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Universität Stuttgart

Adaptive boundary element methods in industrial applications

Günther Of, Olaf Steinbach, Wolfgang Wendland

Outline

- Mixed boundary value problems of potential theory
- Symmetric **Galerkin** boundary integral equation formulation
- Realization of the boundary integral operators by the **Fast Multipole Method**
- Fast Multipole Method for **Gadaptive meshes**
- **Preconditioning** of the matrices of the hypersingular operator and the single layer potential
- Example: spray painting
- **Evaluation** of representation formula **on demand**
- Experimental setup and errors
- **Error estimator**

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Mixed boundary value problem

Laplace equation:

$$
\begin{aligned}
-\Delta u(x) &= 0 && \text{for } x \in \Omega \subset \mathbb{R}^3, \\
u(x) &= g_D(x) && \text{for } x \in \Gamma_D, \\
t(x) := (T_x u)(x) = (\partial_n u)(x) &= g_N(x) && \text{for } x \in \Gamma_N.
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Representation formula:

$$u(x) = \int_\Gamma [U^*(x,y)]^\top t(y) ds_y - \int_\Gamma [T_y U^*(x,y)]^\top u(y) ds_y \qquad \text{for } x \in \Omega.$$

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Calderon projector for the Cauchy data $u(x)$ and $t(x)$:

$$\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} \qquad \text{on } \Gamma$$

Boundary integral operators:

$$(Vt)(x) = \int_\Gamma [U^*(x,y)]^\top t(y) ds_y, \; (Du)(x) = -T_x \int_\Gamma [T_y U^*(x,y)]^\top u(y) ds_y,$$

$$(Ku)(x) = ⨍_\Gamma [T_y U^*(x,y)]^\top u(y) ds_y, \; (K't)(x) = ⨍_\Gamma [T_x U^*(x,y)]^\top t(y) ds_y.$$

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Symmetric boundary integral formulation

Symmetric boundary integral formulation (Sirtori ’79, Costabel ’87):

$$(V\widetilde{t})(x) - (K\widetilde{u})(x) = (\frac{1}{2}I + K)\widetilde{g}_D(x) - (V\widetilde{g}_N)(x) \quad \text{for } x \in \Gamma_D,$$

$$(K'\widetilde{t})(x) + (D\widetilde{u})(x) = (\frac{1}{2}I - K')\widetilde{g}_N(x) - (D\widetilde{g}_D)(x) \quad \text{for } x \in \Gamma_N.$$

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Galerkin discretization with piecewise constant (φ_l) and piecewise linear (ψ_i) trial and test functions leads to a **system of linear equations**:

$$\begin{pmatrix} V_h & -K_h \\ K_h' & D_h \end{pmatrix} \begin{pmatrix} \underline{\widetilde{t}}_h \\ \underline{\widetilde{u}}_h \end{pmatrix} = \begin{pmatrix} \underline{f}_N \\ \underline{f}_D \end{pmatrix}.$$

Single Galerkin blocks for $k, l = 1, \ldots, m$ and $i, j = 1, \ldots, \widetilde{m}$

$$\begin{aligned} V_h[l,k] &= \langle V\varphi_k, \varphi_l \rangle_{\Gamma_D}, \qquad & K_h[l,i] &= \langle K\psi_i, \varphi_l \rangle_{\Gamma_D}, \\ K_h'[j,k] &= \langle K'\varphi_k, \psi_j \rangle_{\Gamma_N}, \qquad & D_h[j,i] &= \langle D\psi_i, \psi_j \rangle_{\Gamma_N}. \end{aligned}$$

Potential and Fast Multipole Method

Double layer potential:

$$(Ku)(x) = \frac{1}{4\pi} \int_\Gamma \frac{\partial}{\partial n_y} \frac{1}{|x-y|} u(y) ds_y \qquad \text{for } x \in \Gamma$$

Single layer potential:

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In the farfield **numerical quadrature**:

$$\frac{1}{4\pi} \sum_{k=1}^{N} \int_{\tau_k} t(y) \frac{1}{|x-y|} ds_y \approx \frac{1}{4\pi} \sum_{k=1}^{N} \sum_{s=1}^{N_g} \underbrace{\Delta_k \omega_{k,s} t(y_{k,s})}_{=q_{k,s}} \frac{1}{|x-y_{k,s}|}.$$

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Fast Multipole Method: [Rohklin 1984; Greengard, Rohklin 1987; ...]
Fast evaluation of potentials in many particle systems:

$$\Phi(x_j) = \sum_{i=1}^{N} \frac{q_i}{|x_j - y_i|} \qquad \text{for } j = 1, \ldots, M.$$

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Idea of the Fast Multipole Method

- Starting point: evaluation of sums in many points x_j

$$\Phi(x_j) = \sum_{i=1}^{N} q_i k(x_j, y_i) \qquad j = 1, \ldots, M.$$

effort: $\mathcal{O}(N \cdot M)$; for $N = M$: $\mathcal{O}(N^2)$

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- Effect on the sums:

$$\Phi(x_j) = f(x_j) \sum_{i=1}^{N} q_i g(y_i) \qquad j = 1, \ldots, M.$$

reduced effort: $\mathcal{O}(N + M)$.

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Realization of the Fast Multipole Method

- Actual realization in the **Fast Multipole Method**:

$$k(x,y) = \frac{1}{|x-y|} \approx \sum_{n=0}^{p} \sum_{m=-n}^{n} |x|^n Y_n^{-m}(\hat{x}) \frac{Y_n^m(\hat{y})}{|y|^{n+1}}$$

- with **spherical harmonics** for $m \geq 0$

$$Y_n^{\pm m}(\hat{x}) = \sqrt{\frac{(n-m)!}{(n+m)!}} (-1)^m \frac{d^m}{d\hat{x}_3^m} P_n(\hat{x}_3)(\hat{x}_1 \pm i\hat{x}_2)^m.$$

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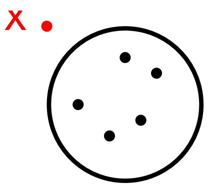
- **Multipol expansion** for $|x| > |y_j|$

$$\sum_{i=1}^{k} \frac{q_i}{|x - y_i|} \approx \sum_{n=0}^{p} \sum_{m=-n}^{n} \sum_{i=1}^{k} q_i |y_i|^n Y_n^{-m}(\hat{y}_i) \frac{Y_n^m(\hat{x})}{|x|^{n+1}}$$



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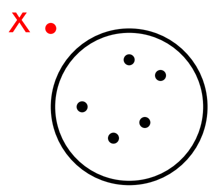

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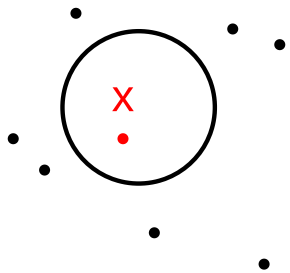

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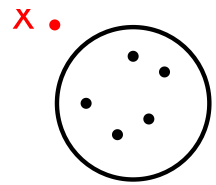

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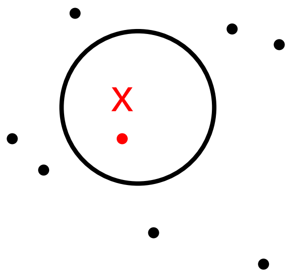

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- **farfield depends on the evaluation point** x_j;
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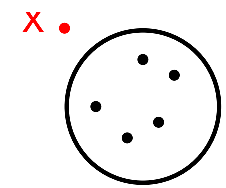

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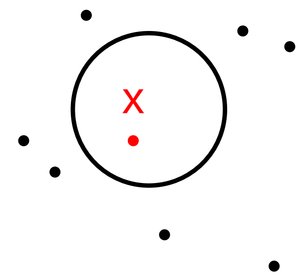

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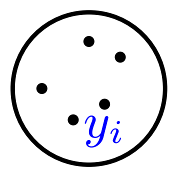

multipole expansion

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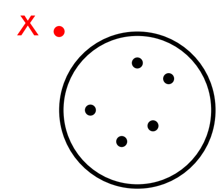

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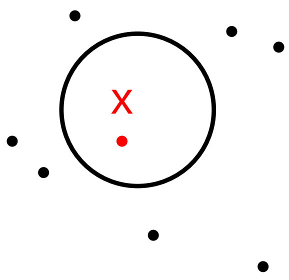

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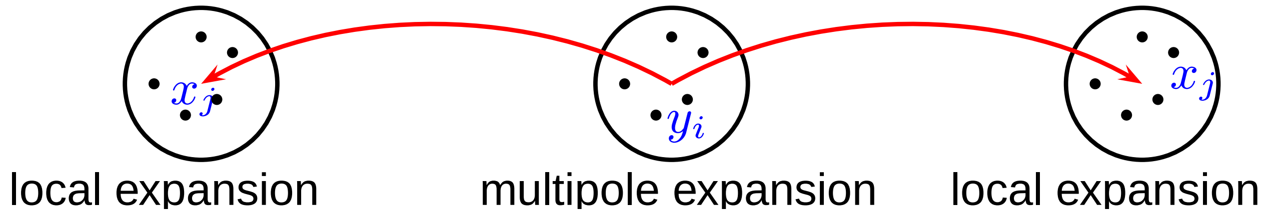

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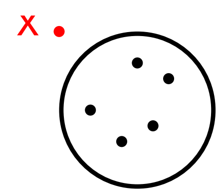

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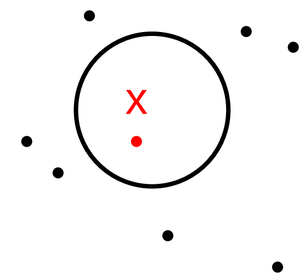

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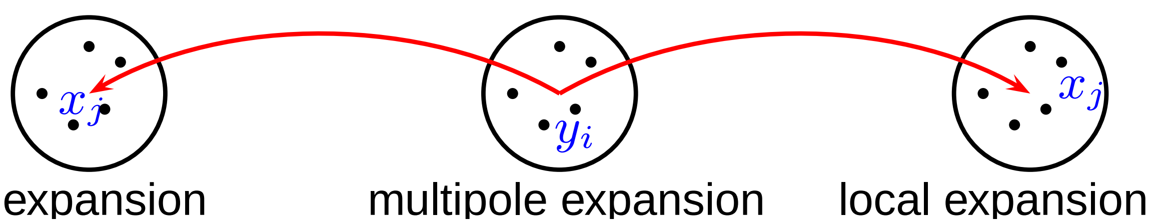

- efficient computation by the use of a **hierarchical structure**
  $\Longrightarrow$ complexity (time and memory) with error control: $\mathcal{O}(N \log^2 N)$ for $N = M$.

Adaptive version of the Fast multipole method

Adaptive versions already exist:

Cheng, Greengard, Rokhlin; Nabors, Korsmeyer, Leighton, White.

Why necessary ?

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$\Longrightarrow$ Condition for the cluster tree: **symmetry of the nearfields**.

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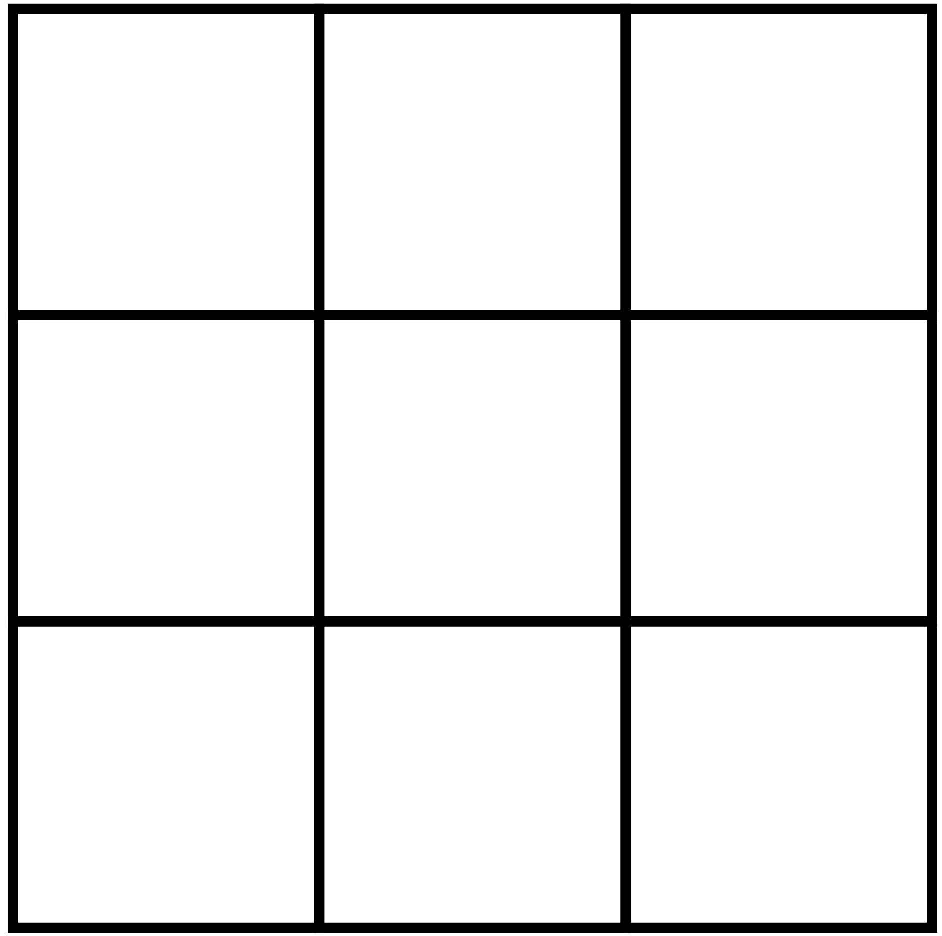
Construction of the adaptive cluster tree

Symmetry of the nearfields and distance control. Example:



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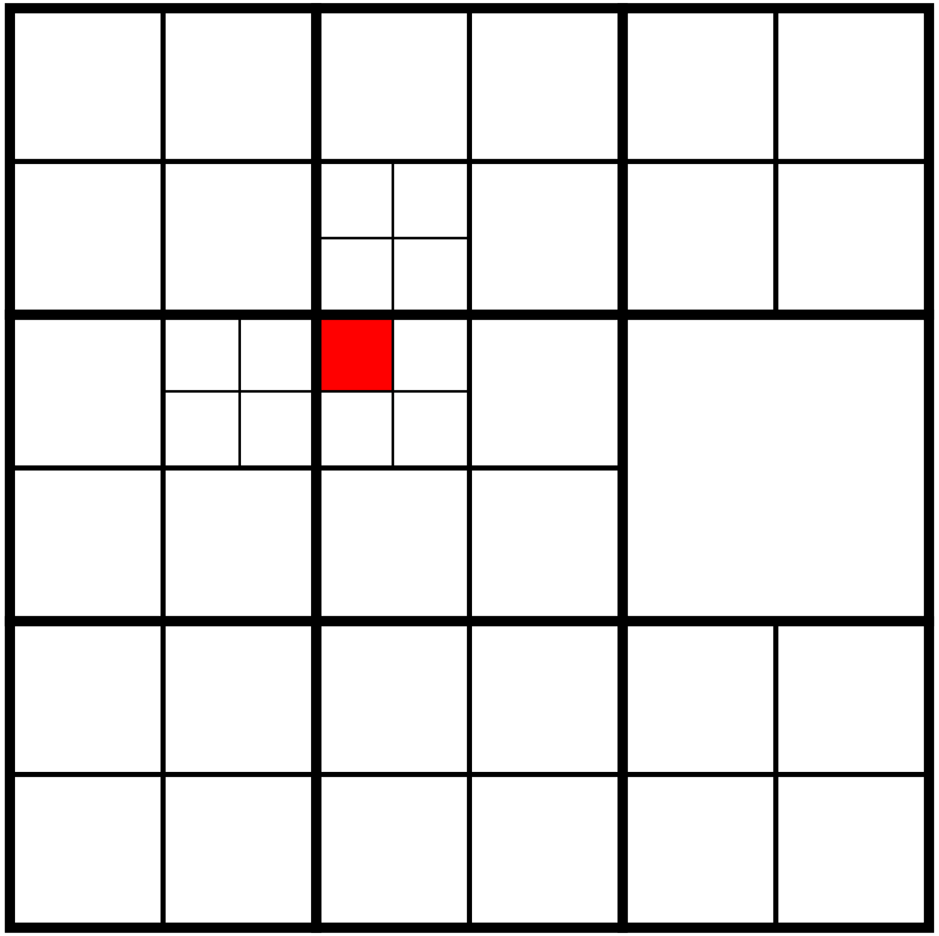

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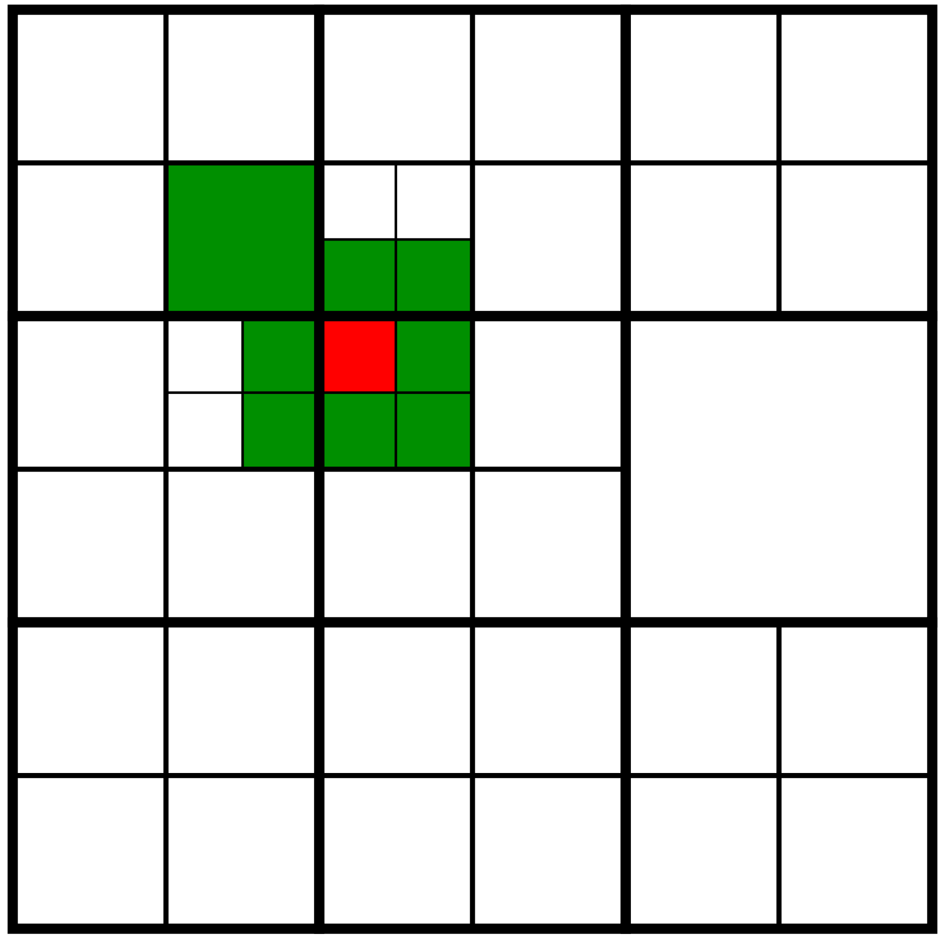

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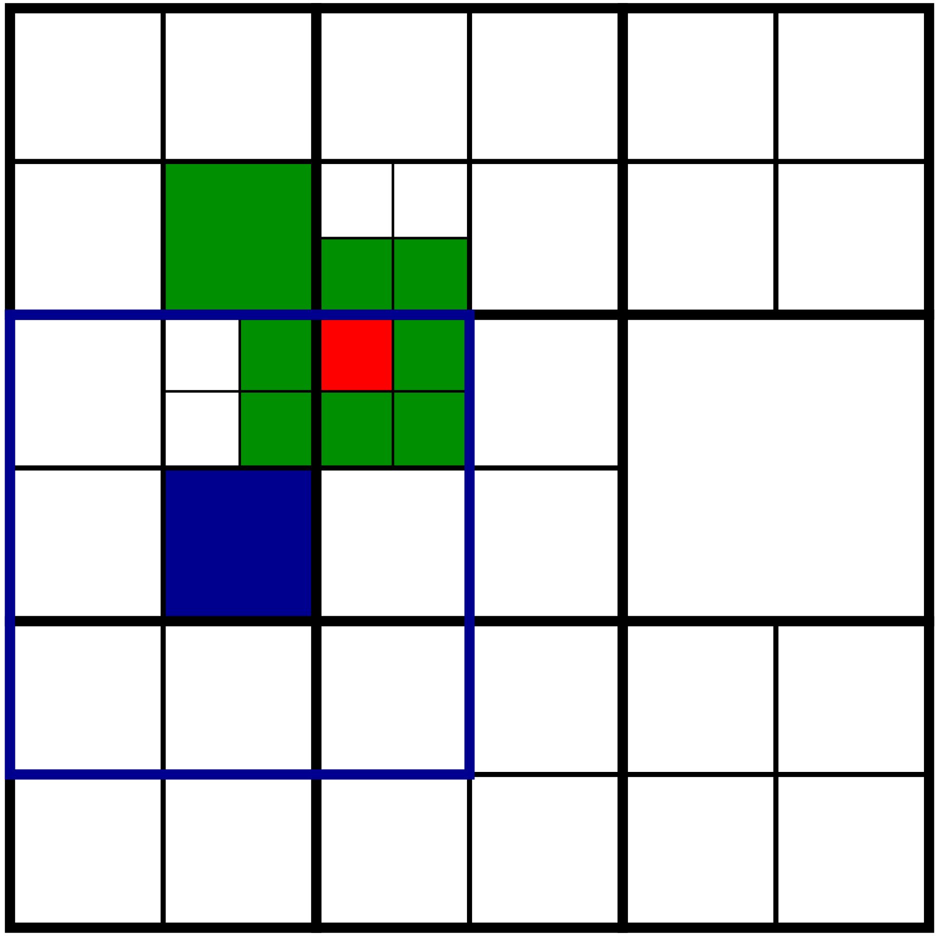

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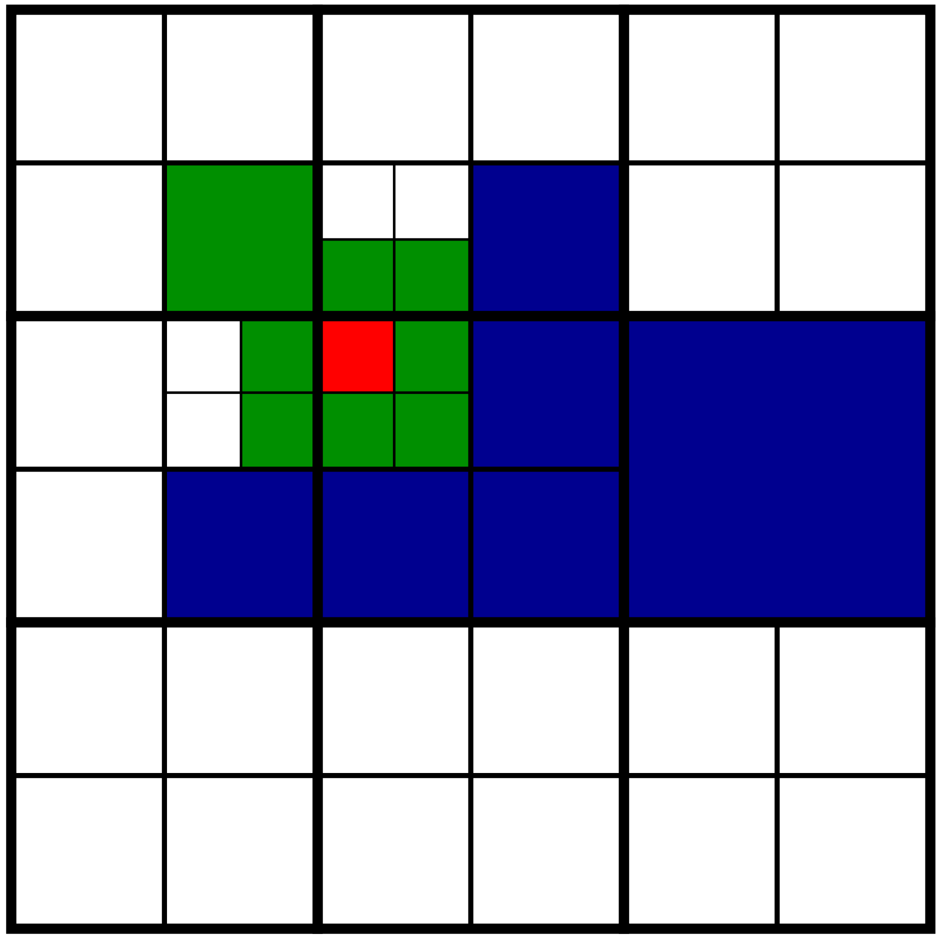

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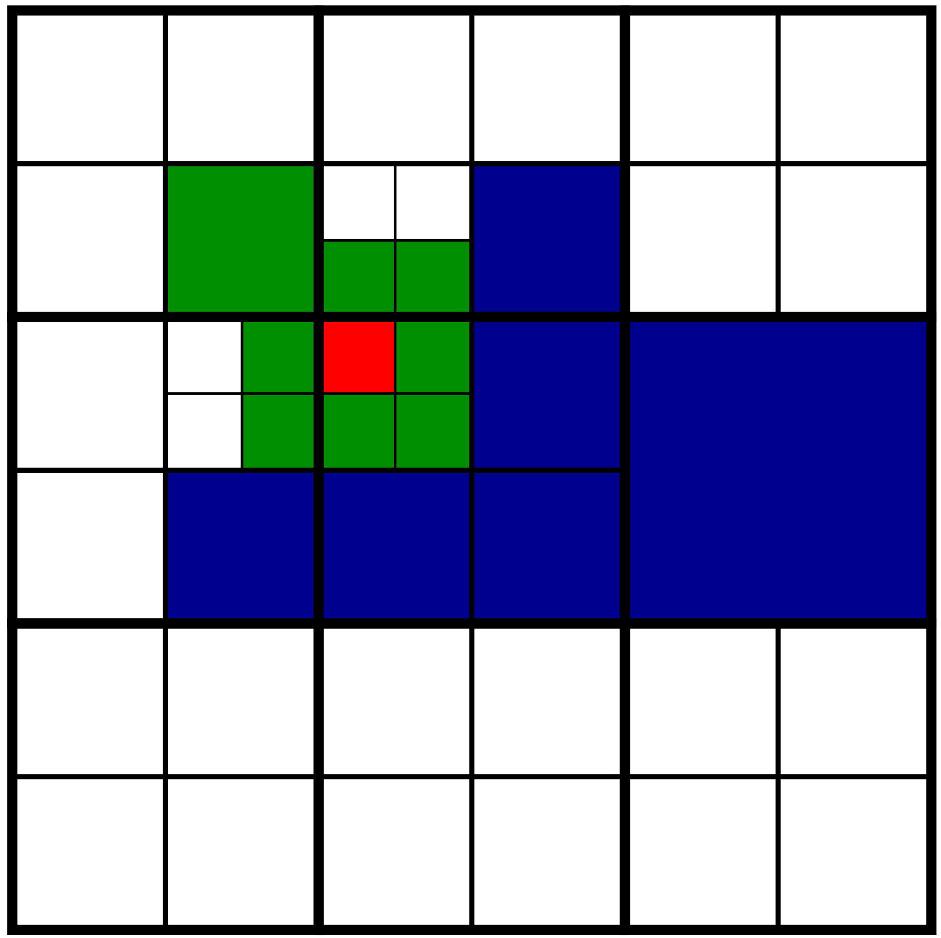
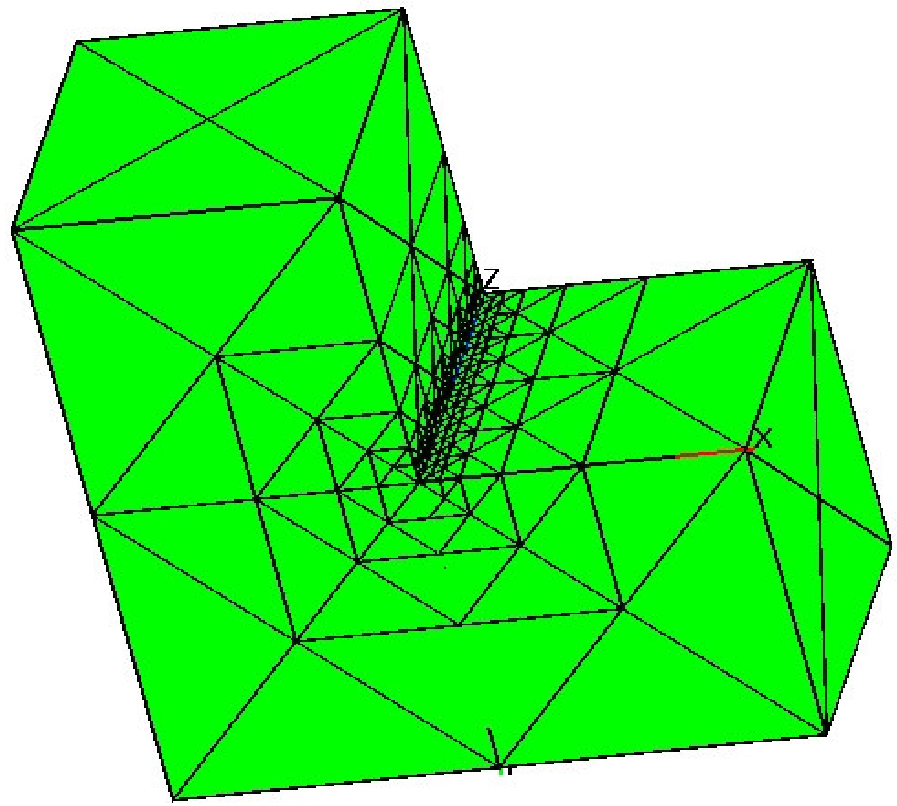


Postprocessing of the nearfields on son levels possible.

Example of an adaptive grid



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Example of an adaptive Fast Multipole Method

level	# elements	setup	solving	nearfield	$\lvert u(x^*) - u_h(x^*) \rvert$
0	338	5	2	26.91 %	1.676e-3
		6	1	40.42 %	1,677e-3
1	1352	20	12	6.22 %	4.101e-4
		58	3	26.43 %	4.103e-4
2	5408	60	42	1.77 %	1.025e-4
		113	13	4.51 %	1.021e-4
3	21632	203	188	0.44 %	2.699e-5
		183	43	0.67 %	2.674e-5
4	86528	720	849	0.11 %	6.414e-6
		556	284	0.16 %	6.561e-6
5	346112			%	
		1909	1596	0.04 %	1.610e-6

Preconditioning of hypersingular operator

- using **boundary integral operators of opposite order**:

  [Steinbach, Wendland 95,98; McLean, Steinbach 99; Of, Steinbach 03]

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$$\frac{c_1^V c_1^D}{1+c_R} \langle V^{-1} v, v\rangle_\Gamma \le \langle \widetilde{D} v, v\rangle_\Gamma \le \frac{1}{4(1-c_R)} \langle V^{-1} v, v\rangle_\Gamma .$$

  - single layer potential with piecewise linear basis functions

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- spectral equivalent **preconditioning matrix** $C_{\widetilde{D}}$ for $\widetilde{D}_h$:

$$C_{\widetilde{D}}[j,i] = \langle V^{-1}\psi_i,\psi_j\rangle_\Gamma \qquad \text{for } i,j=1,\dots,M.$$

Approximation of the preconditioning matrix $C_{\widetilde{D}}$

$$\widetilde{C}_{\widetilde{D}} = M_h^T V_h^{-1} M_h, \widetilde{C}_{\widetilde{D}}^{-1} = M_h^{-1} V_h M_h^{-T}$$

with $\qquad V_h[j,i] = \langle V\psi_i,\psi_j\rangle_\Gamma , \qquad M_h[j,i] = \langle \psi_i,\psi_j\rangle_\Gamma .$

Artificial Multilevel Preconditioner

[Steinbach 2003]

Sequence of nested boundary element spaces (globally quasiuniform)

$$Z_0 \subset Z_1 \subset \ldots Z_J = Z_h \subset H^{-1/2}(\Gamma)$$

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L_2 projection:

$$Q_j w \in Z_j : \langle Q_j w, v_j \rangle_\Gamma = \langle w, v_j \rangle_\Gamma \quad \text{for all } v_j \in Z_j$$

Multilevel operator: [Bramble, Pasciak, Xu 1990]

$$A^s w := \sum_{j=0}^{J} h_j^{-2s} \left(Q_j - Q_{j-1} \right) w$$

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$$c_1 \, \|w\|^2_{H^{-1/2}(\Gamma)} \le \langle A^{-1/2} w, w \rangle_\Gamma \le c_2 J^2 \, \|w\|^2_{H^{-1/2}(\Gamma)} \qquad \text{for all } w \in Z_J .$$

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The sequence of nested boundary element spaces is build from the **clustering of the boundary elements by a hierarchical structure** as used in the Fast BEM.

Extendable to **adaptive grids**.

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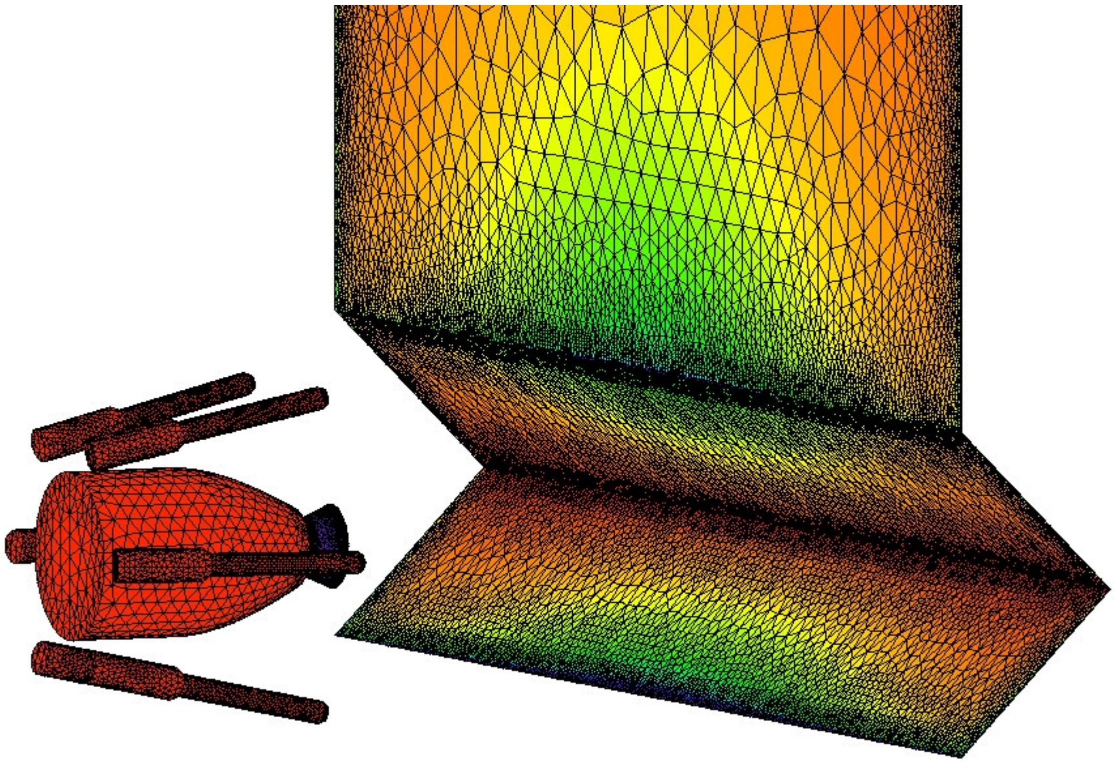
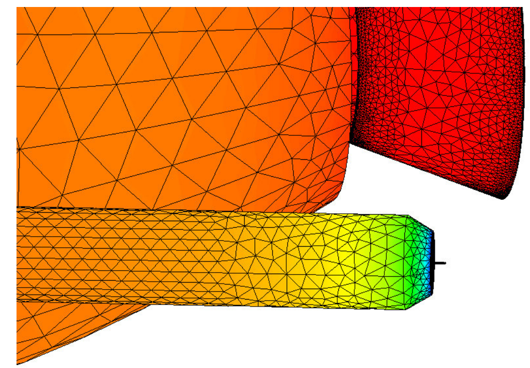
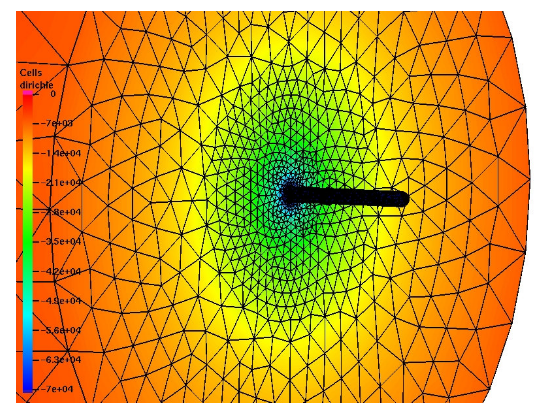
Simulation of spray painting

(with R. Sonnenschein, Daimler Chrysler, Dornier)





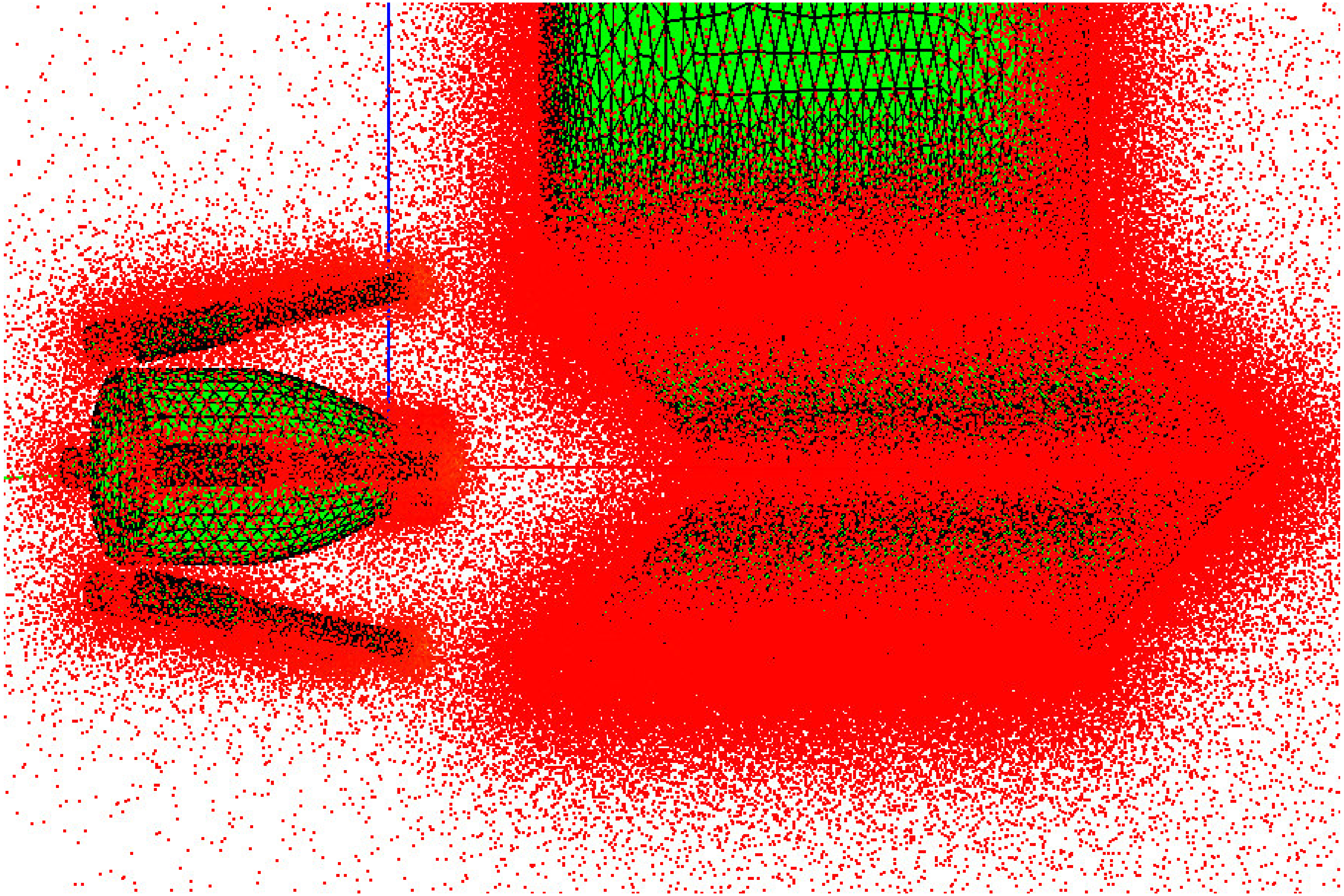


- 112146 boundary elements
- mesh ratio $\frac{h_{\text{max}}}{h_{\text{min}}} \approx 1454,5$
- 570930 evaluation points
- 150 iteration steps

- thickness of the wall: 0.8 mm
- size of the wall: about 1 m
- flux: $-2.3 \cdot 10^5 \ldots 5.5 \cdot 10^8$

Field evaluation

in 570930 points or better interactive on demand. $\Longrightarrow$ Fast Multipole Method



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Evaluation on demand

- Evaluation of the representation formula at many points one by one.
- exact evaluation instead of interpolation on a grid (FEM)
- no grid to construct (FEM)
- build cluster tree without information on the evaluation points
- build separate trees for the geometry and the evaluation.
- new admissibility condition: number of nearfield panels limited
- extra feature: prediction of the point of intersection of the boundary and a straight line (point and direction (electric field))

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- example: 267069 evaluation points

on demand	at once
254 sec	333 sec + 158 sec

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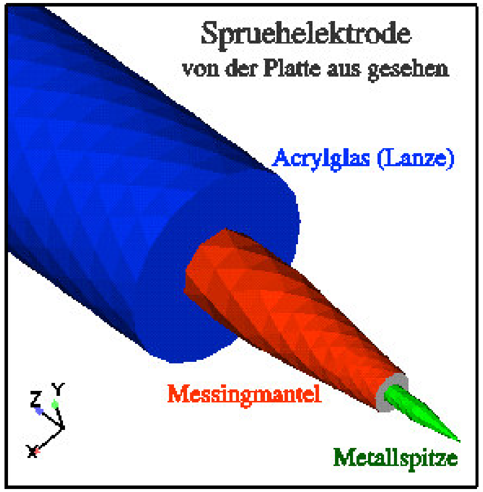
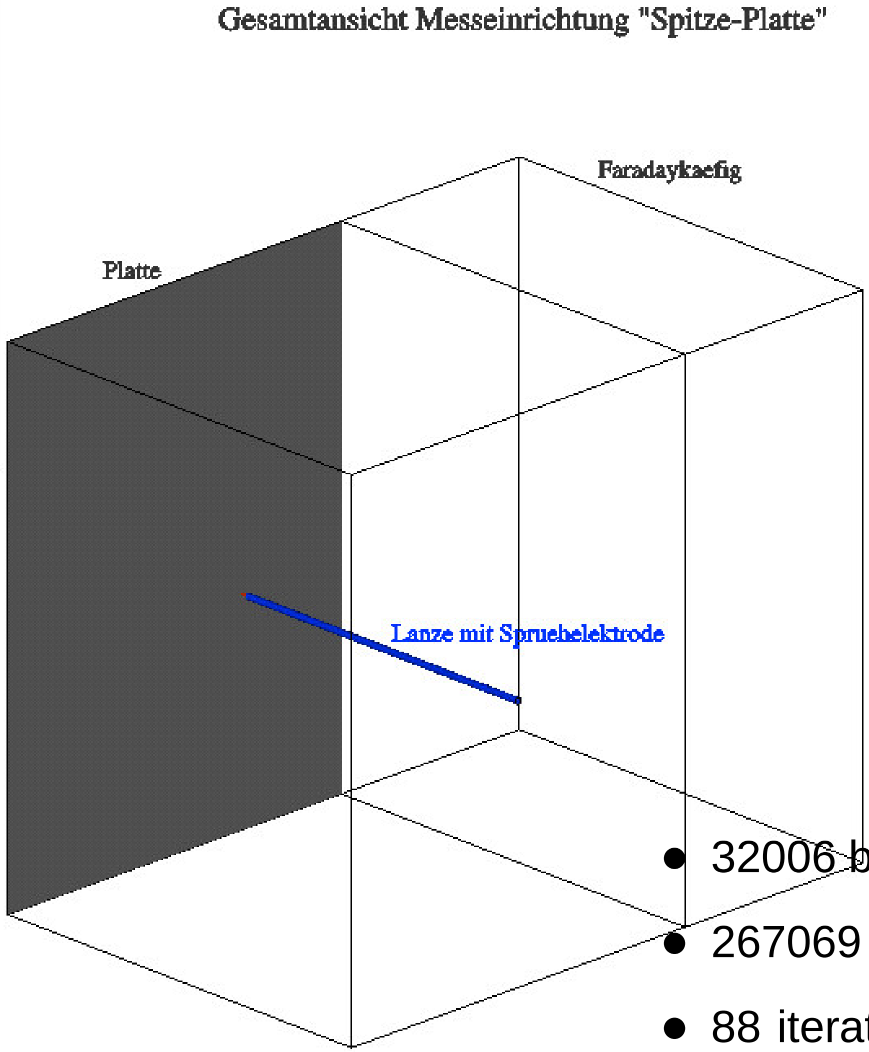
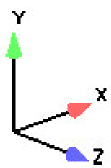
Experimental setup

(with R. Sonnenschein, Daimler Chrysler, Dornier)

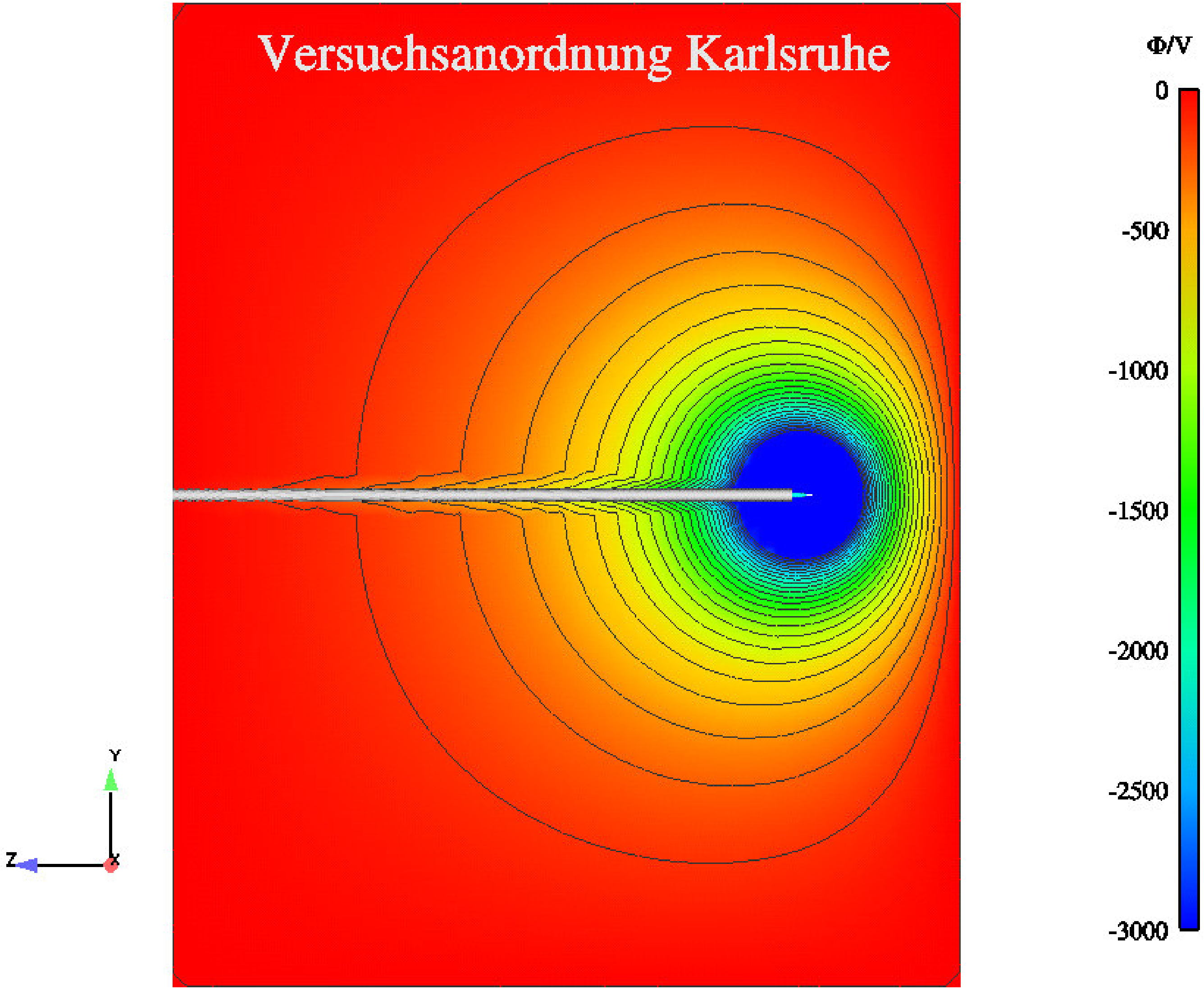






- 32006 boundary elements
- 267069 evaluation points
- 88 iteration steps ($1\times$ refined 92 iteration steps)

Isolines of the potential



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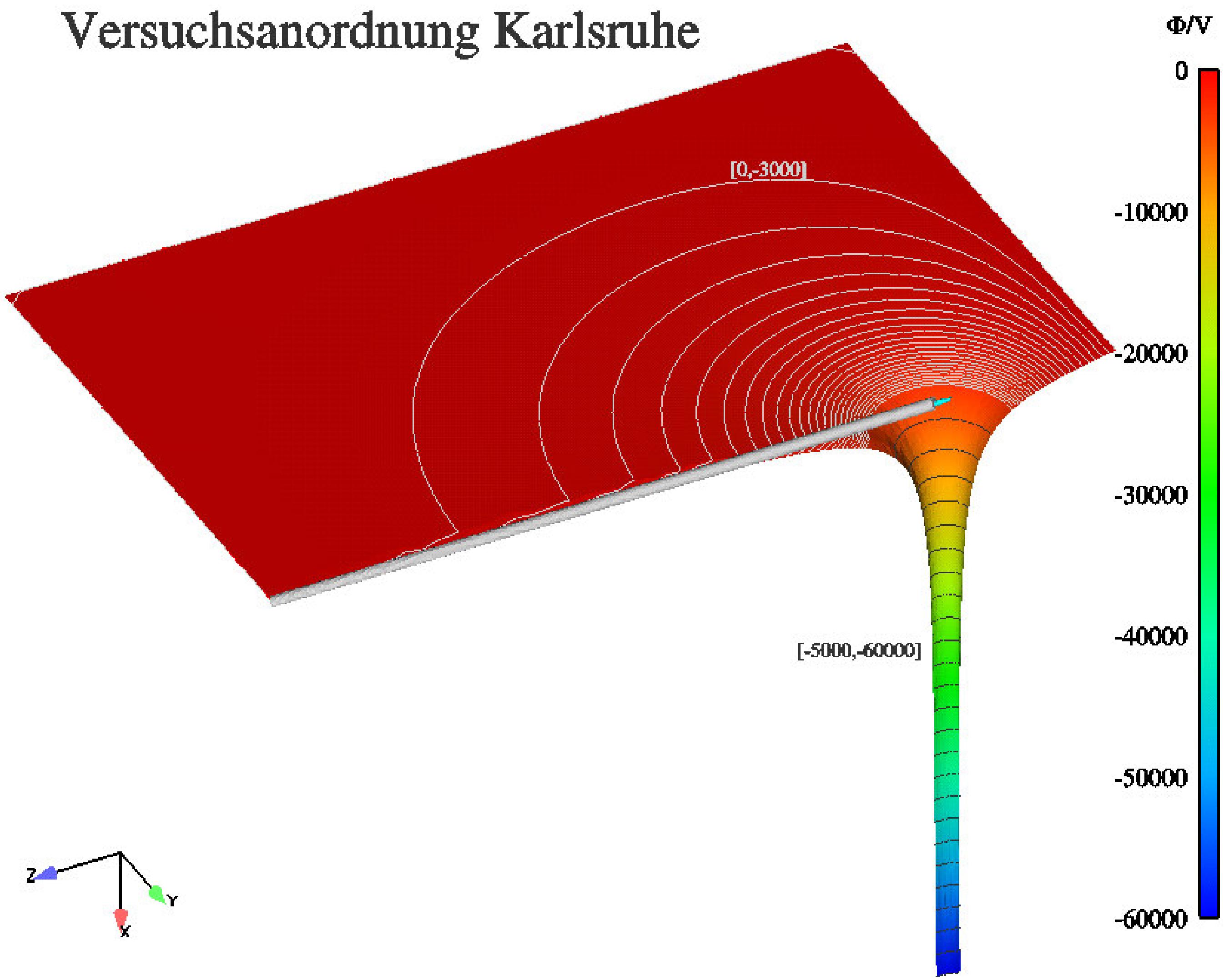

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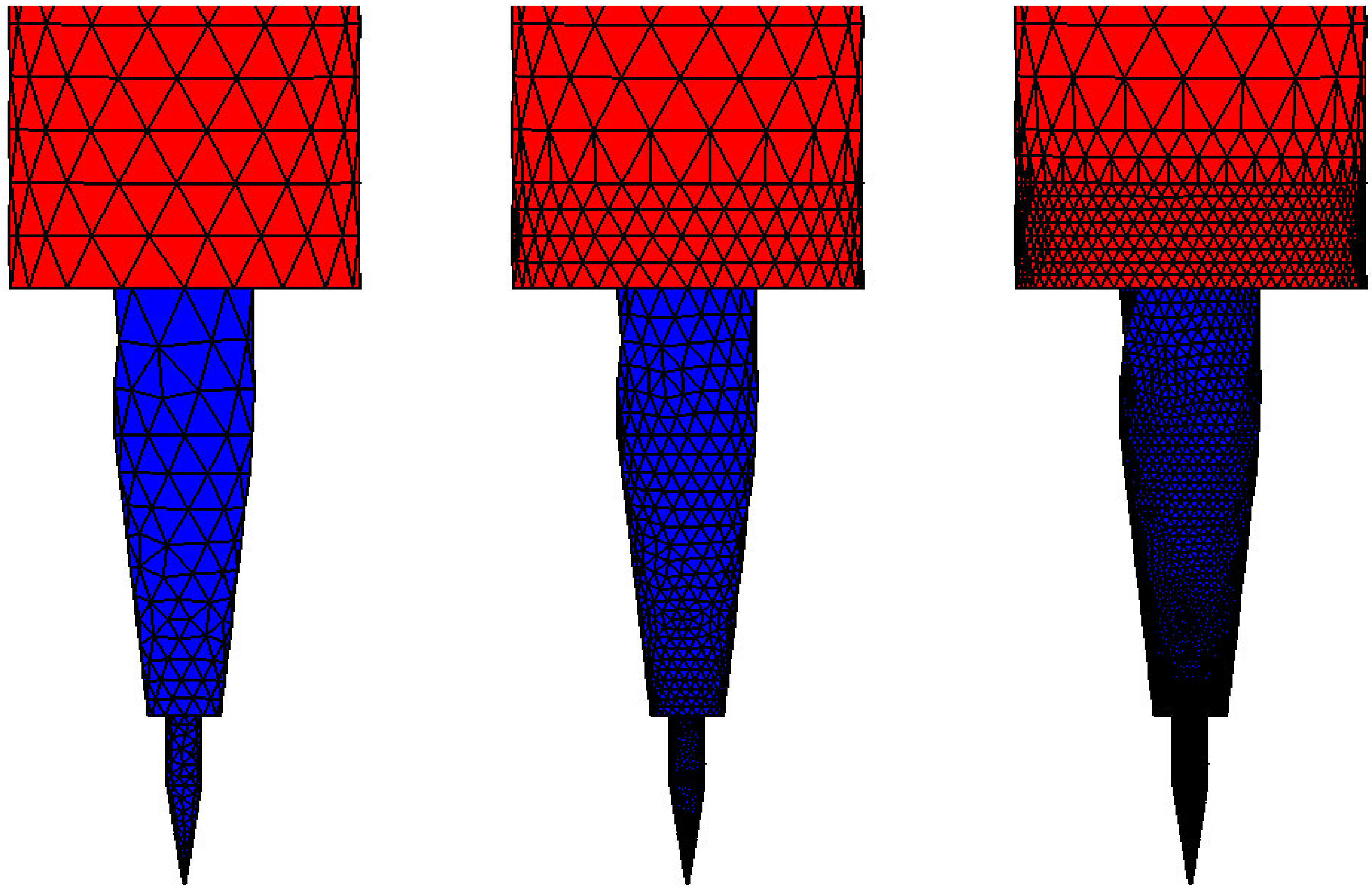
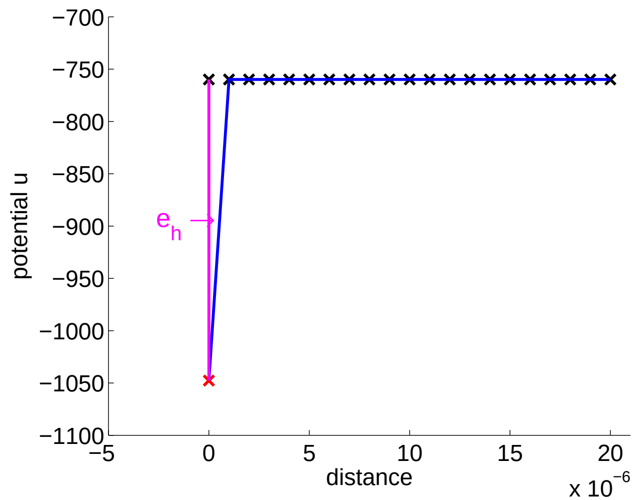
Adaptive meshes



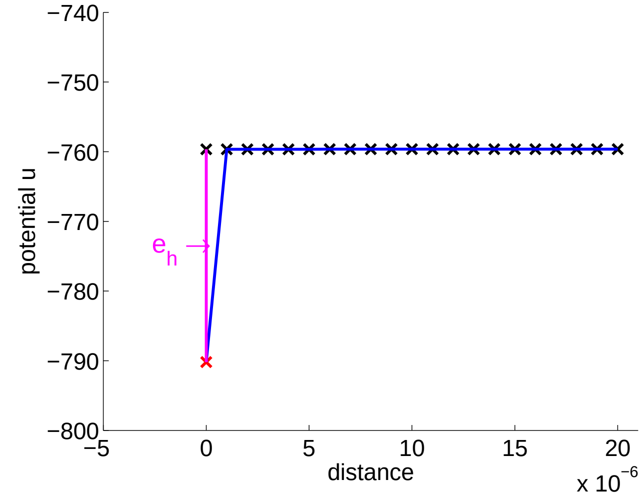
32006 triangles | 34424 triangles | 44168 triangles

Cauchy data and representation formula

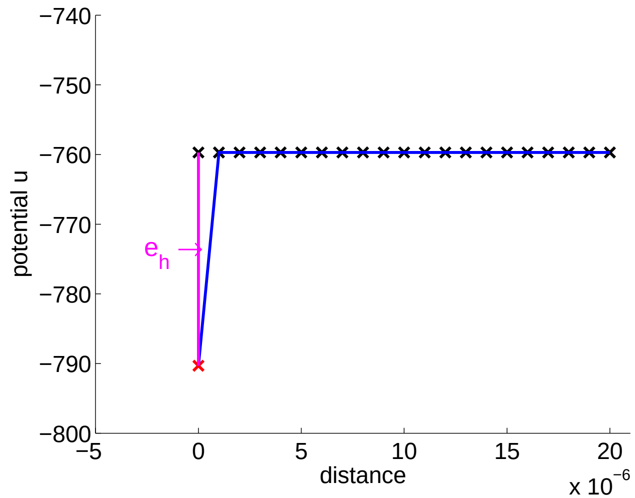
unrefined mesh



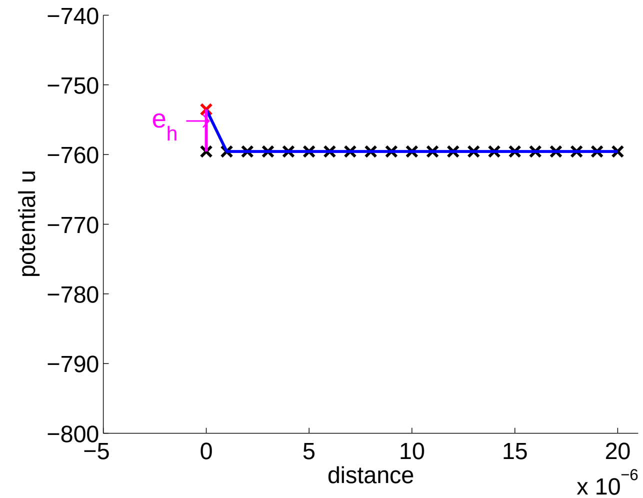
once adaptively refined mesh



uniform refined mesh



twice adaptively refined mesh



A posteriori error estimator

Neumann problem:

$$
\begin{aligned}
-\Delta u(x) &= 0 && \text{for } x \in \Omega \subset \mathbb{R}^3, \\
t(x) := (T_x u)(x) = (\partial_n u)(x) &= g(x) && \text{for } x \in \Gamma.
\end{aligned}
$$

compatibility condition:

$$\int_\Gamma g(x) \cdot 1 \, ds_x = 0$$

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Define $\widetilde{u}$ for a discrete solution u_h:

$$\widetilde{u}(x) = Vg(y) + ((1 - \sigma(x))I - K)u_h(x) \qquad \text{for } x \in \Gamma.$$

Lemma (Schulz,Steinbach). *The* **error** $u - u_h$ *of the approximation* u_h *is a solution of the boundary integral equation*

$$(\sigma(x)I + K)(u - u_h)(x) = (\widetilde{u} - u_h)(x) \qquad \textit{for } x \in \Gamma.$$

A simple **error estimator** e_h:

$$\frac{1}{1 + c_K} \, \|\widetilde{u} - u_h\|_D \leq \|u - u_h\|_D \leq \frac{1}{1 - c_K} \, \|\widetilde{u} - u_h\|_D$$

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Current and future work

- **domain decomposition methods** for spray painting geometry
  $\Longrightarrow$ **parallel** solvers
- Boundary Element Tearing and Interconnecting methods (**BETI**)
- **automatic** generation of domain decompositions
- adaptive meshes based on the **error estimator**
- **industrial applications**
- . . .