n-universal subsets and Newton sequences

Paul-Jean Cahen and Jean-Luc Chabert

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The sequence of integers

The sequence $\{n\}_{n\geq 0}$ is remarkable for integer-valued polynomials.
One can test polynomials of degree at most n on $0, 1, \ldots, n$:

$$f(0), f(1), \ldots, f(n) \in \mathbb{Z} \Longrightarrow f(\mathbb{Z}) \subseteq \mathbb{Z}.$$

Indeed, one can (uniquely) write

$$f = \alpha_0 + \alpha_1 X + \alpha_2 \binom{X}{2} + \ldots + \alpha_n \binom{X}{n},$$

$$\text{where } \binom{X}{k} = \frac{\prod_{0\leq i<k}(X-i)}{k!},$$

and then compute the α_k's in term of $f(0), f(1), \ldots f(n)$.
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Considering $f(X-k)$, one can test f on $n+1$ consecutive integers!

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Unfortunately, in more general settings, there are often no strong Newton sequences (let alone strong ones!), either for

- integer-valued polynomials on the ring $\mathcal{O}_K$ of integers of a number field K, $*$
- integer-valued polynomials on a subset (of $\mathbb{Z}$, or a domain D.)

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Definitions and notations

Notations

Let D be a domain with quotient field K.

- If E is a subset of D,

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}$$

  denotes the ring of *integer-valued polynomials on E* (with respect to D).
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Definition

A subset S of E is said to be an *n-universal subset* of E (with respect to D) if, for each $f \in K[X]$ with $\deg(f) \leq n$,

$$f(S) \subseteq D \Longrightarrow f(E) \subseteq D.$$

That is, $f \in \mathrm{Int}(S, D) \iff f \in \mathrm{Int}(E, D).*$

If S is an n-universal subset S of E then $\mathrm{Card}(S) \geq n+1$.

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A *Newton sequence of length n* of E is a sequence $a_0, \ldots, a_n$ in E such that, for each $k \leq n$,
$\{a_0, \ldots, a_k\}$ is a k-optimal subset of E.

Its terms must be distinct (we assume, $\mathrm{Card}(E) \geq n + 1$).

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What we are up to

We are inspired by [BFS]:
Simultaneous p-orderings and minimising volumes in number fields.
J. Byszewski, M. Frączyk, and A. Szumowicz,
arXiv:1506.02696 [math.NT], 8 Jun. 2015.

They study n-universal subsets of a Dedekind domain D.

We wish to

- generalize their results to n-universal subsets of a subset E of D (rather than D itself),
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1- Generalities

Throughout this section,
E is a subset of a domain D (with quotient field K).

Trivial results

By definition an n-universal subset is k-universal for each $k \leq n$.$*$

Proposition (Transitivity)

Let $T \subseteq S \subseteq E$. Then T is an n-universal subset of E, if and only if T is an n-universal subset of S, and S is an n-universal subset of E.

Corollary

Let S be an n-universal subset of E. Then,

1. *for each $k \leq n$, a k-universal (resp. k-optimal) subset of S, is a k-universal (resp. k-optimal) subset of E.*
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Proposition

Let S be a subset of E. Each one of the following assertions implies the next one.

(i) *S contains a Newton sequence of length n of E.*

(ii) *S contains an n-optimal subset of E.*

(iii) *S is an n-universal subset of E.*

The converse of each implication does not hold in general: $*$ for instance an n-optimal subset may fail to contain a k-optimal subset for some $k < n$.

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Testing polynomials

As in [BFS]:

Proposition

$S = \{a_0, a_1, \ldots, a_{n-1}\}$ is an n-optimal subset of E if and only if, for each k, the Lagrange interpolation polynomial

$$Q_k = \prod_{j \neq k} \frac{X - a_j}{a_k - a_j}$$

is integer-valued on E.

Testing polynomials

Definition

The *generalized binomials* associated to a sequence $a_0, \ldots, a_n$ in E (with distinct terms) are the polynomials

$$\binom{X}{a_0} = 1, \text{ and, for } 1 \leq k \leq n, \binom{X}{a_k} = \prod_{0 \leq i < k} \frac{X - a_i}{a_k - a_i},$$

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Localization

Lemma

*If S is an n-universal subset of E (resp. of D) with respect to D, and if either S is **finite** or D is **Noetherian**, then S is an n-universal subset of E (resp. of $T^{-1}D$) with respect to $T^{-1}D$.*

[BFS] gives it as trivial (without hypothesis).
But there are counterexamples in the general case.

Theorem

S is an n-optimal subset of E (resp. of D) with respect to D if and only if, for each maximal ideal $\mathfrak{m}$ of D, S is an n-optimal subset of E (resp. of $D_{\mathfrak{m}}$) with respect to $D_{\mathfrak{m}}$.

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We say S is an *n-locally Newton orderable* subset of E if, for each maximal ideal $\mathfrak{m}$ of D, S can be ordered as a Newton sequence of length n of E with respect to $D_{\mathfrak{m}}$.

Corollary

An n-locally Newton orderable subset is an n-optimal subset.

The converse holds if D is a Dedekind domain $*$.

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Generalized factorial ideals

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The n^{th} *generalized factorial ideal* of E (with respect to D) is

$$n!_E^D = \{a \in D \mid \forall f \in \mathrm{Int}(E, D), \deg(f) \leq n, af \in D[X]\}.$$

In case $E = D$, one simply writes $n!_D$ for $n!_D^D$.

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1. *The ideals* $n!_E^D$ *form a decreasing sequence, with* $0!_E^D = D$.
2. $n!_E^D \neq (0)$, *if and only if* $\mathrm{Card}(E) \geq n+1$.
3. *If* $S \subseteq E$, *then* $n!_S^D \subseteq n!_E^D$.
4. *If* S *is an* n*-universal subset of* E, *then* $n!_S^D = n!_E^D$.

If moreover D *is Noetherian, then*

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1. *The ideals $n!_E^D$ form a decreasing sequence, with $0!_E^D = D$.*
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If E admits a Newton sequence $a_0, \ldots, a_n$ then

$$n!_E^D = \prod_{0 \le i < n} (a_n - a_i)D.$$

Not difficult to prove using the associated generalized binomials:

$$a \in n!_E^D \iff \forall k \le n,\ a\binom{X}{a_k} \in D[X].$$

Corollary

If E admits a Newton sequence of length n, then, for each $k \le n$, $k!_E^D$ is a principal ideal ($k!_E^D = \prod_{0 \le i < k}(a_k - a_i)D$).

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The *volume* of $S = \{a_0, a_1, \ldots, a_{n-1}\}$ is the principal ideal

$$\mathrm{Vol}(S) = \prod_{0\leq i<j\leq n} (a_j - a_i)D.*$$

Corollary

If S can be ordered as a Newton sequence, then

$$\mathrm{Vol}(S) = 1!_E^D \ldots n!_E^D.$$

Proof. Write $\prod_{0\leq i<j\leq n}(a_j - a_i) = \prod_{1\leq k\leq n}\left(\prod_{0\leq i<k}(a_k - a_i)\right)$. □

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Volume

If D is Noetherian, we can use the good localization properties:

Corollary

Assume D is a Noetherian domain. If S is an n-locally Newton orderable subset of E then

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Let $S = \{a_0, a_1, \ldots, a_{n-1}\} \subseteq E$.
We say that $\mathrm{Vol}(S)$ is *minimal* (in E) if,
for each $T = \{b_0, b_1, \ldots, b_{n-1}\} \subseteq E$, $\mathrm{Vol}(T) \subseteq \mathrm{Vol}(S)$,
that is, $\prod_{0 \le i < j \le n}(a_j - a_i)$ divides $\prod_{0 \le i < j \le n}(b_j - b_i)$.

As [BFS] (for $E = D$ and D a Dedekind domain):

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If $\mathrm{Vol}(S)$ *is minimal, then* S *is an* n*-optimal subset of* E.

Converse? At least in Dedekind domains!.

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2-Dedekind domains

We first consider the local case.

Notations

- V is a discrete valuation domain,
- v the corresponding valuation,
- $\mathfrak{m}$ the maximal ideal of V,
- t a uniformizing element (that is, $\mathfrak{m} = Vt$, and $v(t) = 1$),
- $q = \mathrm{Card}(V/\mathfrak{m})$ the cardinality (finite or infinite) of the residue field,
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Bhargava's v-orderings

Definition

A v-ordering of E of length n (possibly with $n = \infty$)
is a sequence $a_0, a_1, \ldots, a_n$ in E defined inductively as follows:

- a_0 is arbitrarily chosen,
- a_1 is chosen such that $v(a_1 - a_0)$ is minimal, that is

$$\forall x \in E,\ v(a_1 - a_0) \leq v(x - a_0),$$

- and so on,

$$\forall x \in E,\ v\left(\prod_{i=0}^{k-1}(a_k - a_i)\right) \leq v\left(\prod_{i=0}^{k-1}(x - a_i)\right). \quad (1)$$

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$a_0, \ldots, a_n$ are distinct if and only if $\mathrm{Card}(E) \geq n+1$.
(our running assumption).

(1) means that $\binom{X}{a_k} = \prod_{0 \leq i < k} \frac{X - a_i}{a_k - a_i}$ is integer-valued:

- *A v-ordering of length n of E is nothing else than a Newton sequence of length n of E.*
- *E always admits a Newton sequence of length n.*
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Characterization of n-universal subsets (local case)

Proposition

Let S be a subset of E. The following assertions are equivalent.

(i) *S is an n-universal subset of E.*

(ii) *S contains a Newton sequence of length n of E.*

(iii) *S contains an n-optimal subset of E.*

Proof. For (i) $\Longrightarrow$ (ii), consider a Newton sequence of S and use transitivity. All other implications hold in any domain D. $\square$

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Characterization of n-universal subsets (global case)

We now turn to a Dedekind domain D and a subset E of D, always with $\mathrm{Card}(E) \geq n+1$.

Each maximal ideal $\mathfrak{m}$ is associated to a discrete valuation $v_{\mathfrak{m}}$.

Definition

An $\mathfrak{m}$-*ordering of length n of E* is a $v_{\mathfrak{m}}$-ordering $a_0, \ldots, a_n$, that is, a Newton sequence of E with respect to $D_{\mathfrak{m}}$.

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Characterization of n-universal subsets (global case)

We now turn to a Dedekind domain D and a subset E of D, always with $\mathrm{Card}(E) \geq n+1$.
Each maximal ideal $\mathfrak{m}$ is associated to a discrete valuation $v_{\mathfrak{m}}$.

Definition

An $\mathfrak{m}$-*ordering of length n of E* is a $v_{\mathfrak{m}}$-ordering $a_0, \ldots, a_n$, that is, a Newton sequence of E with respect to $D_{\mathfrak{m}}$.

Corollary

Let S be a subset of E. The following assertions are equivalent.

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Theorem

Let S be a subset of E with $\mathrm{Card}(S) = n+1$. The following assertions are equivalent.

(i) *S is an n-optimal subset of E,*

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Let $T = \{b_0, \ldots, b_n\}$ be a subset of E. Then

$$\mathrm{Vol}(T) \subseteq 1!_E^D \ldots n!_E^D.$$

This is a result of Bhargava (in another wording).

Proof. T is obviously an n-optimal subset of itself. Thus T can locally be ordered as a Newton sequence of itself. Therefore

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As $T \subseteq E$, $k!_T^D \subseteq k!_E^D$, for each k. $\square$

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From a previous study of maximal lengths of Newton sequences:

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Second example, a (rank-one) non-discrete valuation domain V.

It is well known that

$\mathrm{Int}(V) = V[X]$, *and for all* n, $\mathfrak{m}$ *is an* n*-universal subset of* V.$*$

For $n \geq 1$, *there is no finite* n*-universal subset of* $\mathfrak{m}$,
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Proof. Let $x_0 \in S$ be such that $\forall x \in S, v(x_0) \leq v(x)$,
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For $n \geq q = \mathrm{Card}(V/\mathfrak{m})$, there is no finite (let alone optimal) n-universal subset of V. (But for $n < q$ there exists a Newton sequence of length n of $V*$).

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Last, $D = \mathbb{F}_2[[x^2, x^3]]$ ($\mathbb{F}_2$ the field with 2 elements).

D is a pseudo-valuation domain (contained in $V = \mathbb{F}_2[[x]]$).
It is a one-dimensional Noetherian local domain,
with maximal ideal $\mathfrak{m} = (x^2, x^3)$.

Set $E = \{0, 1, x^2, x^3\}$.

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3 - Almost strong Newton sequences

In all generality

Definition

A sequence $\{a_n\}$ (finite or infinite) in a subset E of the domain D, is said to be a *strong Newton sequence* of E if, for each k, every set of $k+1$ consecutive terms is a k-optimal subset of E.

Equivalently:

For each r, the truncated sequence $\{a_n\}_{n \geq r}$ is a Newton sequence of E.

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Back to Dedekind domains, we first look at the local case.
Just as a Newton sequence is but a v-ordering,

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As before. In particular, $\mathrm{Card}(V/\mathfrak{m}) = q$, t a uniformizing element: $v(t) = 1$. Moreover, for each $m \in \mathbb{N}$, $v_q(m)$ denotes the largest k such that q^k divides m.

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The following assertions are equivalent:

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Lemma

Recipe to obtain a strong v-ordering $\{a_n\}_{n\geq 0}$ of V:

- *take $a_0 = 0$,*
- *for $0 < n < q$, take $a_n \not\equiv a_{n-1} \pmod{\mathfrak{m}}$, $*$*
- *for $q^k \leq n < q^{k+1}$, take*

$$a_n \equiv a_i t^k + a_r \pmod{\mathfrak{m}^{k+1}},$$

where $n = iq^k + r$, with $r < q^k$(euclidian division) and $i < q*$.

Note this recipe fits for $V/\mathfrak{m}$ infinite. $*$.

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Almost strong Newton sequence

Theorem

Let D be a Dedekind domain. There is a sequence $\{a_n\}_{n\geq 0}$ in D such that,

1. *for each maximal ideal $\mathfrak{m}$ of D, remove at most one term you get a strong $\mathfrak{m}$-ordering!*
2. *Any $n+2$ consecutive terms form an n-universal subset of D.*

Postpone 1, 2 follows:

Proof. Consider $n+2$ consecutive terms of $\{a_n\}_{n\geq 0}$.
For each $\mathfrak{m}$, remove at most one term,
you are left with $n+1$ consecutive terms of a strong Newton sequence of $D_{\mathfrak{m}}$, thus with an n-optimal subset of $D_{\mathfrak{m}}$. $\square$

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Proof of 1. We build $\{a_n\}_{n\geq 0}$ inductively, so that, for each $\mathfrak{m}$, it (almost) meets the congruence conditions of Julie Yeramian's construction. We use the Chinese remainder theorem.

- First take $a_0 = 0$.

As we use the Chinese remainder theorem, choose (arbitrarily) a finite set M_1 of maximal ideals.

- Take a_1 to satisfy Julie's conditions with respect to each $\mathfrak{m} \in M_1$.

Miracle!

*In fact, a_1 is suitable for **all** but finitely many maximal ideals.* $*$

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Let M_2 be the finite set of offending maximal ideals.
Observe that M_2 does not meet M_1.
Discard a_1 for each $\mathfrak{m} \in M_2$.

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Subsets

The situation is more intricate for subsets, even in the local case:

A subset E of a valuation domain V admits a strong v-ordering if and only if it is regular.

The notion of regularity was introduced by Yvette Amice in 1964.
It is a (somewhat technical) property of repartition.
Here is the definition in case the residue field is finite:

Definition

A subset E of V is *regular* when, for each k, each class modulo $\mathfrak{m}^k$ that meets E contains the same number of classes modulo $\mathfrak{m}^{k+1}$ that meets E.

We can generalize Julie's construction to build inductively strong v-orderings of regular subsets by congruence conditions.

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A finite union of classes modulo an ideal is locally regular.

Yet we were able to extend our construction to one class only:

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Prime numbers

We finally consider the set $\mathbb{P}$ formed by the prime numbers in $\mathbb{Z}$.

$\mathbb{P}$ is not locally regular subset, but almost:

For each p, the p-adic closure of $\mathbb{P}$ in $\mathbb{Z}_{(p)}$ is $\{p\} \cup \mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$.

As $\mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$ is a union of classes modulo p, it is regular.

Notation

For each integer m, $\mathbb{P}_{>m} = \{p \in \mathbb{P} \mid p > m\}$.

Proposition

There is a sequence in $\mathbb{P}_{>m}$ such that,for each $n < m$, any $n+2$ consecutive terms form an n-universal subset of $\mathbb{P}_{>m}$.

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For each p, the p-adic closure of $\mathbb{P}$ in $\mathbb{Z}_{(p)}$ is $\{p\} \cup \mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$.

As $\mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$ is a union of classes modulo p, it is regular.

Notation

For each integer m, $\mathbb{P}_{>m} = \{p \in \mathbb{P} \mid p > m\}$.

Proposition

*There is a sequence in $\mathbb{P}_{>m}$ such that,for each $n < m$,
any $n+2$ consecutive terms form an n-universal subset of $\mathbb{P}_{>m}$.*

Prime numbers

A last one!

Proposition

For each n, $\mathbb{P}$ admits an n-universal subset S with

$$\mathrm{Card}(S) = n + \pi(n+1).$$

(As usual, $\pi(n)$ denotes the number of primes $p \leq n$.) $*$

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The end

Thank you for your attention.