Institute of Applied Analysis and Numerical Simulation
University of Stuttgart

Söllerhaus 2004

The coupling of electrical eddy current heat production and air cooling

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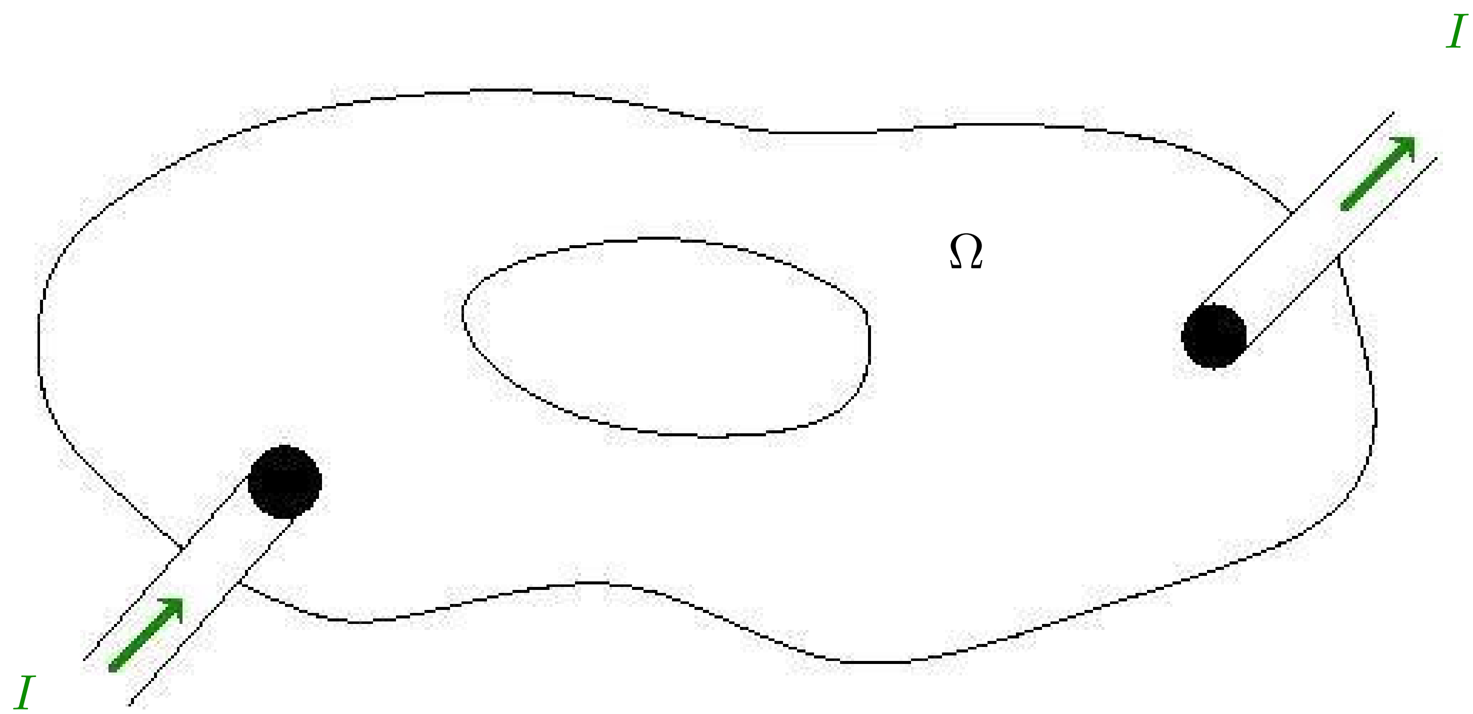
Outline of the talk

1. Formulation of the problem
2. Mathematical tools
3. Modelling of the boundary condition
4. Solution of the eddy current problem
5. Some remarks on the heat transfer
6. Fast BEM via <u>A</u>daptive <u>C</u>ross <u>A</u>pproximation
7. Numerical examples

Introduction



Given data: amperage I
frequency ω

Unknown: current density $\mathbf{j}$ in Ω
$\rightsquigarrow$ elastostatics, elastodynamics
$\rightsquigarrow$ thermodynamics

Model for the current density

Maxwell's equations

$$\mathbf{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \mathrm{div}\,\mathbf{D} = 0,$$
$$\mathbf{curl}\,\mathbf{H} = \mathbf{j} + \frac{\partial D}{\partial t}, \quad \mathrm{div}\,\mathbf{B} = 0.$$

Linear material and Ohm's law

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{j} = \sigma\mathbf{E}.$$

for piecewise homogeneous and isotropic material, i.e. ε, μ and σ are piecewise constant. For simplicity only the constant case is considered in this presentation.

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Time harmonic equations

$$\mathbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H}, \qquad \mathrm{div}\,\varepsilon\mathbf{E} = 0,$$
$$\mathbf{curl}\,\mathbf{H} = (\sigma + i\omega\varepsilon)\mathbf{E}, \quad \mathrm{div}\,\mu\mathbf{H} = 0.$$

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Time harmonic equations and neglection of displacement currents (eddy currents model)

$$\textbf{curl}\,\mathbf{E} = -i\omega\mu\mathbf{H}, \qquad \text{div}\,\varepsilon\mathbf{E} = 0,$$

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Elimination of the magnetic field results in the system

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + \kappa^2\mathbf{E} = 0 \quad \text{in } \Omega,$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = 0 \quad \text{in } \Omega^c,$$

$$\mathrm{div}\,\mathbf{E} = 0 \quad \text{in } \Omega^c \text{ (gauge condition).}$$

with the complex wavenumber $\kappa := (1+i)\frac{\sqrt{2}}{2}\sqrt{\omega\mu\sigma}$.

Mathematical tools

$$\mathbf{H}(\mathbf{curl}\,,\Omega) := \{\mathbf{V} \in \mathbf{L}_2(\Omega) : \mathbf{curl}\,\mathbf{V} \in \mathbf{L}_2(\Omega)\}$$

Energy spaces:

$$\mathbf{W}(\mathbf{curl}\,,\Omega^c) := \left\{ \frac{\mathbf{V}}{\sqrt{1+|\cdot|^2}} \in \mathbf{L}_2(\Omega^c) : \mathbf{curl}\,\mathbf{V} \in \mathbf{L}_2(\Omega^c) \right\}$$

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Trace operators:

$$\gamma_D \mathbf{U} := \mathbf{n} \times (\mathbf{U}_{|\Gamma} \times \mathbf{n}), \quad \gamma_N \mathbf{U} := \mathbf{curl}\,\mathbf{U}_{|\Gamma} \times \mathbf{n}.$$

$$\gamma_\times \mathbf{U} := \mathbf{U}_{|\Gamma} \times \mathbf{n}, \quad \gamma_n \mathbf{U} := \mathbf{U}_{|\Gamma} \cdot \mathbf{n}, \quad \mathsf{R}\mathbf{u} := \mathbf{u} \times \mathbf{n}.$$

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Trace spaces:

$$\mathbf{H}_{||}^{\frac{1}{2}}(\Gamma) := \{\text{tang. fields in } \mathbf{H}_{pw}^{\frac{1}{2}}(\Gamma) \text{ with weakly count. tang. comp. at edges}\}$$

$$\mathbf{H}_{\perp}^{\frac{1}{2}}(\Gamma) := \{\text{tang. fields in } \mathbf{H}_{pw}^{\frac{1}{2}}(\Gamma) \text{ with weakly count. norm. comp. at edges}\}$$

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Surface derivatives:

$$\nabla_\Gamma u := \mathbf{n} \times (\nabla u_{|\Gamma} \times \mathbf{n}), \quad \mathbf{curl}_\Gamma\, u := \nabla_\Gamma u \times \mathbf{n}.$$

$$\mathrm{div}_\Gamma := -{\nabla_\Gamma}^*, \quad \mathrm{curl}_\Gamma := {\mathbf{curl}_\Gamma}^*, \quad \Delta_\Gamma := \mathrm{div}_\Gamma \nabla_\Gamma.$$

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Trace spaces:

$$\mathbf{H}^{-\frac{1}{2}}_{\perp}(\mathrm{curl}_\Gamma\,,\Gamma) := \left\{ \mathbf{V} \in \mathbf{H}^{-\frac{1}{2}}_{\perp}(\Gamma) : \mathrm{curl}_\Gamma\,\mathbf{V} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) \right\}$$

$$\mathbf{H}^{-\frac{1}{2}}_{||}(\mathrm{div}_\Gamma\,,\Gamma) := \left\{ \mathbf{V} \in \mathbf{H}^{-\frac{1}{2}}_{||}(\Gamma) : \mathrm{div}_\Gamma\,\mathbf{V} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) \right\}$$

For details see: A. Buffa, *Some numerical and theoretical problems in computational eletromagnetism*, thesis, 2000.
Buffa, Ciarlet (2001)

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Boundary value problem

Eddy current equations

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + \kappa^2 \mathbf{E} = 0 \quad \text{in } \Omega$$

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = 0 \quad \text{in } \Omega^c,$$

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Continuous tangential component (Dirichlet data)

$$\left[\gamma_D \mathbf{E}\right]_\Gamma := \gamma_D^c \mathbf{E} - \gamma_D \mathbf{E} = 0 \quad \text{on } \Gamma$$

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$$\begin{aligned} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + \kappa^2 \mathbf{E} &= 0 \quad \text{in } \Omega \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} &= 0 \quad \text{in } \Omega^c, \\ \operatorname{div} \mathbf{E} &= 0 \quad \text{in } \Omega^c \end{aligned}$$

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Transmission condition (Neumann data)

$$\left[\frac{1}{\mu}\gamma_N \mathbf{E}\right]_\Gamma = \mathcal{J} = \mathcal{J}(I,\omega) \quad \text{on } \Gamma$$

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Radiation condition

$$\mathbf{E} = \mathcal{O}(|x|^{-1}), \quad \mathbf{curl}\,\mathbf{E} = \mathcal{O}(|x|^{-2}) \quad \text{uniformly for } |x| \to \infty$$

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Gauge condition

$$\int_{\Gamma_k} \mathbf{E} \cdot \mathbf{n}\, dS_x = 0, \quad k = 1, \ldots, Z := \#\ \text{connect. comp. of } \Gamma$$

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$\Rightarrow$ unique solution for *Im* $\kappa > 0$ and continuous right hand side $\langle\mathcal{J},\cdot\rangle$

For details see: R. Hiptmair, *Symmetric coupling for eddy current problems*, SIAM J. Numer. Anal., Vol. 40, No. 1, 2002.

Computation of $\mathcal{J}$

Due to

$$\begin{aligned} \langle \mathsf{div}_\Gamma\,(\gamma_N \mathbf{E}), \phi \rangle &= -\langle \mathbf{curl}\,\mathbf{E} \times \mathbf{n}, \nabla_\Gamma \phi \rangle = \langle \mathbf{curl}\,\mathbf{E}, \nabla_\Gamma \phi \times \mathbf{n} \rangle \\ &= \langle \mathsf{curl}_\Gamma\,\mathbf{curl}\,\mathbf{E}, \phi \rangle = \langle (\mathbf{curl}\,\mathbf{curl}\,\mathbf{E}) \cdot \mathbf{n}, \phi \rangle \end{aligned}$$

each solution of the BVP satisfies

$$\mathsf{div}_\Gamma\,(\gamma_N \mathbf{E}) = -\kappa^2 \gamma_n \mathbf{E}$$

and

$$\mathsf{div}_\Gamma\,(\gamma_N^c \mathbf{E}) = 0$$

respectively.

$\longrightarrow$

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Computation of $\mathcal{J}$:

1. Compute $\gamma_n \mathbf{E}$ from the given data I and ω

$\longrightarrow$

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1. Compute $\gamma_n \mathbf{E}$ from the given data I and ω
2. Compute $\mathcal{J}$ satisfying

$$\mathrm{div}_\Gamma\,\mathcal{J} = \mathrm{div}_\Gamma\,\left[\frac{1}{\mu}\gamma_N \mathbf{E}\right]_\Gamma = \frac{\kappa^2}{\mu}\gamma_n\mathbf{E} = i\omega\sigma\gamma_n\mathbf{E} =: -f$$

$\longrightarrow$

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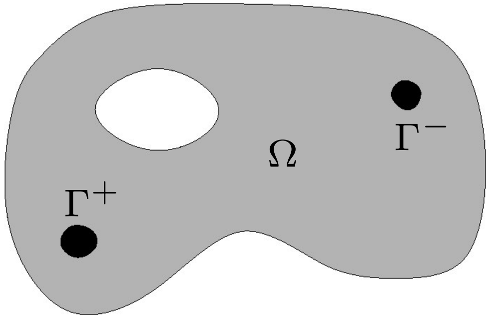
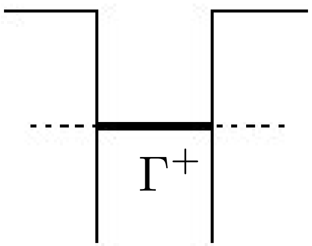
The above relations can also be used to realize a domain decomposition method via a Dirichlet–Neumann map. $\longrightarrow$

First step

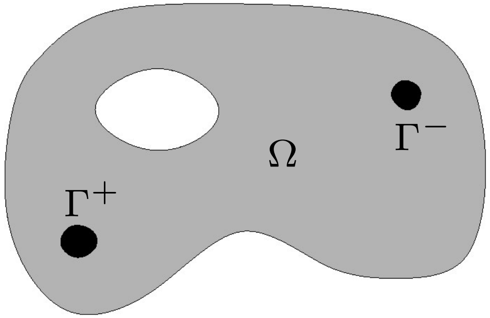


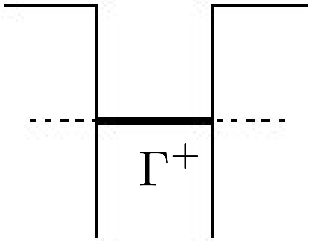


Solve field equations in an infinity cylinder over $\Gamma^\pm$ with the side condition

$$\int_{\Gamma^\pm} \sigma \mathbf{E}_\pm \cdot \mathbf{n}\, dx = I_\pm$$

First step



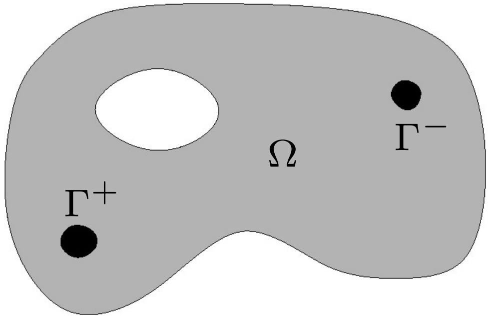


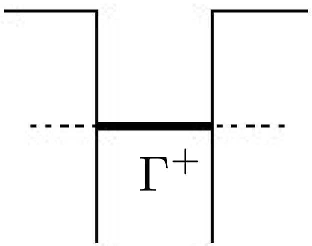
Solve field equations in an infinity cylinder over $\Gamma^{\pm}$ with the side condition

$$\int_{\Gamma^{\pm}} \sigma \mathbf{E}_{\pm} \cdot \mathbf{n} \, dx = I_{\pm}$$

Split the current density into a source field $\mathbf{j}_s := \sigma \mathbf{E}_s$ and reaction field $\mathbf{j}_r := \sigma \mathbf{E}_r$.

First step





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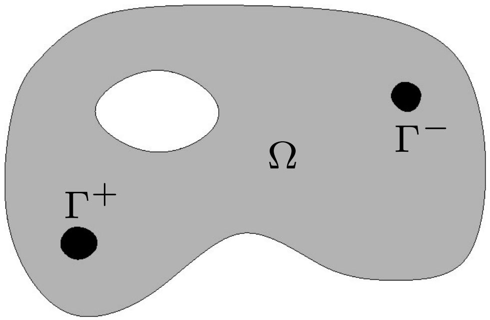
Split the current density into a source field $\mathbf{j}_s := \sigma \mathbf{E}_s$ and reaction field $\mathbf{j}_r := \sigma \mathbf{E}_r$. Exploiting symmetry and Maxwell equations yields the twodimensional scalar system

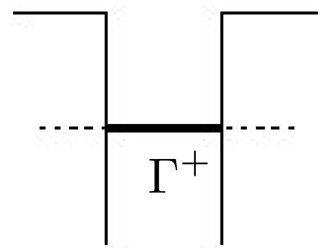
$$\begin{aligned} -\Delta e_r + \kappa^2 e_r &= -i\omega\mu j_s && \text{in } \Gamma \\ -\Delta e_r &= 0 && \text{in } \Gamma^c \\ \left[\gamma_0 e_r\right]_\Gamma &= 0 && \text{on } \partial\Gamma \\ \left[\mu^{-1}\gamma_1 e_r\right]_\Gamma &= 0 && \text{on } \partial\Gamma \end{aligned}$$

From the side condition one can derive

$$j_s = \frac{1}{|\Gamma|}\left(I - \int_\Gamma \sigma e_r\, dx\right).$$

First step





Solve field equations in an infinity cylinder over $\Gamma^{\pm}$ with the side condition

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$$j_s = \frac{1}{|\Gamma|}\left(I - \int_\Gamma \sigma e_r \, dx\right).$$

The system has to be solved on each contact.
$\leadsto$ 2D–BEM or analytical solution for simple geometries

Second step

We use the Hodge decomposition

$$\mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\,,\Gamma) = \nabla_\Gamma(\mathcal{H}(\Gamma)) \oplus \mathbf{curl}_\Gamma\,(H^{\frac{1}{2}}(\Gamma)/\mathbb{C}^Z),$$

using the space

$$\mathcal{H}(\Gamma) := \left\{\phi \in H^1(\Gamma)/\mathbb{C}^Z : \Delta_\Gamma\phi \in H^{-\frac{1}{2}}(\Gamma)/\mathbb{C}^Z\right\}.$$

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$$\mathcal{J} = \nabla_\Gamma \phi + \mathbf{curl}_\Gamma\,\rho$$

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Due to

$$-\langle \mathsf{div}_\Gamma\,\mathcal{J}, 1\rangle = \langle \mathcal{J}, \nabla_\Gamma 1\rangle = 0$$

on each component of Γ we get the solvability condition

$$\langle f, 1\rangle = \int_{\Gamma_k} \mathbf{E}\cdot\mathbf{n}\,dS_x = 0,\ \ k = 1,\ldots,Z \quad (\widehat{=}\ \text{continuity equation})$$

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Second step

Solve the variational problem:

Find $\phi \in H^1(\Gamma)/\mathbb{C}^Z$ with

$$\langle \nabla_\Gamma \phi, \nabla_\Gamma \psi \rangle_{\mathbf{L_2(\Gamma)}} = \langle f, \psi \rangle_{L_2(\Gamma)} \tag{1}$$

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The bilinear form is continuous and elliptic

$$\langle \nabla_\Gamma \phi, \nabla_\Gamma \phi \rangle_{\mathbf{L}_2(\Gamma)} = \|\nabla_\Gamma \phi\|^2_{\mathbf{L}^2(\Gamma)} \geq c\, \|\phi\|^2_{H^1(\Gamma)} \text{ for all } \phi \in H^1(\Gamma)/\mathbb{C}^Z.$$

Using Lax–Milgram we have the existence of a unique solution $\phi \in H^1(\Gamma)/\mathbb{C}^Z$ with $\phi \in \mathcal{H}(\Gamma)$.

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In the continuous case one can get the solution $\mathcal{J}$ from the surface potential

$$\mathcal{J} := \nabla_\Gamma \phi \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$$

The contribution $\mathbf{curl}_\Gamma \, \rho$ corresponds to $\gamma_n \mathbf{E} = 0$.

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The contribution $\mathbf{curl}_\Gamma\, \rho$ corresponds to $\gamma_n \mathbf{E} = 0$.
The discretization $\phi_h \in S^1(\Gamma) \subset H^1(\Gamma)$ is not possible because of $\Delta_\Gamma \phi_h \in H^{-1}(\Gamma)$.

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Second step

Consider the corresponding saddle point problem:

Find $\mathcal{J} \in \mathbf{H}(\mathsf{div}_\Gamma, \Gamma)$ and $\phi \in L_2(\Gamma)/\mathbb{C}^Z$ such that

$$\begin{aligned} \langle \mathcal{J}, \mathcal{K} \rangle_{\mathbf{L}_2(\Gamma)} + \langle \mathsf{div}_\Gamma \mathcal{K}, \phi \rangle_{L_2(\Gamma)} &= 0 \\ -\langle \mathsf{div}_\Gamma \mathcal{J}, \psi \rangle_{L_2(\Gamma)} &= \langle f, \psi \rangle_{L_2(\Gamma)} \end{aligned}$$

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Adding the second equation with $\psi = \mathrm{div}_\Gamma \mathcal{K}$ and $\langle \nabla_\Gamma \phi, \nabla_\Gamma \psi \rangle_{\mathbf{L_2}(\Gamma)} = \langle f, \psi \rangle_{L_2(\Gamma)}$ leads to the modified saddle point problem:

Find $\mathcal{J} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ und $\phi \in H^1(\Gamma)/\mathbb{C}^Z$ such that

$$\langle \mathcal{J}, \mathcal{K} \rangle_{\mathbf{H}(\mathrm{div}_\Gamma, \Gamma)} + \langle \mathrm{div}_\Gamma \mathcal{K}, \phi \rangle_{L_2(\Gamma)} = -\langle f, \mathrm{div}_\Gamma \mathcal{K} \rangle_{L_2(\Gamma)} \quad (2)$$

$$-\langle \mathrm{div}_\Gamma \mathcal{J}, \psi \rangle_{L_2(\Gamma)} + \langle \nabla_\Gamma \phi, \nabla_\Gamma \psi \rangle_{\mathbf{L_2}(\Gamma)} = 2\langle f, \psi \rangle_{L_2(\Gamma)} \quad (3)$$

holds for all $\mathcal{K} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ and all $\psi \in H^1(\Gamma)/\mathbb{C}^Z$.

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System (2,3) is elliptic.

$\Rightarrow$ unique solution $\mathcal{J} \in \mathbf{H}(\mathrm{div}_\Gamma, \Gamma)$ and $\phi \in H^1(\Gamma)/\mathbb{C}^Z$.

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Conforming Galerkin discretization

$\Rightarrow$ unique solution $\mathcal{J}_h \in \mathcal{RT}(\Gamma)$ (Raviart Thomas elements) and $\phi_h \in S^1(\Gamma)$

$\Rightarrow$ quasi optimal error estimates

BEM for eddy currents

Single layer potential (scalar and vectorial)

$$\Psi_V^\kappa[\sigma](x) := \int_\Gamma G_\kappa(x,y)\sigma(y)\,dS_y, \qquad \mathbf{\Psi}_{\mathbf{A}}^\kappa[\mathbf{\Sigma}](x) := \int_\Gamma G_\kappa(x,y)\mathbf{\Sigma}(y)\,dS_y, \quad x \in \mathbb{R}^3 \setminus \Gamma$$

Double layer potential

$$\mathbf{\Psi}_{\mathbf{M}}^\kappa[\mathbf{V}](x) := \mathbf{curl}\,_x \mathbf{\Psi}_{\mathbf{A}}^\kappa(\mathsf{R}\mathbf{V})(x), \quad x \in \mathbb{R}^3 \setminus \Gamma$$

with the fundamental solution of the Helmholtz–type operator,

$$G_\kappa(x,y) := \frac{1}{4\pi}\frac{e^{-\kappa|x-y|}}{|x-y|}.$$

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Scalar boundary integral operators

$$V_\kappa := \gamma_0 \Psi_V^\kappa$$

Vectorial boundary integral operators

$$\mathbf{A}_\kappa := \gamma_D \mathbf{\Psi}_{\mathbf{A}}^\kappa, \quad \mathbf{B}_\kappa := \frac{1}{2}(\gamma_N + \gamma_N^c)\mathbf{\Psi}_{\mathbf{A}}^\kappa, \quad \mathbf{C}_\kappa := \frac{1}{2}(\gamma_D + \gamma_D^c)\mathbf{\Psi}_{\mathbf{M}}^\kappa, \quad \mathbf{N}_\kappa := \gamma_N \mathbf{\Psi}_{\mathbf{M}}^\kappa.$$

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Boundary integral equations $\longrightarrow$

$$\gamma_D \mathbf{E} = \left(\frac{1}{2}\mathbf{I} + \mathbf{C}_\kappa\right)[\gamma_D \mathbf{E}] + \mathbf{A}_\kappa[\gamma_N \mathbf{E}] + (\nabla_\Gamma \circ V_\kappa)[\gamma_n \mathbf{E}]$$

$$\gamma_N \mathbf{E} = \mathbf{N}_\kappa[\gamma_D \mathbf{E}] + \left(\frac{1}{2}\mathbf{I} + \mathbf{B}_\kappa\right)[\gamma_N \mathbf{E}]$$

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BEM for eddy currents

Split the Neumann data $\gamma_N \mathbf{E}$ into known part $\mathcal{J}$ and unknown part λ

$$\gamma_N \mathbf{E} =: \mu(-\mathcal{J} + \boldsymbol{\lambda}), \quad \gamma_N^c \mathbf{E} =: \mu_0 \boldsymbol{\lambda}^c = \mu_0 \boldsymbol{\lambda}$$

then it holds

$$\mathrm{div}_\Gamma \boldsymbol{\lambda} = 0.$$

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We can choose the unknown $\boldsymbol{\lambda}$ in the constraint space

$$\mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma \, \mathbf{0}, \Gamma) := \left\{ \mathbf{V} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma \, , \Gamma) : \mathrm{div}_\Gamma \, \mathbf{V} = 0 \right\}.$$

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Testing the first integral equation

$$\gamma_D \mathbf{E} = \left(\frac{1}{2} \mathbf{I} + \mathbf{C}_\kappa \right) [\gamma_D \mathbf{E}] + \mathbf{A}_\kappa [\gamma_N \mathbf{E}] + (\nabla_\Gamma \circ V_\kappa) [\gamma_n \mathbf{E}]$$

with $\boldsymbol{\theta} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma\, \mathbf{0}, \Gamma)$ it follows from partial integration on the surface

$$\langle (\nabla_\Gamma \circ V_\kappa)[\gamma_n \mathbf{E}], \boldsymbol{\theta} \rangle = -\langle V_\kappa [\gamma_n \mathbf{E}], \mathrm{div}_\Gamma\, \boldsymbol{\theta} \rangle = 0.$$

All the term including the normal trace $\gamma_n \mathbf{E}$ vanish.

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All the term including the normal trace $\gamma_n \mathbf{E}$ vanish.
Finally we set

$$\gamma_D \mathbf{E} = \gamma_D^c \mathbf{E} =: \mathbf{u} \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma\,, \Gamma)$$

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Variational formulation

Find $(\mathbf{u},\boldsymbol{\lambda}) \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma\,,\Gamma) \times \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\,\mathbf{0},\Gamma)$ with

$$\begin{aligned}
\langle \boldsymbol{\theta}, (\mu_0 A_0 + \mu A_\kappa)\boldsymbol{\lambda}\rangle + \langle \boldsymbol{\theta}, (C_0 + C_\kappa)\mathbf{u}\rangle &= \langle \boldsymbol{\theta}, \mu A_\kappa \mathcal{J}\rangle \\
\langle (B_0 + B_\kappa)\boldsymbol{\lambda}, \mathbf{v}\rangle + \left\langle \left(\frac{1}{\mu_0} N_0 + \frac{1}{\mu} N_\kappa\right)\mathbf{u}, \mathbf{v}\right\rangle &= \left\langle \left(-\frac{1}{2}\mathbf{I} + \mathbf{B}_\kappa\right)\mathcal{J}, \mathbf{v}\right\rangle
\end{aligned}$$

for all $(\mathbf{v},\boldsymbol{\theta}) \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma\,,\Gamma) \times \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\,\mathbf{0},\Gamma)$.

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Due to $C_\kappa = -B_\kappa^*$ the problem is skew symmetric.

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Due to $C_\kappa = -B_\kappa^*$ the problem is skew symmetric. Further we have the estimate

$$|\langle \boldsymbol{\lambda}, A_\kappa \boldsymbol{\lambda}\rangle| \geq c\, \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} \qquad \text{for all } \boldsymbol{\lambda} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma),$$

which gives ellipticity of the operator A_κ

$$|\langle \boldsymbol{\lambda}, A_\kappa \boldsymbol{\lambda}\rangle| \geq c\, \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} = c\, \|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} \qquad \text{for all } \boldsymbol{\lambda} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma\, 0, \Gamma).$$

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$$|\langle \boldsymbol{\lambda}, A_\kappa \boldsymbol{\lambda}\rangle| \geq c\,\|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\Gamma)} = c\,\|\boldsymbol{\lambda}\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma, \Gamma)} \qquad \text{for all } \boldsymbol{\lambda} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\, 0, \Gamma).$$

Integration by parts

$$\langle N_\kappa \mathbf{u}, \mathbf{v}\rangle = \kappa^2 \langle \mathsf{R}\mathbf{v}, A_\kappa(\mathsf{R}\mathbf{u})\rangle + \langle V_\kappa(\mathsf{curl}_\Gamma\, \mathbf{u}), \mathsf{curl}_\Gamma\, \mathbf{v}\rangle$$

and ellipticity of V_κ yields the ellipticity of the Operator N_κ on $\mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma, \Gamma)$.

Variational formulation

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The variational problem:

Find $(\mathbf{u}, \boldsymbol{\lambda}) \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma, \Gamma) \times \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\, \mathbf{0}, \Gamma)$ with

$$\begin{aligned} \langle \boldsymbol{\theta}, (\mu_0 A_0 + \mu A_\kappa)\boldsymbol{\lambda}\rangle + \langle \boldsymbol{\theta}, (C_0 + C_\kappa)\mathbf{u}\rangle &= \langle \boldsymbol{\theta}, \mu A_\kappa \mathcal{J}\rangle \\ \langle (B_0 + B_\kappa)\boldsymbol{\lambda}, \mathbf{v}\rangle + \left\langle \left(\frac{1}{\mu_0} N_0 + \frac{1}{\mu} N_\kappa\right)\mathbf{u}, \mathbf{v}\right\rangle &= \left\langle \left(-\frac{1}{2}\mathbf{I} + \mathbf{B}_\kappa\right)\mathcal{J}, \mathbf{v}\right\rangle \end{aligned}$$

for all $(\mathbf{v}, \boldsymbol{\theta}) \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma, \Gamma) \times \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\, \mathbf{0}, \Gamma)$ is

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1. Conforming Galerkin discretization requires discrete subspace $X_h \subset \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma\, 0, \Gamma)$
   (remark: $\ker(\mathsf{div}_\Gamma) \neq Im(\mathsf{curl}_\Gamma)$ for multiply connected domains
   domain decomposition $\rightsquigarrow \mathsf{div}_\Gamma\, \gamma_N \mathbf{E} = 0$ on a part of the boundary)

Variational formulation

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2. Enforce the constraint $\mathrm{div}_{\Gamma}\, \boldsymbol{\lambda} = 0$ via Lagrangian multipliers
   $\rightsquigarrow$ also works for the domain decomposition

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Variational formulation

Using operator notation we get

$$\begin{pmatrix} A & -B^* \\ B & N \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}$$

with the constraint $\mathsf{div}_\Gamma \boldsymbol{\lambda} = 0$. Let the operator D be defined by

$$\langle D\boldsymbol{\lambda}, \psi \rangle := \langle \mathsf{div}_\Gamma \boldsymbol{\lambda}, V\psi \rangle \quad \text{for all } \psi \in H^{-\frac{1}{2}}(\Gamma),$$

then the saddle point problem with Lagrangian multiplier $p \in H^{-\frac{1}{2}}(\Gamma)$ reads as follows

$$\begin{pmatrix} A & -B^* & -D^* \\ B & N & 0 \\ D & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ 0 \end{pmatrix}.$$

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Adding the stabilization term $0 = \langle V(\operatorname{div}_\Gamma \boldsymbol{\lambda}), \operatorname{div}_\Gamma \boldsymbol{\theta} \rangle$ leads to a new operator $\widetilde{A}$ via

$$\left\langle \boldsymbol{\theta}, \widetilde{A}\boldsymbol{\lambda} \right\rangle := \langle \boldsymbol{\theta}, A\boldsymbol{\lambda} \rangle + \langle V(\operatorname{div}_\Gamma \boldsymbol{\lambda}), \operatorname{div}_\Gamma \boldsymbol{\theta} \rangle$$

We then get the stabilized system

$$\begin{pmatrix} \widetilde{A} & -B^* & -D^* \\ B & N & 0 \\ D & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\lambda} \\ \mathbf{u} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ 0 \end{pmatrix}$$

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$$\mathbf{u} = N^{-1}(\mathbf{g}_2 - B\boldsymbol{\lambda})$$

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The Schur complement $S := (\widetilde{A} + B^* N^{-1} B)$ is $\mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$–elliptic

$$\boldsymbol{\lambda} = S^{-1}(\mathbf{g}_1 + B^* \mathbf{g}_2 + D^* p)$$

using the properties of $\widetilde{A}$. For the Lagrangian multiplier we then get the equation

$$DS^{-1}D^* p = -DS^{-1}\mathbf{g}_1 - DS^{-1}B^* N^{-1}\mathbf{g}_2.$$

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From the relation $\ker(DS^{-1}D^*) = \ker(D^*) = \mathsf{lin}\,\{V^{-1}1_z\}$ the invertibility of $DS^{-1}D^*$ on $H^{-\frac{1}{2}}(\Gamma)/\mathbb{C}^Z$ immediately follows.

$\Rightarrow$ unique solution $\boldsymbol{\lambda} \in \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma, \Gamma)$, $\mathbf{u} \in \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma, \Gamma)$ and $p \in H^{-\frac{1}{2}}(\Gamma)/\mathbb{C}^Z$

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Conforming Galerkin discretization

$$\boldsymbol{\lambda}_h \in \mathcal{RT}(\Gamma) \subset \mathbf{H}_{||}^{-\frac{1}{2}}(\mathsf{div}_\Gamma, \Gamma), \qquad p_h \in S^0(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma)/\mathbb{C}^Z,$$

$$\mathbf{u}_h \in \mathcal{RT}^{\times}(\Gamma) \subset \mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathsf{curl}_\Gamma, \Gamma).$$

Discrete inf–sup condition for D on $\mathcal{RT}(\Gamma) \times S^0(\Gamma)$ can be shown.

$\Rightarrow$ unique solution $\boldsymbol{\lambda}_h \in \mathcal{RT}(\Gamma)$, $\mathbf{u}_h \in \mathcal{RT}^{\times}(\Gamma)$ and $p_h \in S^0(\Gamma)$

$\Rightarrow$ quasi optimal error estimate

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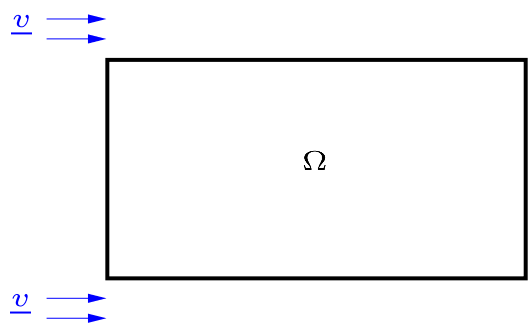
Heat transfer

Conductor produces heat and is cooled by a given air flow.



Full model
$\leadsto$ full Navier–Stokes equations + energy balance in Ω^c

Given velocity dominates the flow
$\leadsto$ compute $\underline{v}$ independent of temperature T

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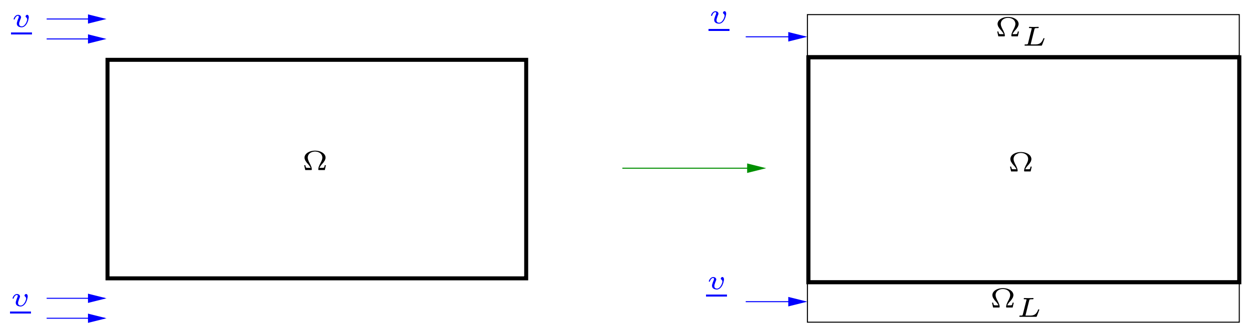

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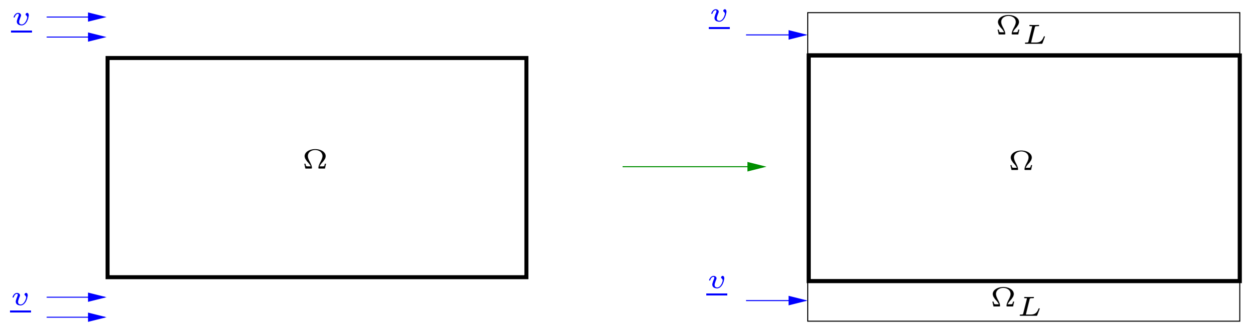

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Energy equation in Ω_L

$$-\mathrm{div}\left(\alpha(T)\nabla T\right) + \underline{v} \cdot \nabla T = q_V(\mathbf{v})$$

Linear heat conduction equation in Ω

$$-\alpha \Delta T = q_V(\mathbf{E})$$

α is the thermal conductivity.
The Joule losses and the dissipation enter as sources on the right hand side.

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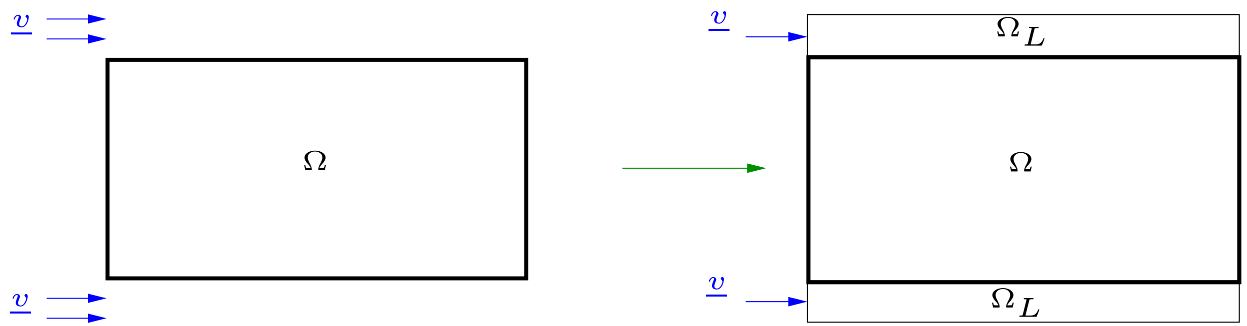

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Substitute $\underline{v} = \infty$ in farfield and $\underline{v} = \underline{0}$ in boundary layer
$\rightsquigarrow$ nonlinear heat conduction equation in Ω_L

Heat conduction

Heat conduction equations

$$-\alpha \Delta T = q_V(\mathbf{E}) \quad \text{in } \Omega_1 := \Omega$$

$$-\mathrm{div}\left(\alpha(T)\nabla T\right) = q_V(\mathbf{v}) \quad \text{in } \Omega_2 := \Omega_L$$

$$\gamma_0 T = T_0 \quad \text{on } \Gamma_D := \partial\Omega_2 \setminus \Gamma_{12}$$

$$\left[\gamma_0 T\right] = 0 \quad \text{on } \Gamma_{12} := \partial\Omega_1 \cap \partial\Omega_2$$

$$\left[\alpha(T)\gamma_1 T\right] = 0 \quad \text{on } \Gamma_{12}$$

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\left[\alpha(T)\gamma_1 T\right] &= 0 && \text{on } \Gamma_{12}
\end{aligned}
$$

Sutherland Law

$$
\alpha(T) = \frac{c_p \mu^*}{\mathcal{P}r} \left(\frac{T}{T^*}\right)^{\frac{3}{2}} \left(\frac{T^* + T_0}{T + T_0}\right) \quad \text{in } \Omega_2
$$

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The Kirchhoff Transformation $u = M(T)$ with $M(T) = \int^T \alpha(\tau)\, d\tau$ gives

$$-\Delta u = q_V(\mathbf{E}) \quad \text{in } \Omega_1$$

$$-\Delta u = q_V(\mathbf{v}) \quad \text{in } \Omega_2$$

$$\gamma_0 u = M(T_0) \quad \text{on } \Gamma_D$$

$$\left[M(\gamma_0 u)\right] = 0 \quad \text{on } \Gamma_{12}$$

$$\left[\gamma_1 u\right] = 0 \quad \text{on } \Gamma_{12}$$

BEM for the heat transfer

Let $\lambda_i := \gamma_1^i u$, $u_i := \gamma_0^i u$, local Steklov–Poincaré operators S_i, Newton potentials N_i.
$\rightsquigarrow$ local Dirichlet–Neumann map

$$\lambda_i = S_i u_i - N_i f$$

The coupling of the Dirichlet data $u_2 = M(u_1)$ is nonlinear
$\rightsquigarrow$ weak coupling of the Dirichlet data
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Standard domain decomposition approach yields the following variational formulation:

$$\langle S_1 u_1, v_1 \rangle_{\Gamma_1} + \langle S_2 u_2, v_2 \rangle_{\Gamma_2} \quad = \quad \sum_{i=1}^{2} \langle N_i f, v_i \rangle_{\Gamma_i}$$

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and $\lambda_{12} \in H^{-\frac{1}{2}}(\Gamma_{12})$ such that

$$\begin{aligned} \langle S_1 u_1, v_1\rangle_{\Gamma_1} + \langle S_2 \widetilde{u}_2, v_2\rangle_{\Gamma_2} - \langle v_2 - M(v_1), \lambda_{12}\rangle_{\Gamma_{12}} &= \sum_{i=1}^{2} \langle N_i f, v_i\rangle_{\Gamma_i} - \langle S_2 \widetilde{M(T_0)}, v_2\rangle_{\Gamma_2} \\ \langle \widetilde{u}_2 - M(u_1), \mu_{12}\rangle_{\Gamma_{12}} &= 0 \end{aligned}$$

for all $(v_1, v_2) \in X$ and all $\mu_{12} \in H^{-\frac{1}{2}}(\Gamma_{12})$.

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and $\lambda_{12} \in H^{-\frac{1}{2}}(\Gamma_{12})$ such that

$$\begin{aligned} \langle S_1 u_1, v_1\rangle_{\Gamma_1} + \langle S_2 \widetilde{u}_2, v_2\rangle_{\Gamma_2} - \langle v_2 - M(v_1), \lambda_{12}\rangle_{\Gamma_{12}} &= \sum_{i=1}^{2} \langle N_i f, v_i\rangle_{\Gamma_i} - \langle S_2 \widetilde{M(T_0)}, v_2\rangle_{\Gamma_2} \\ \langle \widetilde{u}_2 - M(u_1), \mu_{12}\rangle_{\Gamma_{12}} &= 0 \end{aligned}$$

for all $(v_1, v_2) \in X$ and all $\mu_{12} \in H^{-\frac{1}{2}}(\Gamma_{12})$.

Conforming Galerkin discretization

$$u_i \in S^1(\Gamma_i), \quad \lambda_{12} \in S^0(\Gamma_{12})$$

Discrete inf–sup condition for $\langle \cdot - M(\cdot), \cdot\rangle_{\Gamma_{12}}$ on $\big(S^1(\Gamma_1) \times S^1(\Gamma_2)\big) \times S^0(\Gamma_{12})$.

$\Rightarrow$ unique solution $u_i \in S^1(\Gamma_i), \quad \lambda_{12} \in S^0(\Gamma_{12})$

$\Rightarrow$ quasi optimal error estimate

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Fast BEM

Drawback of the BEM are the fully populated matrices

$\rightsquigarrow$ Fast BEM

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$\leadsto$ Fast BEM

Available Methods:

1. Wavelets

   Dahmen, Prössdorf, Schneider (1993, 1994), Schneider (1998), Harbrecht, Schneider (2002)

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3. Algebraic cluster methods (ACA, $\mathcal{H}$–Matrices, . . .)

   Hackbusch (1999, . . .), Bebendorf, Rjasanow (2000, 2003, . . .)

   Simple implementation, just kernel properties needed

Adaptive Cross Approximation

- Matrix partitioning via hierarchical cluster structure

Adaptive Cross Approximation

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- distinguish between admissable and non–admissable blocks

$$\text{diam}\,(C_2) \leq \eta\,\text{dist}\,\{C_1, C_2\},$$

- use full matrices for non–admissible blocks
- approximate admissable blocks by lowrank matrices

$$A \approx UV^T, \quad A \in \mathbb{C}^{m\times n}, U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{n\times p}$$

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- $\rightsquigarrow$ almost linear complexity $\mathcal{O}(N^{1+\epsilon})$

For details see: M. Bebendorf, S. Rjasanow, *Adaptive Low-Rank Approximation of Collocation Matrices*, Computing 70, 2003

Adaptive Cross Approximation

The kernel function

$$k(x,y) := \frac{e^{-\kappa|x-y|}}{|x-y|}$$

is asymptotically smooth for all wavenumbers $\kappa \in \mathbb{C}$ with *Re* $\kappa > 0$ (and $\kappa = 0$).
This is <u>not</u> true for Helmholtz!.

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All kernel functions in the eddy current BEM model are asymptocically smooth with constants independent of κ.
$\leadsto$ The algorithm works for all frequencies the same!

Numerical example: Compression rates (sphere)

Compute scalar single layer potential with piecewise constants

$$V[i,j] = \int_{\tau_i} \int_{\tau_j} \frac{e^{-\kappa|x-y|}}{|x-y|} \, dS_y \, dS_x$$

for different wavenumbers

$$\kappa = (1+i)\tau$$

on a sphere with 2640 boundary elements.

Cluster tree and accuracy for the ACA method are fixed.

τ	N_{full}	N_{comp}	cpr[%]
0	6969600	1511428	21.7
1	6969600	1507491	21.6
10	6969600	1021391	14.7
100	6969600	909733	13.1
200	6969600	909540	13.1
1000	6969600	909540	13.1

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Numerical example: analytic solution (cylinder)

Fom Lax–Milgram and the approximation property of the discrete spaces we get

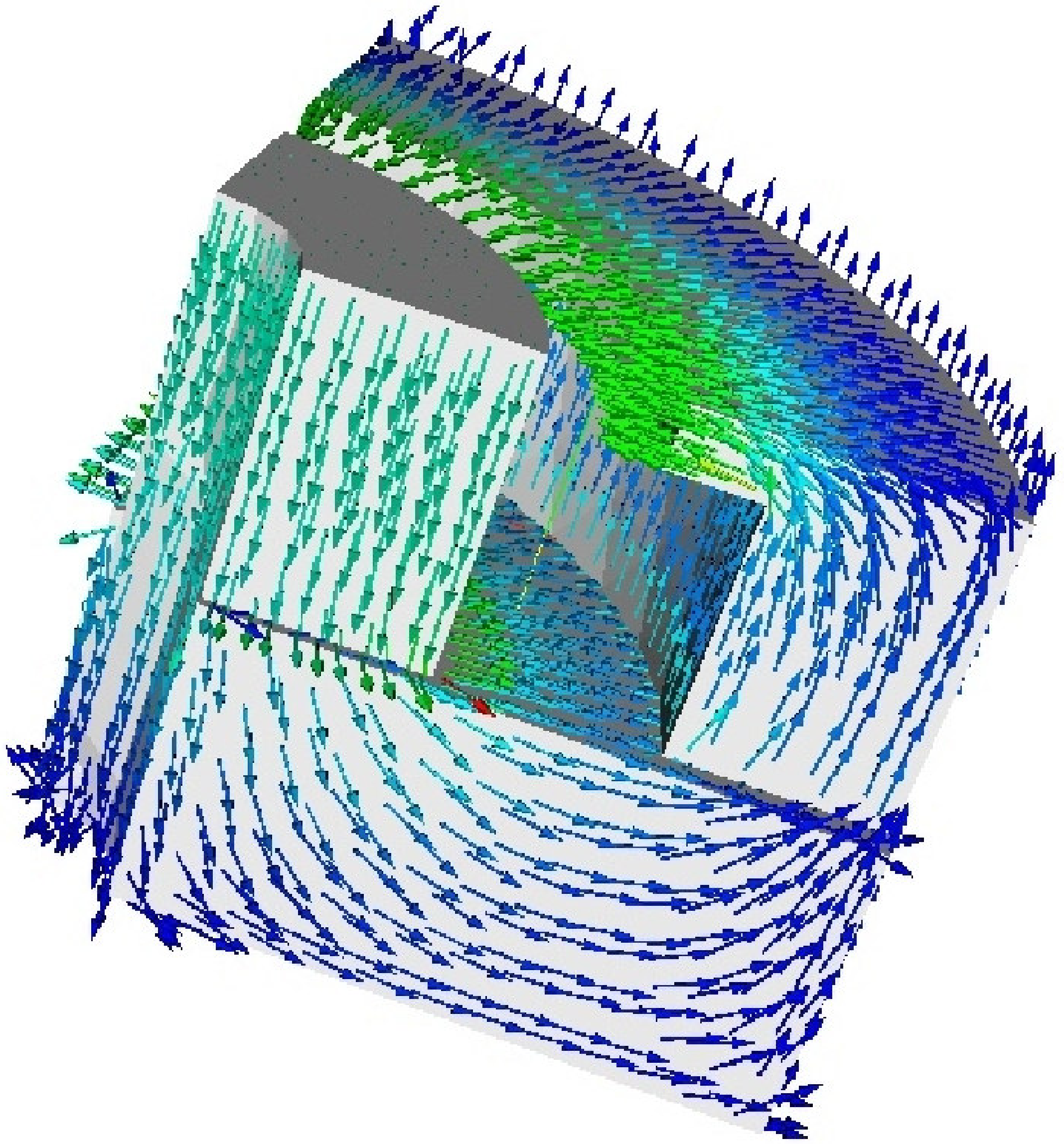
$$\|u - u_h\|^2_{\mathbf{H}_{\perp}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma)} \leq ch^3 \left(\|u\|^2_{\mathbf{H}^1_{pw}(\Gamma)} + \|\mathrm{curl}_\Gamma\, u\|^2_{\mathbf{H}^1_{pw}(\Gamma)} \right)$$

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|^2_{\mathbf{H}_{||}^{-\frac{1}{2}}(\mathrm{div}_\Gamma)} \leq ch^3 \left(\|\boldsymbol{\lambda}\|^2_{\mathbf{H}^1_{pw}(\Gamma)} + \|\mathrm{div}_\Gamma\, \boldsymbol{\lambda}\|^2_{\mathbf{H}^1_{pw}(\Gamma)} \right)$$

so we can expect linear convergence in L^2 for both of the Cauchy data.

N	Dirichlet error	Neumann error	ρ_D	ρ_N
128	9.32 e-02	9.86 e-02	–	–
512	4.26 e-02	5.23 e-02	1.13	0.92
2048	2.21 e-02	2.66 e-02	0.95	0.98
8196	1.13 e-02	1.34 e-02	0.96	0.99
32768	5.40 e-03	6.68 e-03	1.06	1.00

Current density in magnetic valve



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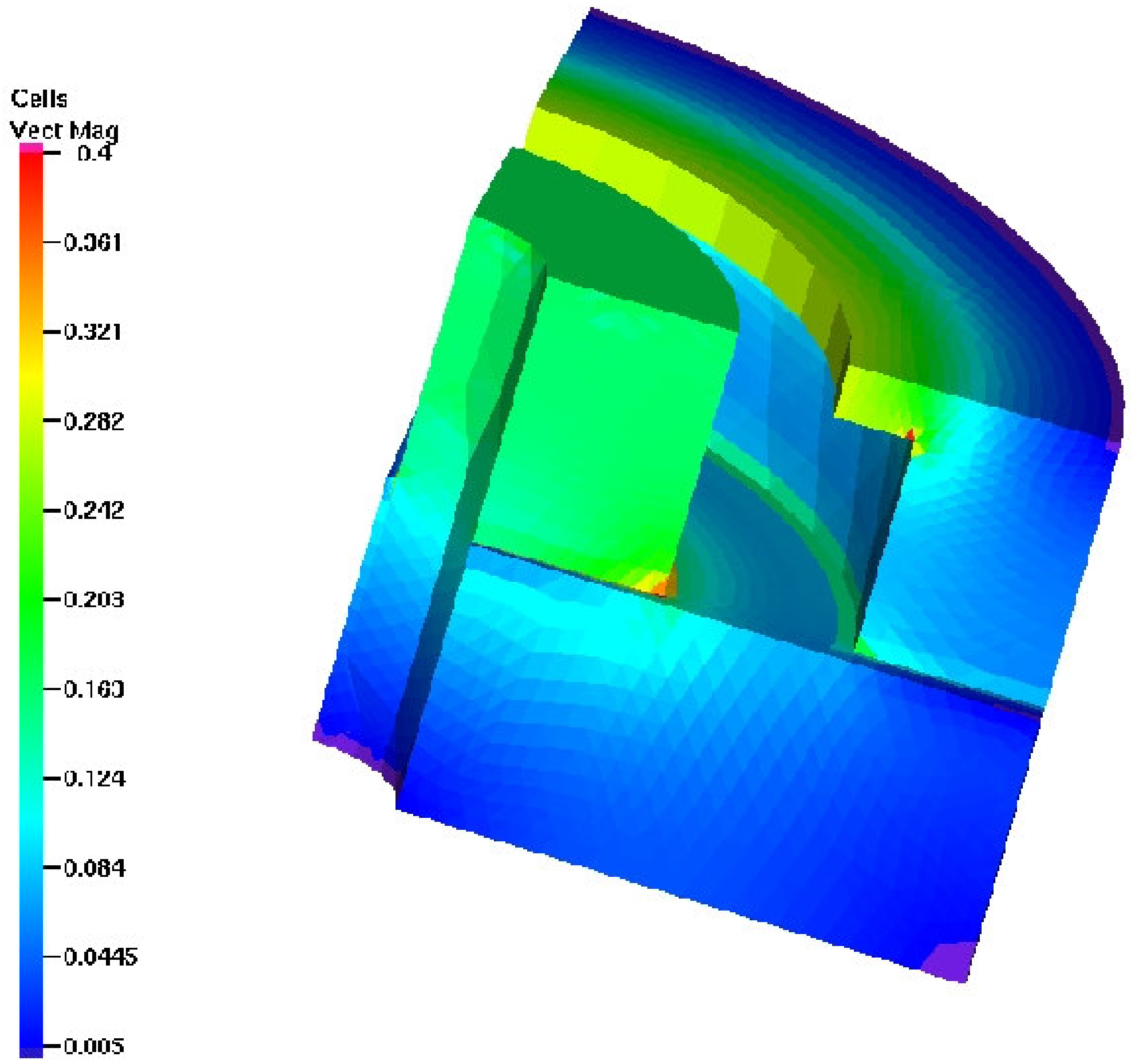
Magnitude of the current density



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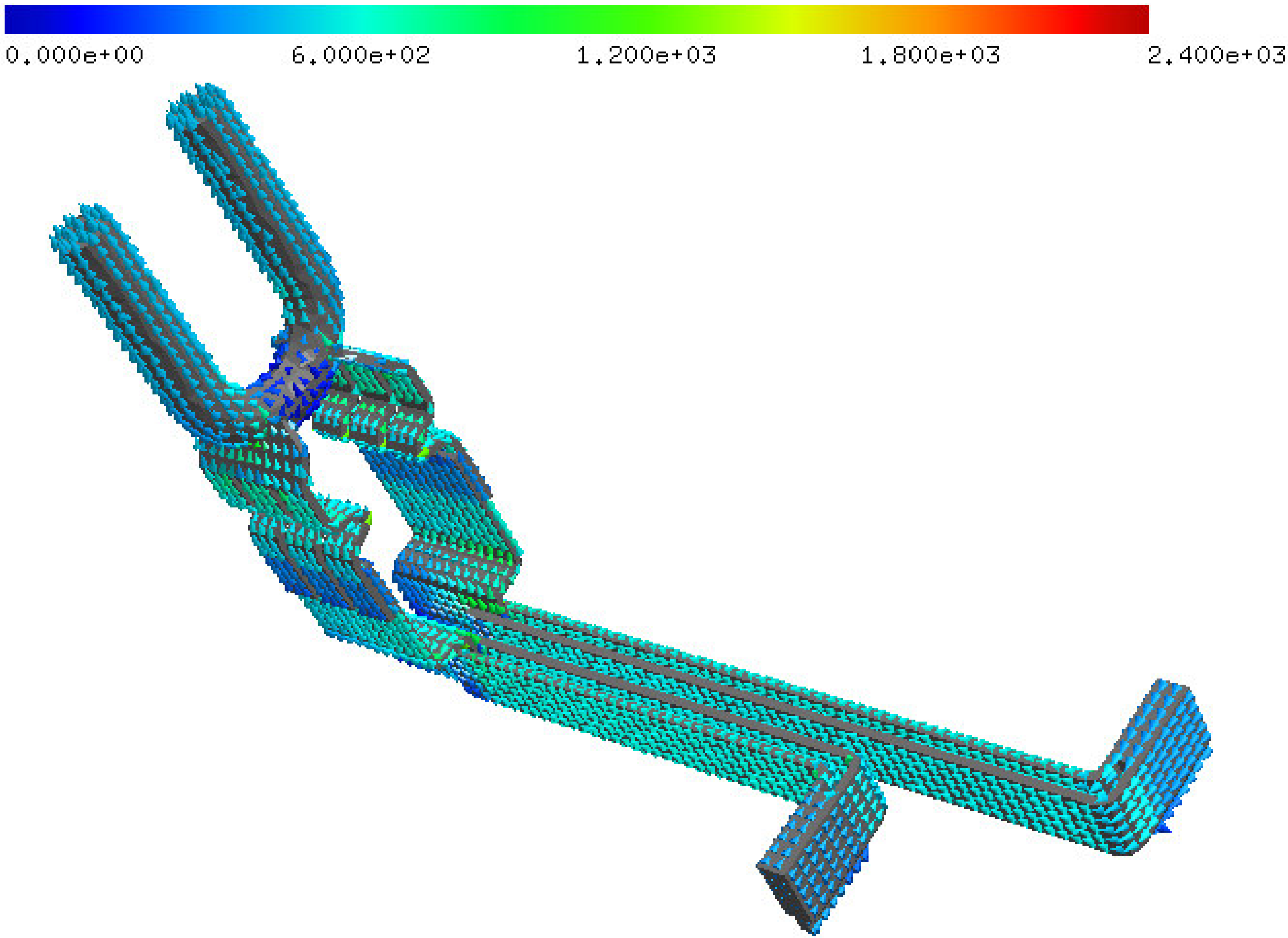
Current density in bus bar



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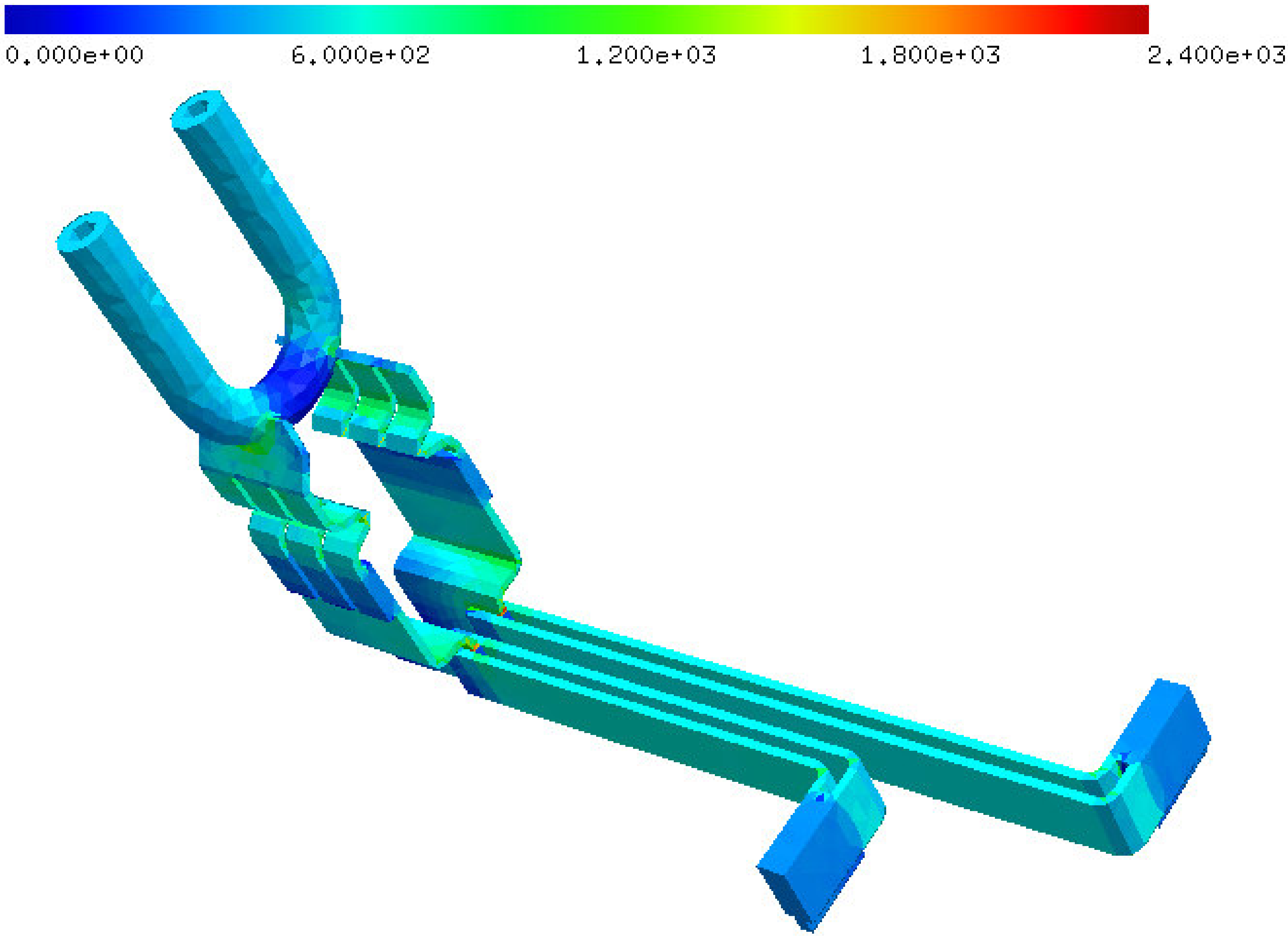

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Thank you!