Finitely generated powers of prime ideals

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1 Preliminaries

1. Preliminaries

2. Powers of maximal ideals

Preliminaries

For a long time Gilmer was interested by this question:

Let P be a prime ideal of a commutative ring R such that P^n is finitely generated for some integer > 1. Is P finitely generated too?

In 1999, with Heinzer and Roitman, they got some positive answers to this question, but also some counter-examples.

Preliminaries

In particular they proved the following theorem:

Theorem

Let R a reduced ring. Assume that each prime ideal has a finitely generated power. Then R is Noetherian.

In 2001, Roitman obtained some other results. In particular he show the following:

Theorem

Let R be a coherent integral domain. Then, each prime ideal P for which P^n is finitely generated for some integer $n > 1$ is finitely generated too.

Preliminaries

In 2015, Mahdou and Zennayi investigated this question when P is a maximal ideal of rings which are built by different ways: trivial ring extension, pullback diagram and amalgamation along an ideal. Some new examples and counter-examples are given.

Powers of maximal ideals

Recall that a ring R is **coherent** if each finitely generated ideal is finitely presented. It is well known that R is coherent if and only if $(0 : r)$ and $A \cap B$ are finitely generated for each $r \in R$ and any two finitely generated ideals A and B.

Theorem

Let R be a coherent ring. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.

Proof.

If $P^n = 0$ and $P^{n-1} \neq 0$ then $P = (0 : r)$ for any $r \in P^{n-1}$. If $P^n \neq 0$ then R/P^n is coherent. □

Powers of maximal ideals

Theorem

Let R be a ring. Suppose that R_P is coherent for each maximal ideal P. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.

A ring R is a **chain ring** if its lattice of ideals is totally ordered by inclusion, and R is **arithmetical** if R_P is a chain ring for each maximal ideal P.

Theorem

Let R be an arithmetical ring. If P is a maximal ideal such that P^n is finitely generated for some integer $n > 0$ then P is finitely generated too.

Powers of maximal ideals

Let R be a ring. For a polynomial $f \in R[X]$, denote by $c(f)$ (the content of f) the ideal of R generated by the coefficients of f. We say that R is **Gaussian** if $c(fg) = c(f)c(g)$ for any two polynomials f and g in $R[X]$. A ring R is said to be a **fqp-ring** if each finitely generated ideal I is projective over $R/(0 : I)$.

Each arithmetical ring is a fqp-ring and each fqp-ring is Gaussian, but the converses do not hold (Abuhlail, Jarrar and Kabbaj (2011)). The following examples show that the previous theorem cannot be extented to the class of fqp-rings and the one of Gaussian rings.

Powers of maximal ideals

Example

Let R be a local ring and P its maximal ideal. Assume that $P^2 = 0$. Then it is easy to see that R is a fqp-ring. But P is possibly not finitely generated.

Example

Let A be a valuation domain (a chain domain), M its maximal ideal generated by m and E a vector space over A/M. Let $R = \{\left(\begin{smallmatrix} a & e \\ 0 & a \end{smallmatrix}\right) \mid a \in A,\ e \in E\}$ be the trivial ring extension of A by E. We easily check that R is a local fqp-ring. Let P be its maximal ideal. Then P^2 is generated by $\left(\begin{smallmatrix} m^2 & 0 \\ 0 & m^2 \end{smallmatrix}\right)$. But, if E is of infinite dimension over A/M then P is not finitely generated over R.

Powers of prime ideals

If R is a coherent integral domain then each prime ideal with a finitely generated power is finitely generated too (Roitman (2001)). The following example shows that this result does not extend to any coherent ring.

Example

Let D be a valuation domain. Suppose there exists a non-zero prime ideal L' which is not maximal. Moreover assume that $L' \neq L'^2$ and let $d \in L' \setminus L'^2$. If $R = D/Dd$ and $L = L'/Dd$, then R is a coherent ring, L is not finitely generated and $L^2 = 0$.

Powers of prime ideals

Remark

Let R be an arithmetical ring. In the previous example we use the fact that each non-zero prime ideal L which is not maximal is not finitely generated. We shall prove that L^n is not finitely generated for each integer $n > 0$ if L is not minimal.

Powers of prime ideals

In the sequel let $\Phi = \mathrm{Max}\ R \cup (\mathrm{Spec}\ R \setminus \mathrm{Min}\ R)$ for any ring R. By adapting the proof of the result of Roitman (2001) we show the following:

Theorem

Let R be a coherent ring. Then, for any $P \in \Phi$, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

Powers of prime ideals

Corollary

Let R be a reduced coherent ring. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

Proof.

If P is a minimal prime then $(0 : P) = (0 : P^n)$ and $(0 : P) \setminus P \neq \emptyset$. So, $P = (0 : (0 : P))$. □

Powers of prime ideals

An exact sequence of R-modules $0 \to F \to E \to G \to 0$ is **pure** if it remains exact when tensoring it with any R-module. Then, we say that F is a **pure** submodule of E. The following proposition is well known.

Powers of prime ideals

Proposition

Let A be an ideal of a ring R. The following conditions are equivalent:

1. *A is a pure ideal of R;*
2. *for each finite family $(a_i)_{1\leq i\leq n}$ of elements of A there exists $t \in A$ such that $a_i = a_i t$, $\forall i$, $1 \leq i \leq n$;*
3. *for all $a \in A$ there exists $b \in A$ such that $a = ab$ (so, $A = A^2$);*
4. *R/A is a flat R-module.*

Moreover:

- *if A is finitely generated, then A is pure if and only if it is generated by an idempotent;*
- *if A is pure, then $R/A = S^{-1}R$ where $S = 1 + A$.*

Powers of prime ideals

If R is a ring, we consider on $\mathrm{Spec}\ R$ the equivalence relation $\mathcal{R}$ defined by $L\mathcal{R}L'$ if there exists a finite sequence of prime ideals $(L_k)_{1\leq k\leq n}$ such that $L = L_1,\ L' = L_n$ and $\forall k,\ 1 \leq k \leq (n-1)$, either $L_k \subseteq L_{k+1}$ or $L_k \supseteq L_{k+1}$. We denote by $\mathrm{pSpec}\ R$ the quotient space of $\mathrm{Spec}\ R$ modulo $\mathcal{R}$ and by $\lambda : \mathrm{Spec}\ R \to \mathrm{pSpec}\ R$ the natural map. The quasi-compactness of $\mathrm{Spec}\ R$ implies the one of $\mathrm{pSpec}\ R$.

Powers of prime ideals

Lemma

Let R be a ring and let C a closed subset of $\mathrm{Spec}\ R$. Then C is the inverse image of a closed subset of $\mathrm{pSpec}\ R$ by λ if and only if $C = V(A)$ where A is a pure ideal. Moreover, in this case, $A = \cap_{P \in C} 0_P$, where $0_P = \ker(R \to R_P)$.

In the sequel, for each $x \in \mathrm{pSpec}\ R$ we denote by $A(x)$ the unique pure ideal which verifies $\overline{\{x\}} = \lambda(V(A(x)))$, where $\overline{\{x\}}$ is the closure of $\{x\}$ in $\mathrm{pSpec}\ R$.

Powers of prime ideals

Theorem

Let R be a ring. Assume that $R/A(x)$ is coherent for each $x \in \mathrm{pSpec}\ R$. Then, for any $P \in \Phi$, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

Corollary

Let R be a reduced ring. Assume that $R/A(x)$ is coherent for each $x \in \mathrm{pSpec}\ R$. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

pf-rings

Now, we consider the rings R for which each prime ideal contains a unique minimal prime ideal. So, the restriction λ' of λ to $\mathrm{Min}\ R$ is bijective. In this case, for each minimal prime ideal L we put
$A(L) = A(\lambda(L))$.
(Couchot (2007)) $\mathrm{pSpec}\ R$ is Hausdorff and λ' is a homeomorphism if and only if $\mathrm{Min}\ R$ is compact.

Proposition

Let R be a ring. Assume that each prime ideal contains a unique minimal prime ideal. Then, for each minimal prime ideal L, $V(L) = V(A(L))$. Moreover, if R is reduced then $A(L) = L$.

pf-rings

We say that a ring R is a **pf-ring** if one of the following equivalent conditions holds:

1. R_P is an integral domain for each maximal ideal P;
2. each principal ideal of R is flat;
3. each cyclic submodule of a flat R-module is flat.

Moreover, if R is a pf-ring then each prime ideal P contains a unique minimal prime ideal P' and $A(P') = P'$.

pf-rings

Corollary

Let R be a coherent pf-ring. Then each prime ideal with a finitely generated power is finitely generated too.

Corollary

Let R be a pf-ring. Assume that R/L is coherent for each minimal prime ideal L. Then each prime ideal with a finitely generated power is finitely generated too.

Corollary

Let R be a reduced arithmetical ring. Then each prime ideal with a finitely generated power is finitely generated too.

pf-rings

Let n be an integer ≥ 0 and G a module over a ring R. We say that pd $G \leq n$ if $\mathrm{Ext}_R^{n+1}(G, H) = 0$ for each R-module H.

Corollary

Let R be a coherent ring. Assume that each finitely generated ideal I satisfies pd $I < \infty$. Then each prime ideal with a finitely generated power is finitely generated too.

Proof.

R_P is an integral domain for any prime ideal P [Bertin (1971)]. □

pf-rings

Corollary

Let A be a ring and $X = \{X_\lambda\}_{\lambda \in \Lambda}$ a set of indeterminates. Consider the polynomial ring $R = A[X]$. Assume that A is reduced and arithmetical. Then each prime ideal of R with a finitely generated power is finitely generated too.

Proof.

"A pf-ring" implies "R pf-ring". For any prime ideal L "A/L Prüfer" implies "$(A/L)[X]$ coherent" [Greenberg and Vasconcelos (1976)] □

pf-rings

Let n be an integer ≥ 0. We say that a ring R is of global dimension $\leq n$ if pd $G \leq n$ for each R-module G.

Corollary

Let A be a ring and $X = \{X_\lambda\}_{\lambda \in \Lambda}$ a set of indeterminates. Consider the polynomial ring $R = A[X]$. Assume that A is of global dimension ≤ 2. Then each prime ideal of R with a finitely generated power is finitely generated too.

Proof.

A_P is a coherent domain for any maximal ideal P and A/L is coherent for each minimal prime L [Le Bihan (1971)]. We end as above by using [Greenberg and Vasconcelos (1976)]. □

locally constant functions

A topological space is called **totally disconnected** if each of its connected components contains only one point. Every Hausdorff topological space X with a base of clopen (closed and open) neighbourhoods is totally disconnected and the converse holds if X is compact.

Proposition

Let R be a ring for which each prime ideal contains only one minimal prime ideal. Let P be a minimal prime ideal such that P^n is finitely generated for some integer $n > 0$. Then $\lambda(P)$ is an isolated point of $\mathrm{pSpec}\ R$.

locally constant functions

Proposition (C(X,O))

Let X be a totally disconnected compact space, let O be a ring with a unique point in $\mathrm{pSpec}\ O$. Let R be the ring of all locally constant maps from X into O. Then, $\mathrm{pSpec}\ R$ is homeomorphic to X and $R/A(z) \cong O$ for each $z \in \mathrm{pSpec}\ R$.

Proposition

Let R be the ring defined in Proposition [C(X,O)]. Assume that O is a reduced coherent ring. Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

locally constant functions

Proposition

Let R be the ring defined in Proposition [C(X,O)]. Assume that O has a unique minimal prime ideal M. Then, every prime ideal of R contains only one minimal prime ideal and $\mathrm{Min}\ R$ is compact. If $M = 0$ then R is a pp-ring, i.e. each principal ideal is projective.

locally constant functions

Corollary

Let R be the ring defined in Proposition [C(X,O)]. Suppose that O has a unique minimal prime ideal M. Assume that O is either coherent or arithmetical and that one of the following conditions holds:

1. *M is either idempotent or finitely generated;*
2. *X contains no isolated point.*

Then, for any prime ideal P, P is finitely generated if P^n is finitely generated for some integer $n > 0$.

locally constant functions

Example

Let R be the ring defined in Proposition [C(X,O)]. Assume that:

- O is either coherent or arithmetical, with a unique minimal prime ideal M;
- M is not finitely generated and $M^k = 0$ for some integer $k > 1$;
- X contains no isolated points (for example the Cantor set).

Then the property "for each prime ideal P, P^n is finitely generated for some integer $n > 0$ implies P is finitely generated" is satisfied by R, but not by $R/A(L)$ for each minimal prime ideal L.

locally constant functions

Corollary

Let R be the ring defined in Proposition [C(X,O)]. Assume that O is local with maximal ideal M. Then each prime ideal of R is contained in a unique maximal ideal, and for each maximal ideal P, $R_P \cong O$. Moreover, if one of the following conditions holds:

1. *O is coherent;*
2. *O is a chain ring;*
3. *X contains no isolated point and M is the sole prime ideal of O.*

then, for each maximal ideal P, P^n finitely generated for some integer $n > 0$ implies P is finitely generated.

locally constant functions

Example

Let R be the ring defined in Proposition [C(X,O)]. Assume that M is the sole prime ideal of O, M is not finitely generated, $M^k = 0$ for some integer $k > 1$ and X contains no isolated points. Then the property "for each maximal ideal P, P^n is finitely generated for some integer $n > 0$ implies P is finitely generated" is satisfied by R, but not by R_L for each maximal ideal L.

Thank you very much!

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