On n-trivial Extensions of Rings

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joint work with
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- all rings considered in this talk are assumed to be commutative with an identity; in particular, R denotes such a ring, and all modules are assumed to be unitary modules.

Definition

The trivial extension of R by an R-module M is the ring denoted by $R \ltimes M$ whose underlying additive group is $R \oplus M$ with multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$.

- In this talk we present a part of a joint work on an extension of the classical trivial extension. For more details see arXiv:1604.01486.
- The paper presents various algebraic aspects of this new ring construction. Here, we present some of them with a special focus on the ideal structure.

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Outline

1. **Motivation, Definition and Examples**
2. Some basic algebraic properties of $R \ltimes_n M$
3. Homogeneous ideals of n-trivial extensions

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Motivation

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▶ Generalized triangular matrix ring.

- The trivial extension is related to some classical ring constructions. Namely, it is related with the following ones:

Motivation

▶ Generalized triangular matrix ring.

Let $\mathscr{R} := (R_i)_{i=1}^n$ be a family of rings and $\mathscr{M} := (M_{i,j})_{1\leq i<j\leq n}$ be a family of modules such that for each $1 \leq i < j \leq n$, $M_{i,j}$ is an (R_i, R_j)-bimodule.
Assume for every $1 \leq i < j < k \leq n$, there exists an (R_i, R_k)-bimodule homomorphism $M_{i,j} \otimes_{R_j} M_{j,k} \longrightarrow M_{i,k}$ denoted multiplicatively such that $(m_{i,j}m_{j,k})m_{k,l} = m_{i,j}(m_{j,k}m_{k,l})$.
Then the set $T_n(\mathscr{R}, \mathscr{M})$ consisting of matrices

$$\begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & \cdots & m_{1,n-1} & m_{1,n} \\ 0 & m_{2,2} & \cdots & \cdots & m_{2,n-1} & m_{2,n} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & m_{n-1,n-1} & m_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 & m_{n,n} \end{pmatrix}, \text{ where } m_{i,i} \in R_i \text{ and}$$

$m_{i,j} \in M_{i,j}$ $(1 \leq i < j \leq n)$, with the usual matrix addition and multiplication is a ring called a *generalized (or formal) triangular matrix* ring.

Motivation

▶ Generalized triangular matrix ring.

The trivial extension $R \ltimes M$ is naturally isomorphic to the subring of the generalized triangular matrix ring

$$T_2((R,R),M) := \begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$$

consisting of matrices $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$ where $r \in R$ and $m \in M$.

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Motivation

▶ Symmetric algebra.

➡ The symmetric algebra associated to an R-module M is the graded ring quotient $S_R(M) := T_R(M)/H$, where $T_R(M)$ is the graded tensor R-algebra with $T_R^n(M) = M^{\otimes n}$ and H is the homogeneous ideal of $T_R(M)$ generated by $\{m \otimes n - n \otimes m | m, n \in M\}$.
Note that $S_R(M) = \bigoplus_{n=0}^{\infty} S_R^n(M)$ is a graded R-algebra with $S_R^0(M) = R$ and $S_R^1(M) = M$ and, in general, $S_R^i(M)$ is the image of $T_R^i(M)$ in $S_R(M)$.

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Motivation

If M is a free R-module with a basis B, then the trivial extension $R \ltimes M$ is naturally isomorphic to the ring quotient $R[\{X_b\}_{b\in B}]/(\{X_b\}_{b\in B})^2$ where $\{X_b\}_{b\in B}$ is a set of indeterminates over R.

In particular, $R \ltimes R \cong R[X]/(X^2)$.

Defintion of n-trivial extension.

Let $M = (M_i)_{i=1}^n$ be a family of R-modules and $\varphi = \{\varphi_{i,j}\}_{\substack{i+j \leq n \\ 1 \leq i,j \leq n-1}}$ be a family of R-module homomorphisms such that each $\varphi_{i,j}$ is written multiplicatively: $\begin{array}{cccc} \varphi_{i,j}: & M_i \otimes M_j & \longrightarrow & M_{i+j} \\ & m_i \otimes m_j & \longmapsto & \varphi_{i,j}(m_i, m_j) := m_i m_j. \end{array}$ such that

- $(m_i m_j) m_k = m_i (m_j m_k)$ for $m_i \in M_i$, $m_j \in M_j$ and $m_k \in M_k$ with $1 \leq i, j, k \leq n-2$ and $i + j + k \leq n$, and
- $m_i m_j = m_j m_i$ for every $m_i \in M_i$ and $m_j \in M_j$ with $1 \leq i, j \leq n-1$ and $i + j \leq n$.

Then, the additive group $R \oplus M_1 \oplus \cdots \oplus M_n$ endowed with the multiplication

$$(m_0, ..., m_n)(m'_0, ..., m'_n) = (\sum_{j+k=i} m_j m'_k)$$

for all $(m_i), (m'_i) \in R \ltimes_\varphi M$, is a ring called the n-φ-trivial extension of R by M or simply the n-trivial extension of R by M. It will be denoted by $R \ltimes_\varphi M_1 \ltimes \cdots \ltimes M_n$ or simply $R \ltimes_\varphi M$.

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Relations with some classical ring constructions.

- Generalized triangular matrix ring. $R \ltimes_n M$ is **naturally isomorphic** to the subring of the generalized triangular matrix ring

$$\begin{pmatrix} R & M_1 & M_2 & \cdots\cdots & M_n \\ 0 & R & M_1 & \cdots & M_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & M_1 \\ 0 & 0 & 0 & \cdots & R \end{pmatrix}$$

consisting of matrice

$$\begin{pmatrix} r & m_1 & m_2 & \cdots\cdots & m_n \\ 0 & r & m_1 & \cdots & m_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & m_1 \\ 0 & 0 & 0 & \cdots & r \end{pmatrix}$$

where $r \in R$ and $m_i \in M_i$ for every $i \in \{1, ..., n\}$.

- **Symmetric algebra.** When, for every $k \in \{1, ..., n\}$, $M_k = S_R^k(M_1)$, the ring $R \ltimes_n M$ is **naturally isomorphic** to $S_R(M_1)/\bigoplus_{k\geq n+1} S_R^k(M_1)$.

- **Polynomial ring.**

In particular, if $M_1 = F$ is a free R-module with a basis B, then the n-trivial extension $R \ltimes F \ltimes S_R^2(F) \ltimes \cdots \ltimes S_R^n(F)$ is also **naturally isomorphic** to $R[\{X_b\}_{b\in B}]/(\{X_b\}_{b\in B})^{n+1}$ where $\{X_b\}_{b\in B}$ is a set of indeterminates over R.

Namely, when $F \cong R$,

$$R \ltimes_n R \ltimes \cdots \ltimes R \cong R[X]/(X^{n+1}).$$

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n-Trivial extensions is not in general 1-trivial extensions.

Remark

- If R_1 and R_2 are two rings and H an (R_1, R_2)-bimodule. Then,

$$T_2((R_1, R_2), H) \cong (R_1, R_2) \ltimes H,$$

where the actions of $R_1 \times R_2$ on H are defined as follows: $(r_1, r_2)h = r_1 h$ and $h(r_1, r_2) = hr_2$ for every $(r_1, r_2) \in R_1 \times R_2$ and $h \in H$.
- Every generalized triangular matrix ring is isomorphic to an n-trivial extension.
- Every generalized triangular matrix ring is isomorphic to generalized triangular matrix ring of order 2.

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➦ However, an *n*-trivial extension with $n \geq 2$ is not necessarily a 1-trivial extension.
For instance, the 2-trivial extension $\mathbb{Z}/2\mathbb{Z} \ltimes_2 \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/2\mathbb{Z}$ cannot be isomorphic to any 1-trivial extension.

Examples

Example

Let $N_1,\dots,N_n$ be R-submodules of an R-algebra T with $N_iN_j \subseteq N_{i+j}$ for $1 \leq i, j \leq n-1$ with $i+j \leq n$. Then, if the module homomorphisms are just the multiplication of L (and so will be, in the sequel, if they are not specified), then $R \ltimes_n N_1 \ltimes \cdots \ltimes N_n$ is an n-trivial extension. The following examples are some special cases:

- Let I be an ideal of R. Then $R \ltimes_n I \ltimes I^2 \ltimes \cdots \ltimes I^n$ is the quotient of the Rees ring $R[It]/(I^{n+1}t^{n+1})$, where $R[It] := \bigoplus_{n \geq 0} I^n t^n$.
- Let T be an R-algebra and $J_1 \subseteq \cdots \subseteq J_n$ ideals of T. Then $R \ltimes_n J_1 \ltimes \cdots \ltimes J_n$ is an example of n-trivial extension since $J_iJ_j \subseteq J_i \subseteq J_{i+j}$ for $i+j \leq n$. For example, we could take $R \ltimes_2 XR[X] \ltimes R[X]$.

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Particular cases of *n*-trivial extensions have been already introduced and used to solve some open questions.

- P. Ara, W. K. Nicholson and M. F. Yousif (2001) introduced a particular case of 2-trivial extensions and they used it in the study of the so-called Faith conjecture.
- Also, a particular case of 2-trivial extensions is introduced by V. Camillo, I. Herzog and P. P. Nielsen, (2007), to give an example of a ring which has a non-self-injective injective hull with compatible multiplication. This gave a negative answer of a question posed by Osofsky.
- Z. Pogorzaly (2005) introduced and studied a particular case of 3-trivial extensions to obtain a Galois coverings for the enveloping algebras of trivial extension algebras of triangular algebras.
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2. Some basic algebraic properties of $R \ltimes_n M$
3. Homogeneous ideals of n-trivial extensions

Notation

Unless specified otherwise, $M = (M_i)_{i=1}^n$ is a family of R-modules with module homomorphisms as indicated in the definition of the n-trivial extension. So $R \ltimes_n M$ is indeed a (commutative) ring with identity $(1, 0, ..., 0)$.

Let S be a nonempty subset of R and $N = (N_i)_{i=1}^n$ be a family of sets such that, for every i, $N_i \subseteq M_i$. Then as a subset of $R \ltimes_n M$, $S \times N_1 \times \cdots \times N_n$ will be denoted by $S \ltimes_n N_1 \ltimes \cdots \ltimes N_n$ or simply $S \ltimes_n N$.

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Unless specified otherwise, $M = (M_i)_{i=1}^n$ is a family of R-modules with module homomorphisms as indicated in the definition of the n-trivial extension. So $R \ltimes_n M$ is indeed a (commutative) ring with identity $(1, 0, ..., 0)$.
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However, if $n \geq 3$ and there is an integer $i \in \{1, ..., n-2\}$ such that, for $j \in \{1, ..., n\}$, $M_j = 0$ if and only if $j \in \{1, ..., i\}$, then $R \ltimes_{n-i} M_{i+1} \ltimes \cdots \ltimes M_n$ has no sense, since (when, for example, $2i+2 \leq n$) $\varphi_{i+1,i+1}(M_{i+1} \otimes M_{i+1})$ is a subset of M_{2i+2} not of M_{i+2}.

- If $M_{2k} = 0$ for every $k \in \mathbb{N}$ with $1 \leq 2k \leq n$, then

$$R \ltimes_n M \cong R \ltimes_1 (M_1 \times M_3 \times \cdots \times M_{2n'+1}),$$

where $2n'+1$ is the biggest odd integer in $\{1, ..., n\}$.

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Observations

Convention. Unless explicitly stated otherwise, when we consider an *n*-trivial extension for a given *n*, then we implicitly suppose that $M_i \neq 0$ for every $i \in \{1, ..., n\}$.

Particular kind of ideals

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Proposition

We have the following (natural) ring extension :
$i_n : R \hookrightarrow R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$. Then, for an ideal I of R, the ideal $I \ltimes_n IM_1 \ltimes \cdots \ltimes IM_n$ of $R \ltimes_n M$ is the extension of I under the ring homomorphism i_n.

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Remark

- In the case of $n = 1$, the ideal structure of $0 \ltimes_1 M_1$ is the same as the R-module structure of $0 \ltimes_1 M_1$.
- However, for $n \geq 2$, the R-module structure of $0 \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ need not be the same as the ideal structure.
- For instance, consider the 2-trivial extension $\mathbb{Z} \ltimes_2 \mathbb{Z} \ltimes \mathbb{Z}$. Then $\mathbb{Z}(0, 1, 1) = \{(0, m, m) | m \in \mathbb{Z}\}$ while the ideal of $\mathbb{Z} \ltimes_2 \mathbb{Z} \ltimes \mathbb{Z}$ generated by $(0, 1, 1)$ is $0 \ltimes_2 \mathbb{Z} \ltimes \mathbb{Z}$.

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Proposition

- For every $m \in \{1, ..., n\}$, $0 \ltimes_n 0 \ltimes \cdots \ltimes 0 \ltimes M_m \ltimes \cdots \ltimes M_n$ is an ideal of $R \ltimes_n M$ and an $R \ltimes_j M_1 \ltimes \cdots \ltimes M_j$-module for every $j \in \{n-m, ..., n\}$ via the action

$$\begin{aligned}(x_0, ..., x_j)(0, ..., 0, y_m, ..., y_n) &:= (x_0, x_1, ..., x_j, 0, ..., 0)(0, ..., 0, y_m, ..., y_n)\\ &= (x_0, x_1, ..., x_{n-m}, 0, ..., 0)(0, ..., 0, y_m, ..., y_n)\end{aligned}$$

- Moreover, the structure of $0 \ltimes_n 0 \ltimes \cdots \ltimes 0 \ltimes M_m \ltimes \cdots \ltimes M_n$ as an ideal of $R \ltimes_n M$ is the same as the $R \ltimes_j M_1 \ltimes \cdots \ltimes M_j$-module structure for every $j \in \{n-m, ..., n\}$.
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Lemma

Every ideal of $R \ltimes_n M$ which contains $0 \ltimes_n M$ has the form $I \ltimes_n M$ for some ideal I of R. In this case, we have the following natural ring isomorphism:

$$R \ltimes_n M / I \ltimes_n M \cong R/I.$$

Theorem

Radical ideals of $R \ltimes_n M$ have the form $I \ltimes_n M$ where I is a radical ideal of R.

In particular, the maximal (resp., the prime) ideals of $R \ltimes_n M$ have the form $\mathcal{M} \ltimes_n M$ (resp, $P \ltimes_n M$) where $\mathcal{M}$ (resp., P) is a maximal (resp., a prime) ideal of R.

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Corollary

The Jacobson radical $J(R \ltimes_n M)$ (resp., the nilradical $Nil(R \ltimes_n M)$) of $R \ltimes_n M$ is $J(R) \ltimes_n M$ (resp., $Nil(R) \ltimes_n M$) and the Krull dimension of $R \ltimes_n M$ is equal to that of R.

Proposition

The following assertions are true.

- The set $Z(R \ltimes_n M)$ of zero divisors of $R \ltimes_n M$ is the set of elements $(r, m_1, ..., m_n)$ such that $r \in Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n)$. Hence $S \ltimes_n M$ where $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$ is the set of regular elements of $R \ltimes_n M$.
- The set of units of $R \ltimes_n M$ is $U(R \ltimes_n M) = U(R) \ltimes_n M$.
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Outline

1. Motivation, Definition and Examples
2. Some basic algebraic properties of $R \ltimes_n M$
3. **Homogeneous ideals of n-trivial extensions**

Graded rings

Graded rings

Let Γ be a commutative additive monoid. Recall that a ring S is said to be a Γ-*graded* ring, if there is a family of subgroups of S, $(S_\alpha)_{\alpha\in\Gamma}$, such that $S = \bigoplus_{\alpha\in\Gamma} S_\alpha$ as an abelian group, with $S_\alpha S_\beta \subseteq S_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$.

Graded rings

The n-trivial extension $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$ may be considered as an $\mathbb{N}_0$-graded ring ($\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), where, in this case we set $M_k = 0$ for all $k \geq n+1$ and $\varphi_{i,j}$ are naturally extended to all $i, j \geq 0$.

Graded rings

Let Γ be a commutative additive monoid and $S = \bigoplus_{\alpha\in\Gamma} S_\alpha$ be a Γ-graded ring. And an S-module N is said to be Γ*-graded* if $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$ (as an abelian group) and $S_\alpha N_\beta \subseteq N_{\alpha+\beta}$ for all $\alpha, \beta \in \Gamma$. Let $N = \bigoplus_{\alpha\in\Gamma} N_\alpha$ be a Γ-graded S-module. For every $\alpha \in \Gamma$, the elements of N_α are said to be *homogeneous of degree* α. A submodule N' of N is said to be *homogeneous* if one of the following equivalent assertions is true.

(1) N' is generated by homogeneous elements,

(2) If $\sum_{\alpha\in G'} n_\alpha \in N'$, where G' is a finite subset of Γ and each n_α is homogeneous of degree α, then $n_\alpha \in N'$ for every $\alpha \in G'$, or

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Graded rings

In particular, an ideal J of $R \ltimes_n M$ is homogeneous if and only if $J = (J \cap R) \oplus (J \cap M_1) \oplus \cdots \oplus (J \cap M_n)$. Note that $I := J \cap R$ is an ideal of R and, for $i \in \{1, ..., n\}$, $N_i := J \cap M_i$ is an R-submodule of M_i which satisfies $IM_i \subseteq N_i$ and $N_i M_j \subseteq N_{i+j}$ for evey $i, j \in \{1, ..., n\}$.

Theorem

1. Let I be an ideal of R and let $C = (C_i)_{i\in\{1,...,n\}}$ be a family of R-modules such that $C_i \subseteq M_i$ for every $i \in \{1,...,n\}$. Then $I \ltimes_n C$ is a (homogeneous) ideal of $R \ltimes_n M$ if and only if $IM_i \subseteq C_i$ and $C_iM_j \subseteq C_{i+j}$ for all $i, j \in \{1,...,n\}$ with $i + j \leq n$.
2. Let J be an ideal of $R \ltimes_n M$ and consider K the projection of J onto R and N_i the projection of J onto M_i for every $i \in \{1,...,n\}$. Then,
   1. K is an ideal of R and N_i is a submodule of M_i for every $i \in \{1,...,n\}$ such that $KM_i \subseteq N_i$ and $N_iM_j \subseteq N_{i+j}$ for every $j \in \{1,...,n\}$ with $i + j \leq n$.
      Thus $K \ltimes_n N_1 \ltimes \cdots \ltimes N_n$ is a homogeneous ideal of $R \ltimes_n M_1 \ltimes \cdots \ltimes M_n$.
   2. $J \subseteq K \ltimes_n N_1 \ltimes \cdots \ltimes N_n$.
   3. The ideal J is homogeneous if and only if $J = K \ltimes_n N_1 \ltimes \cdots \ltimes N_n$.

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Remark and question on homogeneous ideals

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- Every radical (hence prime) ideal of $R \ltimes_n M$ is homogeneous.
- However, it is well-known that the ideals of the classical trivial extensions are not in general homogeneous.
  For instance, consider a quasi-local ring R with maximal m. Then,
  - a proper homogeneous ideal of $R \ltimes R/m$ has either the form $I \ltimes R/m$ or $I \ltimes 0$ where I is a proper ideal of R, and
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When every ideal in a given class $\mathscr{I}$ of ideals of $R \ltimes_n M$ is homogeneous?

- Various particular classes of ideals were treated. Here we present two of them. Namely, we show when every regular ideal is homogeneous and when every regular ideal is homogeneous.

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When every regular ideal of $R \ltimes_n M$ is homogeneous

- An ideal is said to be *regular* if it contains a regular element.
- Thus, an ideal of $R \ltimes_n M$ is regular if and only if it contains an element $(s, m_1, ..., m_n)$ with $s \in R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$.

Theorem

Let $S = R - (Z(R) \cup Z(M_1) \cup \cdots \cup Z(M_n))$. Then the following assertions are equivalent.

1. Every regular ideal of $R \ltimes_n M$ is homogeneous.
2. For every $s \in S$ and $i \in \{1, ..., n\}$, $sM_i = M_i$ (or equivalently, $M_{iS} = M_i$).

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When every ideal of $R \ltimes_n M$ is homogeneous

The question of when every ideal of $R \ltimes_n M$ is homogeneous is still open.
Here, we present a partial answer. For this, we need the following definition:

When every ideal of $R \ltimes_n M$ is homogeneous

Definition

Assume that $n \geq 2$. For $i \in \{1, ..., n-1\}$ and $j \in \{2, ..., n\}$ with $ji \leq n$, M_i is said to be *j-integral* if, for any j elements $m_{i_1}, ..., m_{i_j}$ of M_i, if the product $m_{i_1} \cdots m_{i_j} = 0$, then at least one of the m_{i_k}'s is zero.

When every ideal of $R \ltimes_n M$ is homogeneous

Theorem

Suppose that $n \geq 2$ and R is an integral domain. Assume that M_i is torsion-free, for every $i \in \{1, ..., n-1\}$, and that M_1 is k-integral for every $k \in \{2, ..., n-1\}$. Then the following assertions are equivalent.

1. Every ideal of $R \ltimes_n M$ is homogeneous.
2. The following two conditions are satisfied:
   - i. For every $i \in \{1, ..., n\}$, M_i is divisible, and
   - ii. For every $i \in \{2, ..., n\}$ and every $m_1 \in M_1 - \{0\}$, $M_i = m_1 M_{i-1}$.

Example

Every ideal of the following n-trivial extensions $\mathbb{Z} \ltimes_n \mathbb{Q} \ltimes \cdots \ltimes \mathbb{Q}$ and $\mathbb{Z} \ltimes_n \mathbb{Q} \ltimes \cdots \ltimes \mathbb{Q} \ltimes \mathbb{Q}/\mathbb{Z}$ is homogeneous.

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Thank you!