Torsion pairs generated by a module

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Outline

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- The generalized trace.

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- The torsion pair generated by the quotient field of a commutative domain.

Torsion pairs

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DEFINITION

A torsion pair is a pair $(\mathcal{T}, \mathcal{F})$ of classes of modules which are mutually orthogonal with respect to the Hom_R functor, i.e.:

$$\mathcal{T} = \{T \in \mathrm{Mod}\text{-}R \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } F \in \mathcal{F}\} = {}^{\perp_0}\mathcal{F}$$

$$\mathcal{F} = \{F \in \mathrm{Mod}\text{-}R \mid \mathrm{Hom}_R(T, F) = 0 \text{ for all } T \in \mathcal{T}\} = \mathcal{T}^{\perp_0}.$$

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Closure properties:

- $\mathcal{T}$ is a torsion class if and only if it is closed under epimorphic images, direct sums and extensions.
- $\mathcal{F}$ is a torsion-free class if and only if it is closed under submodules, direct products and extensions.

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The torsion pair $(\mathcal{T}, \mathcal{F})$ generated by a class $\mathcal{M}$ of modules is:
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• $(\mathcal{D}, \mathcal{R})$ torsion pair of divisible - reduced modules.

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Gen $M = \{X \in \mathrm{Mod}\text{-}R \mid X = \mathrm{tr}_M(X)\}$.
Gen $M = \mathcal{T}_M$ iff tr_M is the torsion radical.

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THEOREM (Fuchs '69)

Let $\Phi = \{I \leq R \mid R/I$ is semiartinian $\}$. κ an infinite cardinal such that all ideals of Φ can be generated by less than κ elements, then the Loewy length of every module is at most Ω = the first ordinal of cardinality κ.

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Up to now I know only sufficient conditions.

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Thus $X/\operatorname{tr}(X) \in M^{\perp_0}$, hence $X = \operatorname{tr}(X)$ and $\mathcal{T}_M = \operatorname{Gen} M$.

The tilting case

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PROPOSITION (B-Herbera '07)

A torsion class $\mathcal{T} \subseteq \mathrm{Mod}\text{-}R$ is tilting iff it is of finite type, i.e. there is set $\mathcal{A} = \{A_i\}_{i \in I}$ of finitely presented right R-modules of proj. dim ≤ 1 such that

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• A tilting class is a definable class, i.e. closed under direct products, direct limits and pure submodules.

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PROPOSITION (B'14)

$T^{\perp_0}$ is closed under direct limits iff T is pure projective (direct summand of a direct sum of finitely presented modules).
If R is a commutative ring, then a pure projective tilting module is projective. Hence the tilting torsion class is $\mathrm{Mod}\text{-}R = \mathrm{Gen}\, R$.

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PROPOSITION (B, Herzog, Prihoda, Saroch, Trlifaj '15)

There are examples of pure projective tilting modules not equivalent to classical tilting modules (i.e. not finitely generated).
(T is pure projective with no finitely generated summands).

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$$\mathcal{F} = \{{}_R Y \mid \mathrm{Tor}^1_R(A_i, Y) = 0 \text{ for all } A_i \in \mathcal{A}\}$$

is a cotilting torsion free class (of cofinite type)

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Characterization over commutative rings

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Let $\mathcal{G}$ be a Gabriel topology:

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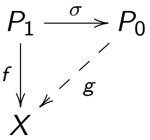

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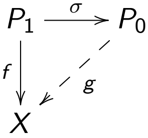

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1. There is a ring R_Σ (universal localization) and a ring epimorphism $f\colon R \to R_\Sigma$ such that f is Σ-inverting, i.e. $\sigma \otimes_R R_\Sigma$ is an isomorphism for every $\sigma \in \Sigma$.
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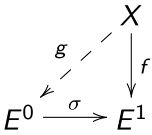

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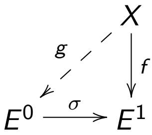

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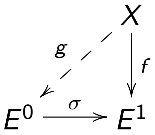

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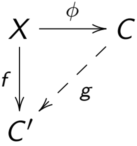

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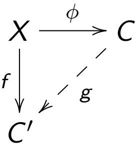

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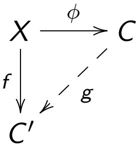

$\mathcal{C}$ is a preenveloping class if every module X admits a $\mathcal{C}$-preenvelope.

- If $\mathcal{C}$ is closed under direct products and pure submodules, then $\mathcal{C}$ is preenveloping (Rada-Saorín '07).
- In particular, definable classes (hence tilting or silting classes) are preenveloping.

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1. $\mathcal{T}$ is Gen M for $M = \bigoplus_{0 \neq r \in R} R/rR$, but R doesn't have a $\mathcal{T}$-preenvelope. ($\mathcal{T}$ is not closed under direct products.)
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Thus, $\mathcal{T} = \mathrm{Gen}\, M$, but Gen $M \not\subseteq M^{\perp_1}$.
In fact, if Gen $M \subseteq M^{\perp_1}$, then M would be a tilting module, hence $\mathcal{T}$ would be a tilting class.

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- Gen $Q = \{D \in \mathrm{Mod}\text{-}R \mid D = \mathrm{tr}_Q(D)\}$.
- The divisible modules in Gen Q are called *h*-divisible.
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If p.d.$Q = 1$, then $T = Q \oplus Q/R$ is a 1-tilting module and $T^{\perp_1} =$ Gen $Q =$ divisible R-modules.

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∂ is a tilting module and Gen ∂ is the class of divisible modules.

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- $\dfrac{D_{\alpha+1}}{D_\alpha} \in \text{Gen}\, Q$. i.e. D is Gen Q-filtered.

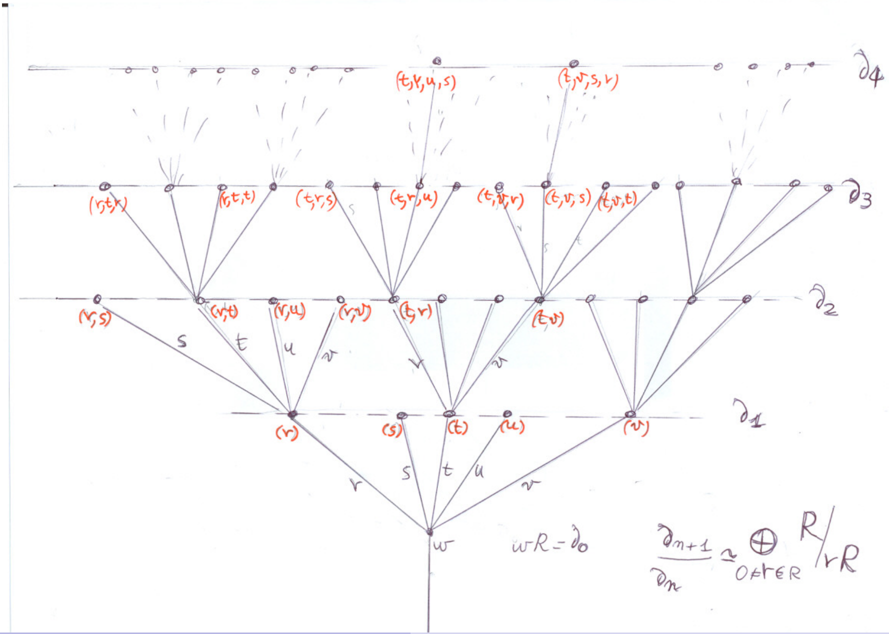
$\mathcal{T}_Q$= class of divisible modules if and only if the Fuchs' divisible module ∂ is in $\mathcal{T}_Q$.

∂ is a tilting module and Gen ∂ is the class of divisible modules.

∂ is generated by $(r_1, \dots, r_n)$, $0 \neq r_i \in R, n \geq 1$
w with $wR \cong R$ with relations

$$(r_1, \dots, r_n)r_n = (r_1, \dots r_{n-1}), n > 1, \quad (r_1)r_1 = w.$$

The Fuchs divisible module ∂



Questions

Assume proj. dim. $Q > 1$.
$\mathcal{T}_Q$ is the class of divisible modules filtered by Gen Q.

The torsion radical τ_Q associated with the torsion pair $(\mathcal{T}_Q, \mathcal{F}_Q)$ commutes with the classical torsion radical.

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QUESTION 1: Is $\mathcal{T}_Q$ closed under direct products?

If every $D \in \mathcal{T}_Q$ is finitely filtered by Gen Q, i.e. if $D = \mathrm{tr}_n(D)$, for some $n \in \mathbb{N}$, then $\mathcal{T}_Q$ is closed under products.

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QUESTION 3 Is $\mathcal{T}_Q = \mathrm{Gen}\, D$, for some $D \in \mathcal{T}_Q$?

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