Unique Factorization in Torsion-free Modules

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Conference on Rings and Polynomials, Graz 2016

Unique Factorization in Torsion-free Modules

Basic Definitions (A.-M. Nicolas 1967-1974)

In the following, R denotes a commutative domain with an identity element 1 and M a torsion-free R-module.
Let $x \in M$.

- $r \in R$ is called *a R-divisor* of x , if $x = ry$ for some $y \in M$; $m \in M$ is called *a M-divisor of* x, if $x = sm$ for some $s \in R$.
- $r \in R$ is called *a greatest R-divisor* of x, if any R-divisor of x divides r. $m \in M$ is called *a smallest M-divisor* of x, if m is a M-divisor of any M-divisor of x.
- x is called *irreducible* if any R-divisor of x is a unit of R.
- x is called *primitive* if $x \neq 0$ and x is a smallest M-divisor of any non-zero element of Rx.
- M is called *atomic*, if any non-zero element of M has an irreducible M-divisor.
- M is called *factorable*, if any non-zero element of M has a smallest M-divisor.

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Proposition (A.-M. Nicolas 1974)

The following conditions are equivalent:

- M is factorable
- Every non-zero element of M has a greatest R-divisor
- Every non-zero element $x \in M$ has a representation $x = ry$ with $r \in R$, y an irreducible element of M and this representation is unique up to a unit of R
- M is atomic and every irreducible element of M is primitive
- Every maximal rank 1 submodule of M is free.

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Corollaries

- If M has rank 1, then M is factorable if and only if M is free.
- Any vector space is factorable.
- Let R be a Noetherian integrally closed domain and M a finitely generated factorable R-module. Then M is reflexive.
- Let $n \geq 2$. R^n is factorable if and only if R is a GCD-domain.

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Factorial Modules (A.-M. Nicolas 1967, D. D. Anderson and S. Valdes-Leon 1997)

M is called *factorial*, if $M \neq 0$ and every non-zero element x of M has a representation $x = r_1 \cdots r_n y$ with atoms r_i $(i = 1, ..., n)$ of R, y an irreducible element of M and this representation is unique up to units of R.

The following conditions are equivalent:

- M is factorial
- R is atomic and M is factorial
- R is a factorial domain, $M \neq 0$ and M is factorable

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Factorial Modules: Corollaries

In the following, let R be a factorial domain.
Any reflexive R-module is factorable.
For any family $(M_i)_{i \in I}$ of R-modules, the following assertions are equivalent:

- M_i is factorable for each $i \in I$
- $\prod_{i \in I} M_i$ is factorable
- $\bigoplus_{i \in I} M_i$ is factorable.

In particular, for any set I, R^I, $R^{(I)}$ and any projective R-module is factorable.

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New Results

- Any locally projective R-module is factorable.
- Any flat Mittag-Leffler R-module is factorable.
- Any torsion-free content R-module is factorable.

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New Results

Let R be a factorial domain and $R \subseteq S$ be an extension of rings such that S is a torsionfree factorable R-module.
Then the following conditions are equivalent:

- Every prime element of R is a prime element of S or is a unit of S
- The product of greatest R-divisors of any two elements x, y of S has the same prime divisors as any greatest R-divisor of xy
- The product of any two irreducible elements of the R-module S is irreducible.

Let $R \subseteq S$ be an extension of domains such that S is factorial.
Then the following conditions are equivalent:

- R is inert in S
- $S^\times = R^\times$ and S is a factorial R-module such that the product of any two irreducible elements of the R-module S is irreducible.