Rings of Integer-Valued Polynomials in a Valued Field which are Prüfer Domains

Jean-Luc Chabert Université de Picardie France

Graz, July 3-8, 2016

1- Introduction

Prüfer domains and Integer-Valued Polynomials

1- Introduction

Prüfer domains and Integer-Valued Polynomials (Chabert 1972)

1- Introduction

Prüfer domains and Integer-Valued Polynomials (Chabert 1972)

K a number field, $\mathcal{O}_K$ the ring of integers

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\} \quad \text{is Prüfer}$$

1- Introduction

Prüfer domains and Integer-Valued Polynomials (Chabert 1972)

K a number field, $\mathcal{O}_K$ the ring of integers

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\} \quad \text{is Prüfer}$$

***

D a domain with quotient field K :

When is $\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(D) \subseteq D\}$ a Prüfer domain?

1- Introduction

Prüfer domains and Integer-Valued Polynomials (Chabert 1972)

K a number field, $\mathcal{O}_K$ the ring of integers

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[X] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\} \quad \text{is Prüfer}$$

***

D a domain with quotient field K :

When is $\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(D) \subseteq D\}$ a Prüfer domain?

Prop. *Assume D is Noetherian. Then, $\mathrm{Int}(D)$ is Prüfer if and only if D is a Dedekind domain with finite residue fields.*

Local case:

If V *a valuation domain, then* $\mathrm{Int}(V)$ *is Prüfer if and only if the maximal ideal is principal and the residue field is finite.*

Local case:

If V a valuation domain, then $\mathrm{Int}(V)$ is Prüfer if and only if the maximal ideal is principal and the residue field is finite.

Global case:

Theorem (Loper 1998, $\mathrm{char}(D) = 0$)

The ring $\mathrm{Int}(D)$ is Prüfer if and only if

1. *D is an almost Dedekind domain with finite residue fields,*
2. $\forall p \in \mathbb{P} \left\{ \begin{array}{l} E_p = \{v_{\mathfrak{m}}(p) \mid \mathfrak{m} \in \mathrm{Max}(D),\ p \in \mathfrak{m}\} \\ F_p = \{[D/\mathfrak{m} : \mathbb{Z}/p\mathbb{Z}] \mid \mathfrak{m} \in \mathrm{Max}(D),\ p \in \mathfrak{m}\} \end{array} \right.$ *are finite.*

Local case:

If V a valuation domain, then $\mathrm{Int}(V)$ is Prüfer if and only if the maximal ideal is principal and the residue field is finite.

Global case:

Theorem (Loper 1998, $\mathrm{char}(D) = 0$)

The ring $\mathrm{Int}(D)$ is Prüfer if and only if

1. *D is an almost Dedekind domain with finite residue fields,*
2. $\forall p \in \mathbb{P}$ $\left\{ \begin{array}{l} E_p = \{v_{\mathfrak{m}}(p) \mid \mathfrak{m} \in \mathrm{Max}(D),\ p \in \mathfrak{m}\} \\ F_p = \{[D/\mathfrak{m} : \mathbb{Z}/p\mathbb{Z}] \mid \mathfrak{m} \in \mathrm{Max}(D),\ p \in \mathfrak{m}\} \end{array} \right.$ *are finite.*

Cornerstone:

If $\mathrm{Int}(D) \subseteq R[X]$ where $D \subseteq R \subsetneq K$, then $\mathrm{Int}(D)$ is not Prüfer.

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985]

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

If S is not a D-fractional subset, then $\mathrm{Int}(S, D) = D$
[S is a D-fractional subset:$= \exists d \in D$ such that $dS \subseteq D$]

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

If S is not a D-fractional subset, then $\mathrm{Int}(S, D) = D$
[S is a D-fractional subset:= $\exists d \in D$ such that $dS \subseteq D$]
$\longrightarrow$ we assume S is a D-fractional subset

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S, D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S, D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

If S is not a D-fractional subset, then $\mathrm{Int}(S, D) = D$
[S is a D-fractional subset:= $\exists d \in D$ such that $dS \subseteq D$]
$\longrightarrow$ we assume S is a D-fractional subset

$$f(X) \in \mathrm{Int}(S, D) \mapsto f\left(\frac{1}{d}X\right) \in \mathrm{Int}(dS, D)$$

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S,D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S,D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

If S is not a D-fractional subset, then $\mathrm{Int}(S,D) = D$
[S is a D-fractional subset:$= \exists d \in D$ such that $dS \subseteq D$]
$\longrightarrow$ we assume S is a D-fractional subset

$$f(X) \in \mathrm{Int}(S,D) \mapsto f\left(\frac{1}{d}X\right) \in \mathrm{Int}(dS,D) \longrightarrow \text{we assume } S \subseteq D$$

2- Integer-valued polynomials on subsets

D a domain with quotient field K and S a subset of K

$$\mathrm{Int}(S,D)=\{f(X)\in K[X]\mid f(S)\subseteq D\}$$

Under which hypotheses is $\mathrm{Int}(S,D)$ Prüfer?

D Prüfer is necessary $\longrightarrow$ we assume D is Prüfer

S finite is then sufficient [McQuillan 1985] $\longrightarrow$ we assume S is infinite

If S is not a D-fractional subset, then $\mathrm{Int}(S,D)=D$
[S is a D-fractional subset$:=\exists d\in D$ such that $dS\subseteq D$]
$\longrightarrow$ we assume S is a D-fractional subset

$$f(X)\in\mathrm{Int}(S,D)\mapsto f\left(\tfrac{1}{d}X\right)\in\mathrm{Int}(dS,D)$$ $\longrightarrow$ we assume $S\subseteq D$

$\mathrm{Int}(S,D)$ Prüfer $\Rightarrow\forall\mathfrak{p}\in\mathrm{Spec}(D)\ \ \mathrm{Int}(S,D_{\mathfrak{p}})$ Prüfer
$\longrightarrow$ we assume D is a valuation domain

What is the question?

V is a valuation domain and S is an infinite subset of V

Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

What is the question?

V is a valuation domain and S is an infinite subset of V
Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

$\mathrm{Int}(V)$ is Prüfer $\Leftrightarrow$ the valuation is discrete and the residue field finite

What is the question?

V is a valuation domain and S is an infinite subset of V
Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

$\mathrm{Int}(V)$ is Prüfer $\Leftrightarrow$ the valuation is discrete and the residue field finite
$\Leftrightarrow \widehat{V}$ is compact

What is the question?

V is a valuation domain and S is an infinite subset of V
Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

$\mathrm{Int}(V)$ is Prüfer $\Leftrightarrow$ the valuation is discrete and the residue field finite
$\Leftrightarrow \widehat{V}$ is compact

S precompact $\Rightarrow \mathrm{Int}(S, V)$ Prüfer [Cahen, C., Loper 2001]
[S *precompact* $:= \widehat{S}$ compact]

What is the question?

V is a valuation domain and S is an infinite subset of V
Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

$\mathrm{Int}(V)$ is Prüfer $\Leftrightarrow$ the valuation is discrete and the residue field finite
$\Leftrightarrow \widehat{V}$ is compact

S precompact $\Rightarrow \mathrm{Int}(S, V)$ Prüfer [Cahen, C., Loper 2001]
[S *precompact* := $\widehat{S}$ compact]

To what extend is the precompactness of S necessary for $\mathrm{Int}(S, V)$ to be Prüfer?

What is the question?

V is a valuation domain and S is an infinite subset of V
Under which hypotheses is $\mathrm{Int}(S, V)$ Prüfer?

$\mathrm{Int}(V)$ is Prüfer $\Leftrightarrow$ the valuation is discrete and the residue field finite
$\Leftrightarrow$ $\widehat{V}$ is compact

S precompact $\Rightarrow$ $\mathrm{Int}(S, V)$ Prüfer [Cahen, C., Loper 2001]
[S *precompact* $:=$ $\widehat{S}$ compact]

To what extend is the precompactness of S necessary for $\mathrm{Int}(S, V)$ to be Prüfer?

Proposition (Partial answers)

The precompactness is a necessary (and sufficient) condition for the Prüfer property under one of the following hypotheses:

1. *the valuation is discrete* [C. C. L. 2001]
2. S is a subgroup of $(V, +)$ [Park 2015]

Remark (Park, 2015)

If S is not precompact and $\mathrm{Int}(S, V)$ is Prüfer, then there exists a height-one prime ideal $\mathfrak{p}$ of V, and then:

$$S \text{ is not precompact, } \mathrm{Int}(S, V_{\mathfrak{p}}) \text{ is Prüfer and } \dim(V_{\mathfrak{p}}) = 1.$$

REMARK (PARK, 2015)

If S is not precompact and $\mathrm{Int}(S, V)$ is Prüfer, then there exists a height-one prime ideal $\mathfrak{p}$ of V, and then:

S is not precompact, $\mathrm{Int}(S, V_{\mathfrak{p}})$ is Prüfer and $\dim(V_{\mathfrak{p}}) = 1$.

From now on,

V is a rank-one valuation domain, S is a non-precompact subset of V.

We are looking for necessary conditions on S for $\mathrm{Int}(S, V)$ to be Prüfer.

Remark (Park, 2015)

If S is not precompact and $\mathrm{Int}(S, V)$ is Prüfer, then there exists a height-one prime ideal $\mathfrak{p}$ of V, and then:

$$S \text{ is not precompact, } \mathrm{Int}(S, V_{\mathfrak{p}}) \text{ is Prüfer and } \dim(V_{\mathfrak{p}}) = 1.$$

From now on,

V is a rank-one valuation domain, S is a non-precompact subset of V.

We are looking for necessary conditions on S for $\mathrm{Int}(S, V)$ to be Prüfer.

Proposition (Loper and Werner, 2016)

There exist non precompact subsets S of V such that $\mathrm{Int}(S, V)$ is Prüfer: for instance, the set formed by the elements of a pseudo-convergent sequence of transcendental type.

3- Prüfer property and polynomial closure

Polynomial closure of S : $\overline{S} = \{a \in V \mid \forall f \in \mathrm{Int}(S, V)\ f(a) \in V\}$

$$\mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V)$$

3- Prüfer property and polynomial closure

Polynomial closure of S : $\overline{S} = \{a \in V \mid \forall f \in \mathrm{Int}(S,V)\ f(a) \in V\}$

$$\mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V)$$

Notation:

$$B(a,\gamma) = \{y \in V \mid v(a-y) \geq \gamma\} \quad (a \in V, \gamma \in \mathbb{R}) \ \text{ball} \begin{cases} \text{center } a \\ \text{radius } e^{-\gamma} \end{cases}$$

3- Prüfer property and polynomial closure

Polynomial closure of S : $\overline{S} = \{a \in V \mid \forall f \in \mathrm{Int}(S, V)\ f(a) \in V\}$

$$\mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V)$$

Notation:

$$B(a, \gamma) = \{y \in V \mid v(a - y) \geq \gamma\} \quad (a \in V, \gamma \in \mathbb{R}) \text{ ball} \begin{cases} \text{center } a \\ \text{radius } e^{-\gamma} \end{cases}$$

Lemma (Cornerstone)

If $\overline{S}$ contains a ball $B(a, \gamma)$, then $\mathrm{Int}(S, V)$ is not Prüfer.

Proof.

$\mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V) \subseteq \mathrm{Int}(B(a, \gamma), V)$

3- Prüfer property and polynomial closure

Polynomial closure of S : $\overline{S} = \{a \in V \mid \forall f \in \mathrm{Int}(S,V)\ f(a) \in V\}$

$$\mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V)$$

Notation:

$$B(a,\gamma) = \{y \in V \mid v(a-y) \geq \gamma\} \quad (a \in V, \gamma \in \mathbb{R}) \ \text{ball} \begin{cases} \text{center } a \\ \text{radius } e^{-\gamma} \end{cases}$$

Lemma (Cornerstone)

If $\overline{S}$ contains a ball $B(a,\gamma)$, then $\mathrm{Int}(S,V)$ is not Prüfer.

Proof.

$$\mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V) \subseteq \mathrm{Int}(B(a,\gamma),V)$$

$$\left.\begin{array}{l} f(X) \in \mathrm{Int}(S,V) \\ v(t) = \gamma \end{array}\right\} \Rightarrow g(Y) = f(a+tY) \in \mathrm{Int}(V)$$

3- Prüfer property and polynomial closure

Polynomial closure of S : $\overline{S} = \{a \in V \mid \forall f \in \mathrm{Int}(S,V)\ f(a) \in V\}$

$$\mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V)$$

Notation:

$B(a,\gamma) = \{y \in V \mid v(a-y) \geq \gamma\} \quad (a \in V, \gamma \in \mathbb{R})$ ball $\begin{cases} \text{center } a \\ \text{radius } e^{-\gamma} \end{cases}$

Lemma (Cornerstone)

If $\overline{S}$ contains a ball $B(a,\gamma)$, then $\mathrm{Int}(S,V)$ is not Prüfer.

Proof.

$\mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V) \subseteq \mathrm{Int}(B(a,\gamma),V)$

$$\left.\begin{array}{l} f(X) \in \mathrm{Int}(S,V) \\ v(t) = \gamma \end{array}\right\} \Rightarrow g(Y) = f(a+tY) \in \mathrm{Int}(V) = V[Y]$$

$\mathrm{Int}(S,V) \subseteq V[\frac{X-a}{t}]$ □

Proposition ($\overline{S}$ is obtained by adjunction to S of)

the limits, the pseudo-limits and the corresponding closed balls (C. 2010)

Proposition ($\overline{S}$ is obtained by adjunction to S of)

the limits, the pseudo-limits and the corresponding closed balls (C. 2010)

Definition

1. [Ostrowski, 1935] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is *pseudo-convergent* if
$$\forall\, n \quad v(x_n - x_{n-1}) < v(x_{n+1} - x_n)$$

Proposition ($\overline{S}$ is obtained by adjunction to S of)

the limits, the pseudo-limits and the corresponding closed balls (C. 2010)

Definition

1. [Ostrowski, 1935] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is *pseudo-convergent* if
$$\forall\, n \quad v(x_n - x_{n-1}) < v(x_{n+1} - x_n)$$
2. [Kaplansky, 1942] x is *a pseudo-limit* of a sequence $\{x_n\}_{n\in\mathbb{N}}$ if
$$\forall\, n \quad v(x - x_n) < v(x - x_{n+1})$$

Proposition ($\overline{S}$ is obtained by adjunction to S of)

the limits, the pseudo-limits and the corresponding closed balls (C. 2010)

Definition

1. [Ostrowski, 1935] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is *pseudo-convergent* if
$$\forall\, n \quad v(x_n - x_{n-1}) < v(x_{n+1} - x_n)$$
2. [Kaplansky, 1942] x is *a pseudo-limit* of a sequence $\{x_n\}_{n\in\mathbb{N}}$ if
$$\forall\, n \quad v(x - x_n) < v(x - x_{n+1})$$

A pseudo-convergent sequence $\{x_n\}_{n\geq 0}$ does not always admit a pseudo-limit, but if it admits a pseudo-limit x, the *accuracy* of the sequence is:

$$\delta = \lim_{n\to+\infty} v(x - x_n).$$

Proposition ($\overline{S}$ is obtained by adjunction to S of)

the limits, the pseudo-limits and the corresponding closed balls (C. 2010)

Definition

1. [Ostrowski, 1935] A sequence $\{x_n\}_{n\in\mathbb{N}}$ is *pseudo-convergent* if
$$\forall\, n \quad v(x_n - x_{n-1}) < v(x_{n+1} - x_n)$$
2. [Kaplansky, 1942] x is *a pseudo-limit* of a sequence $\{x_n\}_{n\in\mathbb{N}}$ if
$$\forall\, n \quad v(x - x_n) < v(x - x_{n+1})$$

A pseudo-convergent sequence $\{x_n\}_{n\geq 0}$ does not always admit a pseudo-limit, but if it admits a pseudo-limit x, the *accuracy* of the sequence is:

$$\delta = \lim_{n\to+\infty} v(x - x_n).$$

- $\delta = +\infty \;\Leftrightarrow\; x = \lim_{n\to+\infty} x_n$
- $\delta < +\infty \;\Rightarrow\; \forall y \in B(x,\delta)\;\; y$ is a pseudo-limit of $\{x_n\}$.

Definition

x is said to be a generalized pseudo-limit *of* $\{x_n\}_{n\in\mathbb{N}}$ *if there exists* n_0 *such that, for* $n \geq n_0$*, the sequence* $\{v(x - x_n)\}$ *is:*

1. *either strictly increasing,*
2. *or strictly decreasing,*
3. *or stationary*

Definition

x is said to be a generalized pseudo-limit *of $\{x_n\}_{n\in\mathbb{N}}$ if there exists n_0 such that, for $n \geq n_0$, the sequence $\{v(x - x_n)\}$ is:*

1. *either strictly increasing,*
2. *or strictly decreasing,*
3. *or stationary with value δ, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies the following condition: $n_0 \leq n < m \Rightarrow v(x_n - x_m) = \delta$.*

Definition

x is said to be a generalized pseudo-limit *of* $\{x_n\}_{n\in\mathbb{N}}$ *if there exists* n_0 *such that, for* $n \geq n_0$*, the sequence* $\{v(x - x_n)\}$ *is:*

1. *either strictly increasing,*
2. *or strictly decreasing,*
3. *or stationary with value* δ*, and the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *satisfies the following condition:* $n_0 \leq n < m \Rightarrow v(x_n - x_m) = \delta$*.*

$\{x_n\}$ is *pseudo-convergent*, *pseudo-divergent*, or *pseudo-stationary*

Definition

x is said to be a generalized pseudo-limit *of $\{x_n\}_{n\in\mathbb{N}}$ if there exists n_0 such that, for $n \geq n_0$, the sequence $\{v(x - x_n)\}$ is:*

1. *either strictly increasing,*
2. *or strictly decreasing,*
3. *or stationary with value δ, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies the following condition: $n_0 \leq n < m \ \Rightarrow v(x_n - x_m) = \delta$.*

$\{x_n\}$ is *pseudo-convergent*, *pseudo-divergent*, or *pseudo-stationary*
the *accuracy* of the sequence is defined by:

$$\delta = \lim_{n\to+\infty} v(x - x_n).$$

Definition

x is said to be a generalized pseudo-limit *of $\{x_n\}_{n\in\mathbb{N}}$ if there exists n_0 such that, for $n \geq n_0$, the sequence $\{v(x - x_n)\}$ is:*

1. *either strictly increasing,*
2. *or strictly decreasing,*
3. *or stationary with value δ, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies the following condition: $n_0 \leq n < m \;\Rightarrow v(x_n - x_m) = \delta$.*

$\{x_n\}$ is *pseudo-convergent*, *pseudo-divergent*, or *pseudo-stationary*
the *accuracy* of the sequence is defined by:

$$\delta = \lim_{n\to+\infty} v(x - x_n).$$

Proposition (C. 2010)

If x is a generalized pseudo-limit of a sequence of elements of S with accuracy δ, then the closed ball $B(x, \delta)$ is contained in $\overline{S}$.

Corollary

If there exists $x \in V$ *which is a generalized pseudo-limit (but not a limit) of a sequence of elements of* S*, then* $\mathrm{Int}(S, V)$ *is not Prüfer.*

Corollary

If there exists $x \in V$ which is a generalized pseudo-limit (but not a limit) of a sequence of elements of S, then $\mathrm{Int}(S, V)$ is not Prüfer.

Proof. Let δ be the accuracy of the sequence. Then $B(x, \delta) \subseteq \overline{S}$
$\Rightarrow \mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V) \subseteq \mathrm{Int}(B(x, \delta), V) \subseteq V\left[\frac{X-x}{t}\right]$

Corollary

If there exists $x \in V$ which is a generalized pseudo-limit (but not a limit) of a sequence of elements of S, then $\mathrm{Int}(S, V)$ is not Prüfer.

Proof. Let δ be the accuracy of the sequence. Then $B(x, \delta) \subseteq \overline{S}$
$\Rightarrow \mathrm{Int}(S, V) = \mathrm{Int}(\overline{S}, V) \subseteq \mathrm{Int}(B(x, \delta), V) \subseteq V\left[\frac{X-x}{t}\right]$

Corollary

If $\mathrm{Int}(S, V)$ is Prüfer, then $\overline{S}$ is equal to the topological closure of S.

Because S does not contain any generalized pseudo-limit.

Corollary

If there exists $x \in V$ which is a generalized pseudo-limit (but not a limit) of a sequence of elements of S, then $\mathrm{Int}(S,V)$ is not Prüfer.

Proof. Let δ be the accuracy of the sequence. Then $B(x,\delta) \subseteq \overline{S}$
$\Rightarrow \mathrm{Int}(S,V) = \mathrm{Int}(\overline{S},V) \subseteq \mathrm{Int}(B(x,\delta),V) \subseteq V\left[\frac{X-x}{t}\right]$

Corollary

If $\mathrm{Int}(S,V)$ is Prüfer, then $\overline{S}$ is equal to the topological closure of S.

Because S does not contain any generalized pseudo-limit.

Corollary

If $\mathrm{Int}(S,V)$ is Prüfer, then S admits v-orderings.

Otherwise, S would contain a pseudo-limit of a pseudo-divergent sequence.

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma)$ (S-ball)

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma)$ (S-ball)

$$q_\gamma = \mathrm{Card}(S \bmod \gamma)$$

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma)$ (S-ball)

$$q_\gamma = \text{Card}(S \bmod \gamma) \quad \text{and} \quad \gamma_\infty = \sup\{\,\gamma \mid q_\gamma \text{ finite}\,\}$$

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma)$ (S-ball)

$$q_\gamma = \mathrm{Card}(S \bmod \gamma) \quad \text{and} \quad \gamma_\infty = \sup\{\,\gamma \mid q_\gamma \text{ finite}\,\}$$

***

S precompact $= \widehat{S}$ compact $\Leftrightarrow$ all the q_γ's are finite $\Leftrightarrow$ $\gamma_\infty = +\infty$

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma)$ (S-ball)

$$q_\gamma = \mathrm{Card}(S \bmod \gamma) \quad \text{and} \quad \gamma_\infty = \sup\{\,\gamma \mid q_\gamma \text{ finite}\,\}$$

***

S precompact $= \widehat{S}$ compact $\Leftrightarrow$ all the q_γ's are finite $\Leftrightarrow\ \gamma_\infty = +\infty$
Thus, here

$$\gamma_\infty < +\infty$$

4- Consequences of the non-compactness of S

Notation. For $\gamma \in \mathbb{R}$

$$x \equiv y \pmod{\gamma} := \quad v(x-y) \geq \gamma$$

$$x \bmod \gamma = \{y \in S \mid v(x-y) \geq \gamma\} = S \cap B(x,\gamma) := S(x,\gamma) \text{ (S-ball)}$$

$$q_\gamma = \mathrm{Card}(S \bmod \gamma) \quad \text{and} \quad \gamma_\infty = \sup\{\,\gamma \mid q_\gamma \text{ finite}\,\}$$

***

S precompact $= \widehat{S}$ compact $\Leftrightarrow$ all the q_γ's are finite $\Leftrightarrow$ $\gamma_\infty = +\infty$
Thus, here

$$\gamma_\infty < +\infty$$

Two cases:

- either γ_∞ is a maximum and q_{γ_∞} is finite,
- or γ_∞ is not a maximum and q_{γ_∞} is infinite.

FIRST CASE: q_{γ_∞} IS FINITE

There exists $x \in S$ such that, for every $\delta > \gamma_\infty$, $S(x, \gamma_\infty)$ contains infinitely many classes modulo δ.

FIRST CASE: q_{γ_∞} IS FINITE

There exists $x \in S$ such that, for every $\delta > \gamma_\infty$, $S(x, \gamma_\infty)$ contains infinitely many classes modulo δ.

Such an x is then a generalized pseudo-limit of
• either a pseudo-stationary sequence with accuracy γ_∞
• or a pseudo-divergent sequence with accuracy γ_∞

in both cases, $B(x, \gamma_\infty) \subseteq \overline{S}$

First case: q_{γ_∞} is finite

There exists $x \in S$ such that, for every $\delta > \gamma_\infty$, $S(x, \gamma_\infty)$ contains infinitely many classes modulo δ.

Such an x is then a generalized pseudo-limit of
- either a pseudo-stationary sequence with accuracy γ_∞
- or a pseudo-divergent sequence with accuracy γ_∞

in both cases, $B(x, \gamma_\infty) \subseteq \overline{S}$

Proposition

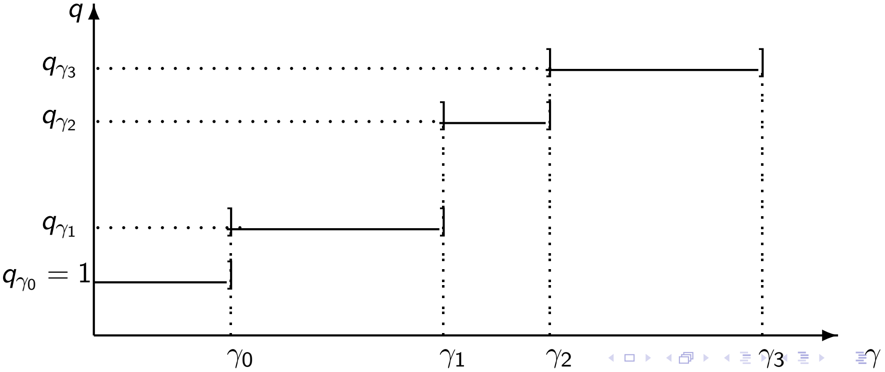
If $\mathrm{Int}(S, V)$ is Prüfer, then q_{γ_∞} is infinite.

Second case: q_{γ_∞} is infinite

There is a sequence $\{\gamma_k\}_{k\geq 0}$ of *critical valuations* of S characterized by:

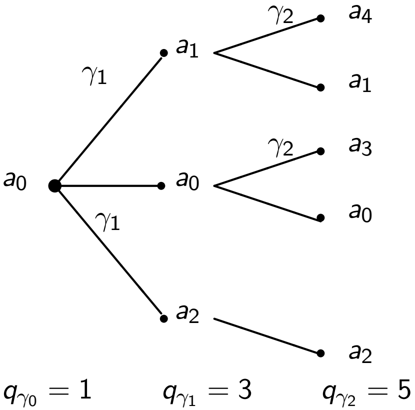
$$\text{for } k \geq 1 \;:\; \gamma_{k-1} < \gamma \leq \gamma_k \;\Leftrightarrow\; q_\gamma = q_{\gamma_k} \quad (\gamma_0 = \sup_{q_\gamma = 1} \gamma)$$

$\gamma_\infty = \lim_{k\geq 0} \gamma_k$

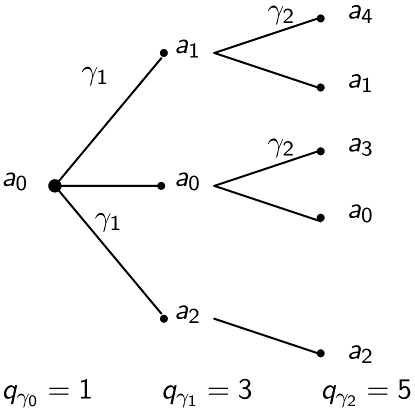


We construct inductively on k a sequence $\{a_n\}_{n\geq 0}$ of elements of S s.t. $a_0, a_1, \ldots, a_{q_{\gamma_k}-1}$, is a complete set of representatives of S mod γ_k

We construct inductively on k a sequence $\{a_n\}_{n\geq 0}$ of elements of S s.t. $a_0, a_1, \ldots, a_{q_{\gamma_k}-1}$, is a complete set of representatives of S mod γ_k



We construct inductively on k a sequence $\{a_n\}_{n\geq 0}$ of elements of S s.t. $a_0, a_1, \ldots, a_{q_{\gamma_k}-1}$, is a complete set of representatives of S mod γ_k



Since $q_{\gamma_\infty} = +\infty$, one may find a branch $\{y_k\}_{k\geq 0}$ such that from each vertex of this branch one can reach infinitely many leaves at the level γ_∞.

First subcase: $\{y_k\}_{k\geq 0}$ is ultimately stationary

Let $y \in S$ denote the constant value of the stationary sequence $\{y_k\}_{k\geq k_0}$
y is a pseudo-limit of a pseudo-convergent sequence with accuracy γ_∞
Thus, $B(y,\gamma_\infty) \subseteq \overline{S}$ and $\mathrm{Int}(S,V)$ is not Prüfer

FIRST SUBCASE: $\{y_k\}_{k\geq 0}$ IS ULTIMATELY STATIONARY

Let $y \in S$ denote the constant value of the stationary sequence $\{y_k\}_{k\geq k_0}$
y is a pseudo-limit of a pseudo-convergent sequence with accuracy γ_∞
Thus, $B(y, \gamma_\infty) \subseteq \overline{S}$ and $\mathrm{Int}(S, V)$ is not Prüfer

Application.
S is said to be a *regular subset* if, at each level k, all the vertices have the same number of edges (namely $q_{\gamma_k}/q_{\gamma_{k+1}}$)

First subcase: $\{y_k\}_{k\geq 0}$ is ultimately stationary

Let $y \in S$ denote the constant value of the stationary sequence $\{y_k\}_{k\geq k_0}$
y is a pseudo-limit of a pseudo-convergent sequence with accuracy γ_∞
Thus, $B(y, \gamma_\infty) \subseteq \overline{S}$ and $\mathrm{Int}(S, V)$ is not Prüfer

Application.
S is said to be a *regular subset* if, at each level k, all the vertices have the same number of edges (namely $q_{\gamma_k}/q_{\gamma_{k+1}}$)
If S is regular, from every vertex one can reach infinitely many leaves at the level γ_∞.

First subcase: $\{y_k\}_{k\geq 0}$ is ultimately stationary

Let $y \in S$ denote the constant value of the stationary sequence $\{y_k\}_{k\geq k_0}$
y is a pseudo-limit of a pseudo-convergent sequence with accuracy γ_∞
Thus, $B(y, \gamma_\infty) \subseteq \overline{S}$ and $\mathrm{Int}(S, V)$ is not Prüfer

Application.
S is said to be a *regular subset* if, at each level k, all the vertices have the same number of edges (namely $q_{\gamma_k}/q_{\gamma_{k+1}}$)
If S is regular, from every vertex one can reach infinitely many leaves at the level γ_∞. Consequently,

Theorem

If S is a regular, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact.

First subcase: $\{y_k\}_{k\geq 0}$ is ultimately stationary

Let $y \in S$ denote the constant value of the stationary sequence $\{y_k\}_{k\geq k_0}$
y is a pseudo-limit of a pseudo-convergent sequence with accuracy γ_∞
Thus, $B(y, \gamma_\infty) \subseteq \overline{S}$ and $\mathrm{Int}(S, V)$ is not Prüfer

Application.
S is said to be a *regular subset* if, at each level k, all the vertices have the same number of edges (namely $q_{\gamma_k}/q_{\gamma_{k+1}}$)
If S is regular, from every vertex one can reach infinitely many leaves at the level γ_∞. Consequently,

Theorem

If S is a regular, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact.

In particular, if S is an additive subgroup of V, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact [Park, 2015]

(any subgroup is a regular subset)

SECOND SUBCASE: $\{y_k\}_{k\geq 0}$ IS NOT STATIONARY

One may extract a subsequence $\{y_{k_n}\}_{n\geq 0}$ *such that* $v(y_{k_{n+1}} - y_{k_n}) = \gamma_{k_n}$.

This sequence $\{y_{k_n}\}_{n\geq 0}$ is *pseudo-convergent* with accuracy γ_∞.

SECOND SUBCASE: $\{y_k\}_{k\geq 0}$ IS NOT STATIONARY

One may extract a subsequence $\{y_{k_n}\}_{n\geq 0}$ *such that* $v(y_{k_{n+1}} - y_{k_n}) = \gamma_{k_n}$.

This sequence $\{y_{k_n}\}_{n\geq 0}$ is *pseudo-convergent* with accuracy γ_∞.

PROPOSITION (KAPLANSKY 1942)

Every pseudo-convergent sequence admits a pseudo-limit in an immediate extension L of K

Second subcase: $\{y_k\}_{k\geq 0}$ is not stationary

One may extract a subsequence $\{y_{k_n}\}_{n\geq 0}$ such that $v(y_{k_{n+1}} - y_{k_n}) = \gamma_{k_n}$.

This sequence $\{y_{k_n}\}_{n\geq 0}$ is *pseudo-convergent* with accuracy γ_∞.

Proposition (Kaplansky 1942)

Every pseudo-convergent sequence admits a pseudo-limit in an immediate extension L of K

Application. Recall that a valuation domain is said to be *maximally complete* if it does not admit any proper immediate extension.

Second subcase: $\{y_k\}_{k\geq 0}$ is not stationary

One may extract a subsequence $\{y_{k_n}\}_{n\geq 0}$ such that $v(y_{k_{n+1}} - y_{k_n}) = \gamma_{k_n}$.

This sequence $\{y_{k_n}\}_{n\geq 0}$ is *pseudo-convergent* with accuracy γ_∞.

Proposition (Kaplansky 1942)

Every pseudo-convergent sequence admits a pseudo-limit in an immediate extension L of K

Application. Recall that a valuation domain is said to be *maximally complete* if it does not admit any proper immediate extension.

Theorem

If the completion $\widehat{V}$ of V is maximally complete, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact.

SECOND SUBCASE: $\{y_k\}_{k\geq 0}$ IS NOT STATIONARY

One may extract a subsequence $\{y_{k_n}\}_{n\geq 0}$ *such that* $v(y_{k_{n+1}} - y_{k_n}) = \gamma_{k_n}$.

This sequence $\{y_{k_n}\}_{n\geq 0}$ is *pseudo-convergent* with accuracy γ_∞.

PROPOSITION (KAPLANSKY 1942)

Every pseudo-convergent sequence admits a pseudo-limit in an immediate extension L of K

Application. Recall that a valuation domain is said to be *maximally complete* if it does not admit any proper immediate extension.

THEOREM

If the completion $\widehat{V}$ of V is maximally complete, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact.

In particular, if V is a discrete valuation domain, then $\mathrm{Int}(S, V)$ is Prüfer if and only if S is precompact [Cahen, C., Loper, 2001].

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V.

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

Theorem (Loper and Werner, 2016)

Let $\{a_n\}_{n\geq 0}$ be a pseudo convergent sequence and $T = \{a_n \mid n \geq 0\}$.

1. *If the sequence is of algebraic type, then $\mathrm{Int}(T, V)$ is not Prüfer.*
2. *If the sequence is of transcendental type, then $\mathrm{Int}(T, V)$ is Prüfer.*

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

Theorem (Loper and Werner, 2016)

Let $\{a_n\}_{n\geq 0}$ be a pseudo convergent sequence and $T = \{a_n \mid n \geq 0\}$.

1. *If the sequence is of algebraic type, then $\mathrm{Int}(T, V)$ is not Prüfer.*
2. *If the sequence is of transcendental type, then $\mathrm{Int}(T, V)$ is Prüfer.*

algebraic type: $\exists f(X) \in V[X]$ s.t. $v(f(a_n))$ is ultimately strictly increasing

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

Theorem (Loper and Werner, 2016)

Let $\{a_n\}_{n\geq 0}$ be a pseudo convergent sequence and $T = \{a_n \mid n \geq 0\}$.

1. *If the sequence is of algebraic type, then $\mathrm{Int}(T, V)$ is not Prüfer.*
2. *If the sequence is of transcendental type, then $\mathrm{Int}(T, V)$ is Prüfer.*

algebraic type: $\exists f(X) \in V[X]$ s.t. $v(f(a_n))$ is ultimately strictly increasing

If $\mathrm{Int}(S, V)$ is Prüfer, then S does not contain

- *any pseudo-limit of a pseudo-divergent sequence*
- *any pseudo-stationary sequence*
- *any pseudo-convergent sequence of algebraic type.*

WORK IN PROGRESS

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

THEOREM (LOPER AND WERNER, 2016)

Let $\{a_n\}_{n\geq 0}$ be a pseudo convergent sequence and $T = \{a_n \mid n \geq 0\}$.

1. *If the sequence is of algebraic type, then $\mathrm{Int}(T, V)$ is not Prüfer.*
2. *If the sequence is of transcendental type, then $\mathrm{Int}(T, V)$ is Prüfer.*

algebraic type: $\exists f(X) \in V[X]$ s.t. $v(f(a_n))$ is ultimately strictly increasing

IF $\mathrm{Int}(S, V)$ IS PRÜFER, THEN S DOES NOT CONTAIN

- *any pseudo-limit of a pseudo-divergent sequence*
- *any pseudo-stationary sequence*
- *any pseudo-convergent sequence of algebraic type.*

IS IT SUFFICIENT?

Work in progress

In the previous results, all the pseudo-convergent, divergent, or stationary sequences admit pseudo-limits in V. But, the pseudo-limits of a pseudo-convergent sequence could belong to a proper immediate extension.

Theorem (Loper and Werner, 2016)

Let $\{a_n\}_{n\geq 0}$ be a pseudo convergent sequence and $T = \{a_n \mid n \geq 0\}$.

1. *If the sequence is of algebraic type, then $\mathrm{Int}(T, V)$ is not Prüfer.*
2. *If the sequence is of transcendental type, then $\mathrm{Int}(T, V)$ is Prüfer.*

algebraic type: $\exists f(X) \in V[X]$ s.t. $v(f(a_n))$ is ultimately strictly increasing

If $\mathrm{Int}(S, V)$ is Prüfer, then S does not contain

- *any pseudo-limit of a pseudo-divergent sequence*
- *any pseudo-stationary sequence*
- *any pseudo-convergent sequence of algebraic type.*

IS IT SUFFICIENT? (That is my question)