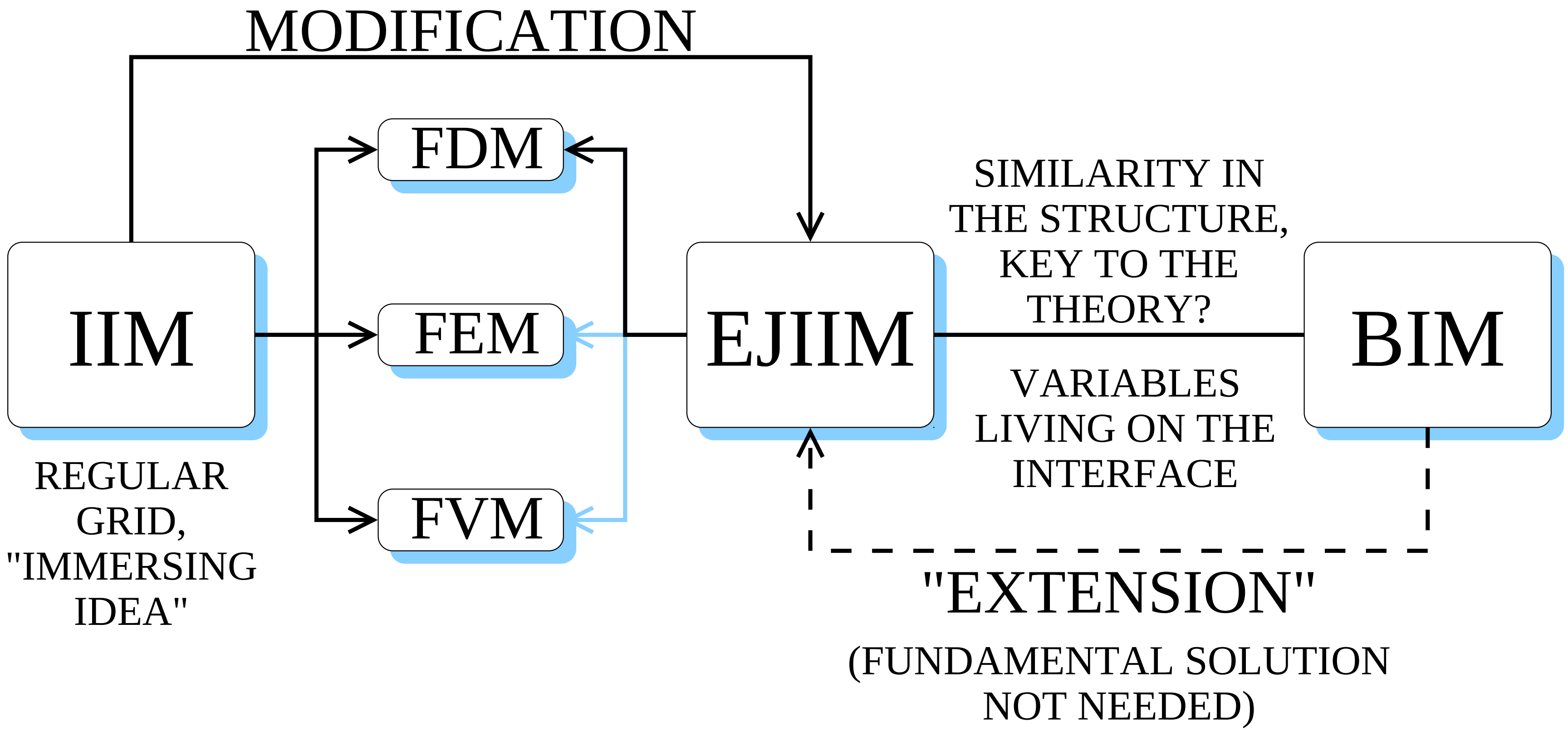
stress tensor: $\boldsymbol{\sigma} = C \boldsymbol{\varepsilon}$

Boundary conditions:

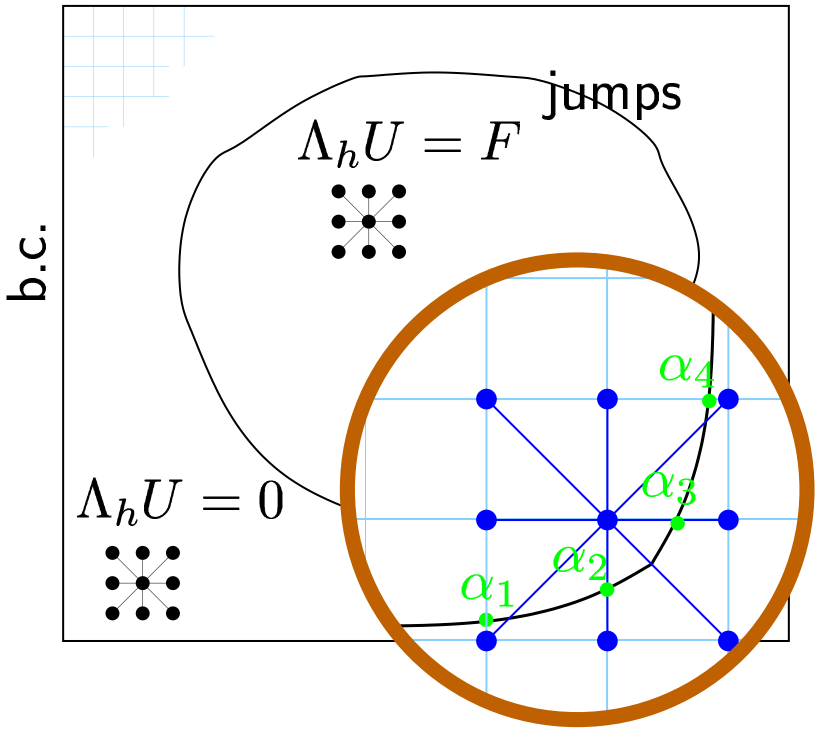
given displacements : $\mathbf{u} = \mathbf{u}_\Gamma$

given acting force : $\boldsymbol{\sigma}\mathbf{n} = \mathbf{g}$

Immersed Interface Methods



EJIIM Discretization



1. Embed the original domain in a "box". B.C.$\Rightarrow$ Jumps
2. Standard FD in the regular points

Irregular points:

$$\Lambda_h U + \text{correction} = F$$

$$\text{correction} = \sum_s \sum_{m=0}^{2} \psi_{m,s} [\partial^m u]_{\alpha_s}$$

Extrapolation + B.C. $\Rightarrow$

$$[\partial^m u]_{\alpha_s} = \tilde{F}_s - \sum_{i \in \text{grid}} d_{s,i} U_i$$

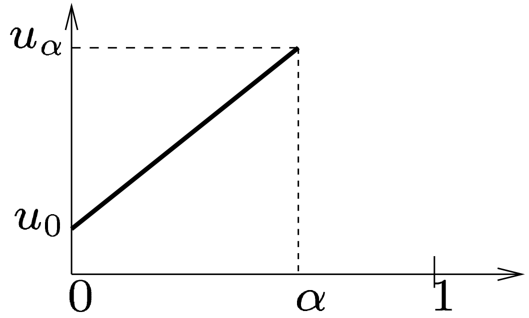
EJIIM system

$$\begin{pmatrix} \mathbf{A} & \mathbf{\Psi} \\ \mathbf{D} & \mathbf{I} \end{pmatrix} \begin{pmatrix} U \\ J \end{pmatrix} = \begin{pmatrix} F \\ \tilde{F} \end{pmatrix}$$

U	:	grid function	J	:	jumps
$\mathbf{A}$	:	standard FD matrix	F	:	extended rhs
$\mathbf{\Psi}$	:	corrections	$\mathbf{I}$	:	identity
$\mathbf{D}$, $\tilde{F}$	:	extrapolation			

EJIIM as BIM

1D problem [Wiegmann 99]



$$u_{xx} = 0,\ u(0) = u_0,\ u(\alpha) = u_\alpha$$

Continuous $(j := [u_x] = -u_x^-(\alpha))$

Discrete $(J := [u_x])$

$$u(x) + \int_0^1 G(x,y)\delta(y-\alpha)\frac{d}{dx}u(x)\,\mathrm{d}y \qquad (\mathbf{I} - \mathbf{A}^{-1}\mathbf{\Psi}_1\mathbf{D})U$$

$$= \int_0^1 G(x,y)\left(\delta'(y-\alpha)u_\alpha + \delta'(y-0)u_0\right)\,\mathrm{d}y \qquad = -\mathbf{A}^{-1}\mathbf{\Psi}_0[u]$$

$$j + \frac{d}{dx}\int_0^1 G(x,y)\delta(y-\alpha)j\,\mathrm{d}y \qquad (\mathbf{I} - \mathbf{D}\mathbf{A}^{-1}\mathbf{\Psi}_1)J$$

$$= -\frac{d}{dx}\int_0^1 G(x,y)(\delta'(y-\alpha)u_\alpha + \delta'(y-0)u_0)\,\mathrm{d}y \qquad = -\mathbf{D}\mathbf{A}^{-1}\mathbf{\Psi}_0[u]$$

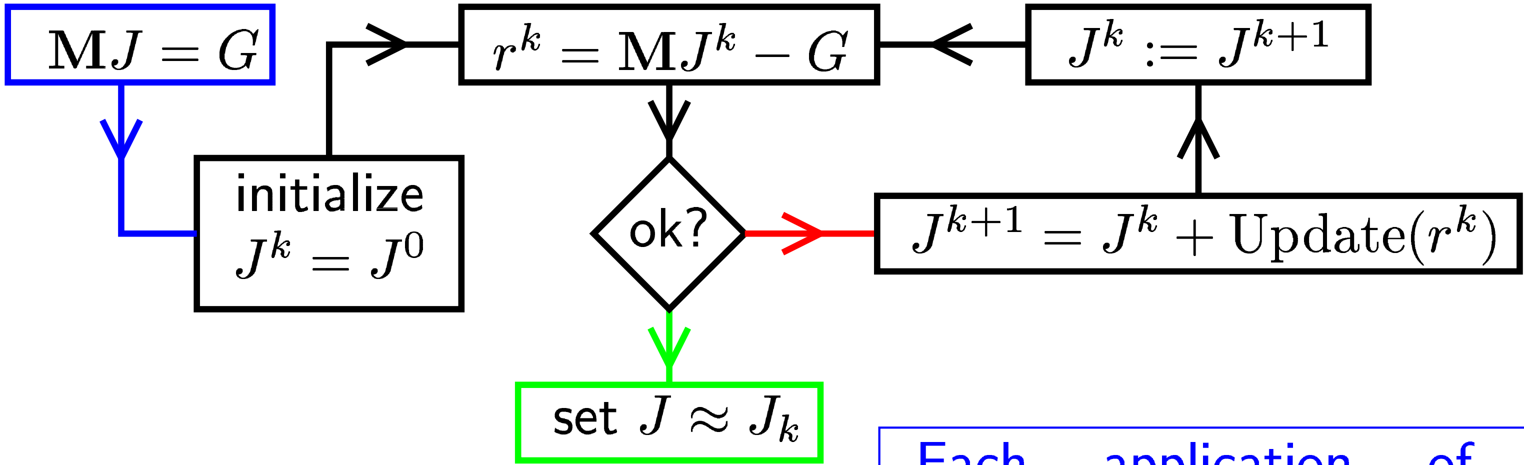
Fast Solver

[Li 95, Wiegmann 99]

Schur complement for jumps:

$$\begin{pmatrix} \mathbf{A} & \mathbf{\Psi} \\ \mathbf{D} & \mathbf{I} \end{pmatrix} \begin{pmatrix} U \\ J \end{pmatrix} = \begin{pmatrix} F \\ \tilde{F} \end{pmatrix} \Rightarrow (\mathbf{I} - \mathbf{D}\mathbf{A}^{-1}\mathbf{\Psi})J = \tilde{F} - \mathbf{D}\mathbf{A}^{-1}F$$

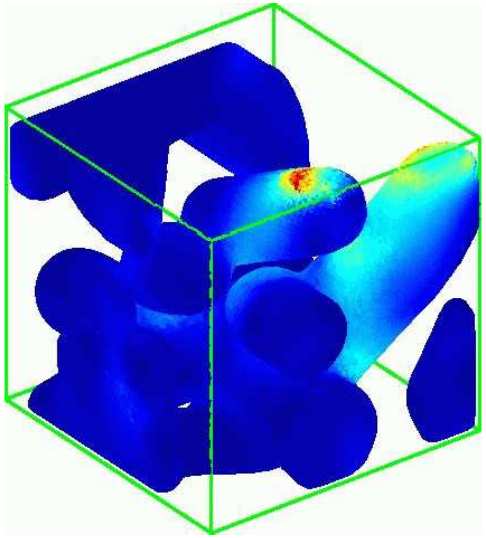
Iterative solver (e.g. BiCGSTAB) for Schur complement:



Displacements: $U = \mathbf{A}^{-1}(F - \mathbf{\Psi}J)$

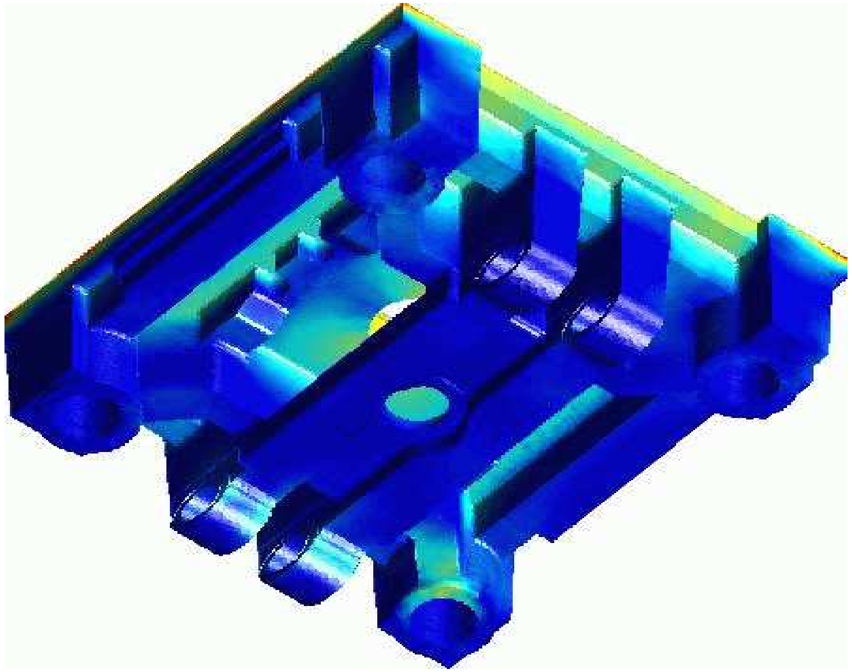
Each application of $\mathbf{M}$ requires application of $\mathbf{A}^{-1}$ to a vector. This is done in $N \log N$ time using FFT

Code



INPUT:

Geometry information,
material parameters



Geometry data prepared by
I.Matei and A.K.Vaikuntam

OUTPUT:

Deformations,
stresses and strains

(MATLAB implementation)

Summary

- **EJIIM:**
  - standard FD used + correction terms + embedding of the domain in a "box" $\Rightarrow$ enable FFT **fast solvers**
  - 2-nd order convergence in max-norm in displacements
  - Geometry information provided by the *Level Set Method* and/or some engineering programms.

- **Open problems:**
  - Convergence theory and error estimates
  - Stability analysis
  - Improvements in the algorithm (mostly in computational memory requirements!)