A model structure approach to the Tate-Vogel cohomology

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Cotorsion pairs

- Let $\mathcal{D}$ be an abelian category and $\mathcal{H}$ a subcategory of $\mathcal{D}$.
- For an object $M \in \mathcal{D}$, write
$$M \in {}^{\perp}\mathcal{H} \text{ (resp. } M \in {}^{\perp_1}\mathcal{H})$$
if $\mathrm{Ext}^{\geq 1}_{\mathcal{D}}(M,X) = 0$ (resp. $\mathrm{Ext}^{1}_{\mathcal{D}}(M,X) = 0$) for each $X \in \mathcal{H}$.
- Dually, we can define $M \in \mathcal{H}^{\perp}$ and $M \in \mathcal{H}^{\perp_1}$.
- A *cotorsion pair* (cotorsion theory) is a pair $(\mathcal{A},\mathcal{B})$ of classes of objects in $\mathcal{D}$ such that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$.

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- A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *complete* if it has enough projectives and injectives, i.e., for any object $X \in \mathcal{D}$, there are exact sequences

$$0 \to B \to A \to X \to 0$$

and

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respectively with $B, B' \in \mathcal{B}$ and $A, A' \in \mathcal{A}$.

- A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *hereditary* if $\mathrm{Ext}^i_{\mathcal{D}}(A, B) = 0 \; \forall \; A \in \mathcal{A}, B \in \mathcal{B}$ and $i \geq 1$.

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Model categories

- A **model structure** on a category $\mathcal{C}$ is three classes of maps called *weak equivalences*, *fibrations*, and *cofibrations* subject to the following axioms.

  A *trivial cofibration* (resp. *trivial fibration*) is both a cofibration (resp. fibration) which is a weak equivalence.

  1. (2-out-of-3) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that any two of f, g, or gf are weak equivalences, then so is the third.

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  1. (2-out-of-3) Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ so that any two of f, g, or gf are weak equivalences, then so is the third.

Model categories

2. (Retracts) A map f in $\mathcal{C}$ is a **retract** of a map g in $\mathcal{C}$ if there is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"'] & A \arrow[d, "f"'] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

with the horizontal composites identities.

The three classes of maps are closed under retracts.

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Model categories

3. (Lifting) Every commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dotted, "h"] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration has a solution h so that $hi = f$ and $ph = g$ if i or p is a weak equivalence.

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4. (Factorization) Any map $f : X \to Y$ can be factored in two ways:

1. $\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & Z \arrow[ur, "q"'] & \end{tikzcd}$, where i is a cofibration and q is a trivial fibration.
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- A **model category** is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.
- An model category has an initial object 0 and a terminal object 1.
- An object A of a model category $\mathcal{C}$ is said to be **cofibrant** if $0 \to A$ is a cofibration and **fibrant** if $A \to 1$ is a fibration.
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  1. A map is a cofibration if and only if it is a monomorphism with cofibrant cokernel.
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Envelopes and covers

Given a class $\mathcal{F}$ of objects in $\mathcal{G}$ and an object $X \in \mathcal{G}$. An $\mathcal{F}$-envelope of X is a morphism $\psi : X \to F$ with $F \in \mathcal{F}$ such that:

(a) Any diagram
$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d] & F \arrow[dl, dotted] \\ F' & \end{tikzcd}$$
with $F' \in \mathcal{F}$ can be completed.

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- If ψ satisfies (a) and perhaps not (b), it is called an $\mathcal{F}$*-preenvelope*.
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Complexes

- The category of chain complexes of left R-modules is denoted by $C(R)$.
- A chain complex

$$\cdots \xrightarrow{\partial_{n+2}^X} X_{n+1} \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots$$

will be denoted by (X, ∂) or simply X.

- $\mathrm{Z}_n(X)$=Ker(∂_n^X), the nth cycle module.
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Let X be a complex.

- $\mathrm{C}_n(X) = \mathrm{coker}(\partial_{n+1}^X)$.
- $\mathrm{sup}X = \sup\{s \in \mathbb{Z} | \mathrm{H}_s(X) \neq 0\}$.
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- X is called *homologically bounded above* (resp., *homologically bounded below*) if $\mathrm{sup}X < \infty$ (resp., $\mathrm{inf}X > -\infty$).
- By convention, $\mathrm{sup}X = -\infty$ and $\mathrm{inf}X = \infty$ if X is exact.

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- Given two objects $X, Y \in C(\mathcal{D})$, then $\mathrm{Hom}(X, Y)$ denotes the complex with

$$\mathrm{Hom}(X, Y)_n = \prod_{i \in Z} \mathcal{G}(X_i, Y_{n+i}),$$

where $(\partial_n f)_m = \partial^Y_{n+m} f_m - (-1)^n f_{m-1} \partial^X_m$ for $f \in \mathrm{Hom}(X, Y)_n$.

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- A morphism $\alpha : X \to Y$ of complexes induces homomorphisms $\mathrm{H}_n(\alpha) : \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ for all $n \in \mathbb{Z}$, and α is a **quasi-isomorphism** when each $\mathrm{H}_n(\alpha)$ is an isomorphism.
- The complexes X and Y are *equivalent*, denoted by $X \simeq Y$, if they can be linked by a sequence of quasi-isomorphisms with arrows in the alternating directions.

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Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in Mod(R) and X an R-complex.

1. X is called an $\mathcal{A}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{A}$ for all n.
2. X is called a $\mathcal{B}$ complex if it is exact and $\mathrm{Z}_n(X) \in \mathcal{B}$ for all n.
3. X is called a $dg\mathcal{A}$ complex if $X_n \in \mathcal{A}$ for all n, and $\mathrm{Hom}(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.
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- In what follows,

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- If $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, \mathrm{Mod}(R))$, then a complex $X \in dg\widetilde{\mathcal{P}}$ is called dg-projective.
- If $(\mathcal{A}, \mathcal{B}) = (\mathrm{Mod}(R), \mathcal{I})$, then a complex $X \in dg\widetilde{\mathcal{I}}$ is called dg-injective.
- If $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$, then a complex $X \in dg\widetilde{\mathcal{F}}$ is called dg-flat, and a complex $Y \in dg\widetilde{\mathcal{C}}$ is called dg-cotorsion.
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- Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in Mod(R). Then the induced cotorsion pairs $(\widetilde{\mathcal{A}}, dg\widetilde{\mathcal{B}})$ and $(dg\widetilde{\mathcal{A}}, \widetilde{\mathcal{B}})$ in $\mathrm{C}(R)$ are both complete and hereditary.
- Furthermore, $dg\widetilde{\mathcal{A}} \cap \mathcal{E} = \widetilde{\mathcal{A}}$ and $dg\widetilde{\mathcal{B}} \cap \mathcal{E} = \widetilde{\mathcal{B}}$, where $\mathcal{E}$ is the class of exact complexes.

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There exists a model structure on $\mathrm{C}(R)$, denoted by $\mathrm{C}(R)_{\mathcal{M}_{\mathcal{A}}^{\mathcal{B}}}$, satisfying:

1. the weak equivalences are the quasi-isomorphisms;
2. the cofibrations (resp. trivial cofibrations) are the monomorphisms whose cokernels are in $dg\widetilde{\mathcal{A}}$ (resp. $\widetilde{\mathcal{A}}$);
3. the fibrations (resp. trivial fibrations) are the epimorphisms whose kernels are in $dg\widetilde{\mathcal{B}}$ (resp. $\widetilde{\mathcal{B}}$).

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Let M be a complex.

- M has a **cofibrant replacement**

$$p_M : QM \to M$$

in $C(R)_{\mathcal{M}^{\mathcal{B}}_{\mathcal{A}}}$, where QM is cofibrant and p_M is a trivial fibration.

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Tate-Vogel cohomology for complexes

Definition 3.1

Let $(\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathrm{Mod}(R)$ and M a complex.

1. A *cofibrant-fibrant* resolution of M is a diagram

$$\mathcal{QR}M \xrightarrow{p_{\mathcal{R}M}} \mathcal{R}M \xleftarrow{i_M} M$$

of morphisms of complexes with $p_{\mathcal{R}M}$ a cofibrant replacement in $\mathrm{C}(R)_{\mathcal{M}_{\mathcal{A}}^{\mathcal{B}}}$ and i_M a fibrant replacement in $\mathrm{C}(R)_{\mathcal{M}_{\mathcal{A}}^{\mathcal{B}}}$.

2. A *fibrant-cofibrant* resolution of M is a diagram

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Remark 3.2

1. We note that both $\mathcal{QRM}$ and $\mathcal{RQM}$ in the above definition are in $dg\widetilde{\mathcal{A}} \cap dg\widetilde{\mathcal{B}}$.
2. A cofibrant replacement p_M is exactly a special $dg\widetilde{\mathcal{A}}$-precover of M.
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Definition 3.3

Let $\mathbf{A} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in Mod(R), and let M and N be complexes.

There are two cofibrant-fibrant resolutions

$\mathcal{QR}M \to \mathcal{R}M \leftarrow M$ and

$\mathcal{QR}N \to \mathcal{R}N \leftarrow N$ of M and N, respectively.

Let $\overline{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N) \leqslant \mathrm{Hom}(\mathcal{QR}M, \mathcal{QR}N)$ with components $\overline{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N)_n$

$$= \{(\varphi_i) \in \mathrm{Hom}(\mathcal{QR}M, \mathcal{QR}N)_n \mid \varphi_i = 0 \text{ for all } i \gg 0\},$$

and $\widetilde{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N)$

$$= \mathrm{Hom}(\mathcal{QR}M, \mathcal{QR}N)/\overline{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N).$$

We define the nth *Tate-Vogel cohomology group*, denoted by $\widetilde{\mathrm{Ext}}^n_{\mathbf{A}}(M, N)$, as

$$\widetilde{\mathrm{Ext}}^n_{\mathbf{A}}(M, N) = \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N)).$$

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$$\widetilde{\mathrm{Ext}}^n_{\mathbf{A}}(M, N) = \mathrm{H}_{-n}(\widetilde{\mathrm{Hom}}_R(\mathcal{QR}M, \mathcal{QR}N)).$$

Definition 3.3

Let $\mathbf{A} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in Mod(R), and let M and N be complexes.

There are two cofibrant-fibrant resolutions

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$$= \{(\varphi_i) \in \mathrm{Hom}(\mathcal{QR}M, \mathcal{QR}N)_n \mid \varphi_i = 0 \text{ for all } i \gg 0\},$$

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Remark 3.5

One can see that $\widetilde{\mathrm{Ext}}^n_{\mathbf{A}}(-,-)$ and $\widetilde{\mathrm{ext}}^n_{\mathbf{A}}(-,-)$ are cohomological functors for each integer n, independent of the choice of cofibrant replacements and fibrant replacements.

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Remark 3.6

Let $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, \mathrm{Mod}(R))$ (resp., $(\mathrm{Mod}(R), \mathcal{I})$) in the definition above, then one easily checks that the Tate-Vogel cohomology group for complexes by cofibrant-fibrant (resp., fibrant-cofibrant) resolutions defined here is exactly the cohomology group for complexes defined by Asadollahi and Salarian in "*Cohomology theories for complexes*, J. Pure Appl. Algebra 210 (2007) 771-787".

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Definition 3.7

Assume that $(\mathcal{A}, \mathcal{B})$ is a complete hereditary cotorsion pair in Mod(R). Let M and N be complexes.

1. The $\mathcal{A}$ dimension of M, denoted by $\mathcal{A}$-dim$M$, is defined as
$$\mathcal{A}\text{-dim}M = \inf\{\sup\{i \mid A_{-i} \neq 0\} \mid M \simeq A \text{ with } A \in dg\widetilde{\mathcal{A}}\};$$
2. The $\mathcal{B}$ dimension of N, denoted by $\mathcal{B}$-dim$N$, is defined as
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Cf. X. Y. Yang and N. Q. Ding, On a question of Gillespie, Forum Mathematicum 27 (6) (2015), 3205-3231.

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Theorem 3.8

Let $\mathbf{A} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathrm{Mod}(R)$.

1. Let M be a homologically bounded above complex. TFAE:
   1. $\mathcal{A}$-dim$M < \infty$;
   2. $\widetilde{\mathrm{Ext}}^i_{\mathbf{A}}(M, Y) = 0$ for all integers i and any complex Y;
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2. Let N be a homologically bounded below complex. TFAE:
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Let $\mathbf{A} = (\mathcal{A}, \mathcal{B})$ be a complete hereditary cotorsion pair in $\mathrm{Mod}(R)$.

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Specializing Theorem 3.8 to the case $(\mathcal{A}, \mathcal{B}) = (\mathcal{F}, \mathcal{C})$, we have

Corollary 3.9

Let $\mathbf{F} = (\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair in $\mathrm{Mod}(R)$ and M a homologically bounded above complex. TFAE:

1. $\mathrm{fd}_R(M) < \infty$;
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TFAE for any ring R:

1. $wD(R) < \infty$;
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Let (R, m, k) be a commutative Noetherian local ring. Note that R is regular if and only if $\mathrm{fd}_R(k) < \infty$.

Corollary 3.11

Let (R, m, k) be a commutative Noetherian local ring. TFAE:

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Corollary 3.12

Let R be a left and right Noetherian ring and $\mathbf{GI} = ({}^{\perp_1}\mathcal{GI}, \mathcal{GI})$ a cotorsion pair. TFAE:

1. R is Gorenstein;
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5. $\widetilde{\mathrm{ext}}^0_{\mathbf{F}}(M, M) = \widetilde{\mathrm{Ext}}^0_{\mathbf{GP}}(N, N) = 0$ for all R-modules M and N, where $\mathbf{F} = (\mathcal{F}, \mathcal{C})$ is the flat cotorsion pair and $\mathbf{GP} = (\mathcal{GP}, \mathcal{GP}^{\perp_1})$ is a cotorsion pair.

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Remark 3.13

Let $\mathbb{A} = (\mathcal{A}, \mathcal{B}) = (\mathcal{P}, \mathrm{Mod}(R))$ and M be an R-module with infinite projective dimension. Then $\widetilde{\mathrm{Ext}}^{0}_{\mathbb{A}}(M, M) \neq 0$. It is easy to check that $\widetilde{\mathrm{ext}}^{0}_{\mathbb{A}}(M, M) = 0$. This implies that

$$\widetilde{\mathrm{Ext}}^{0}_{\mathbb{A}}(M, M) \not\cong \widetilde{\mathrm{ext}}^{0}_{\mathbb{A}}(M, M)$$

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Applications to the finitistic dimension

- Recall the finitistic dimensions of a ring R.

$$\mathrm{Findim}(R) = \sup\{\mathrm{pd}(P) \mid P \in \mathcal{L}\},$$
$$\mathrm{findim}(R) = \sup\{\mathrm{pd}(P) \mid P \in \mathcal{P}^{<\infty}\},$$

where $\mathcal{L}$ (resp., $\mathcal{P}^{<\infty}$) is the class of arbitrary (resp., finitely generated) left R-modules with finite projective dimension.

- Finitistic dimension conjectures. Let Λ be a finite dimensional algebra over a field k.

(I) $\mathrm{Findim}(\Lambda) = \mathrm{findim}(\Lambda)$.

(II) $\mathrm{findim}(\Lambda) < \infty$.

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- Conjecture (II) has been proved for monomial relation algebras Λ (e.g. if $\mathrm{pd}J_\Lambda^3 < \infty$), and in a couple of other cases, but it remains open in general.
- In 1991, Auslander and Reiten proved that Conjecture (II) holds true in case Λ is a finite dimensional algebra such that $\mathcal{P}^{<\infty}$ is precovering (contravariantly finite) in mod-Λ.
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Let R be a ring and $\mathbf{X} = (\mathcal{X}, \mathcal{Y})$ the cotorsion pair cogenerated by $\mathcal{P}^{<\infty}$. TFAE:

1. $\operatorname{findim}(R)$ is finite;
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Complexes with finite Gorenstein flat dimemsion

It is known that if R is a right coherent ring, then

$$\mathrm{fd}_R(M) = \mathrm{Gfd}_R(M)$$

whenever M is a complex with $\mathrm{fd}_R(M) < \infty$.

Inspired by this, we consider the following:

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- Recall that a left R-module M is called *Gorenstein flat* if there is an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat left R-modules with $M = \ker(F^0 \to F^1)$ such that $E \otimes -$ leaves the sequence exact whenever E is an injective right R-module.

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Lemma 5.1

Let R be a right coherent ring and M a left R-module. TFAE:

1. M is Gorenstein flat;
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Definition 5.2

Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair in $\mathrm{Mod}(R)$, and let R be a right coherent ring and M a complex.

A *complete flat resolution* of M is a diagram

$$T \xrightarrow{\tau} \mathcal{QR}M \longrightarrow \mathcal{R}M \longleftarrow M$$

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Let R be right coherent and M a complex. TFAE for each integer n:

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Theorem 5.4

Let R be a right coherent ring and M a complex with $\mathrm{Gfd}_R(M) < \infty$, and let $\mathbf{F} = (\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair in $\mathrm{Mod}(R)$. TFAE:

1. $\mathrm{fd}_R(M) = \mathrm{Gfd}_R(M)$;
2. $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(M, N) = 0$ for all integers i and any complex N;
3. $\widetilde{\mathrm{Ext}}^i_{\mathbf{F}}(M, X) = 0$ for all integers i and any bounded below dg-cotorsion complex X;
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Corollary 5.5

Let R be a left and right Noetherian ring. TFAE:

1. R is Gorenstein;
2. Every homologically bounded above complex M of left R-modules or right R-modules satisfies $\mathrm{Gfd}_R(M) < \infty$;
3. Every homologically bounded above complex M of left R-modules or right R-modules with finite injective dimension satisfies $\mathrm{Gfd}_R(M) < \infty$ (or $\mathrm{fd}_R(M) < \infty$).

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