Some new constructions of spectral spaces and a topological version of Hilbert's Nullstellensatz

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this talk is based on recent joint works with C.A. Finocchiaro and D. Spirito

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§1. Notation and Basic Definitions

Hilbert's Nullstellensatz (*) establishes a fundamental relationship between geometry and algebra, relating algebraic sets in affine spaces to radical ideals in polynomial rings over algebraically closed fields.

On the other hand, for any ring R, the set of radical ideals of R can be thought as a set of representatives of the closed sets of $X := \mathrm{Spec}(R)$, in the sense that the map $\mathcal{J}$, sending a closed set C of X to the radical ideal $\mathcal{J}(C) := \bigcap\{P \mid P \in C\}$, is a natural order-reversing bijection, having as inverse the map $\mathtt{V}$ defined by sending a radical ideal H of R to the Zariski closed subspace $\mathtt{V}(H) := \{P \in \mathrm{Spec}(R) \mid H \subseteq P\}$ of X.

One of the goals of the present talk is to put into a topological perspective the relationship between the geometry of $\mathrm{Spec}(R)$ and the ideal theory of R, shedding new light onto the Nullstellensatz-type correspondence established by the maps $\mathcal{J}$ and $\mathtt{V}$.

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- Given a spectral space X, M. Hochster in 1969 introduced a new topology on X, that we call here the *inverse topology*, by defining a Kuratowski closure operator, for each subset Y of X, as follows:

$$\mathtt{Cl}^{\mathrm{inv}}(Y) := \bigcap \{U \mid \; U \text{ quasi-compact open in } X, \; U \supseteq Y\}.$$

- If we denote by X^{inv} *the set X equipped with the inverse topology*, Hochster proved that X^{inv} *is still a spectral space and the partial order on X induced by the inverse topology is the opposite order of that induced by the given topology on X.*

In particular, *the closure under generizations $\{x\}^{\mathrm{gen}} := \{x' \in X \mid x' \leq x\}$ of a singleton $\{x\}$ is closed in the inverse topology of X*, since

$$\{x\}^{\mathrm{gen}} = \bigcap \{U \mid U \subseteq X \text{ quasi-compact open}, \, x \in U\}.$$

On the other hand, it is trivial, by the definition, that *the closure under specializations of a singleton $\{x\}^{\mathrm{sp}}$ is closed in the given topology of X*, since $\{x\}^{\mathrm{sp}} = \mathtt{Cl}(\{x\})$.

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Given a spectral space X, let

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If $X = \mathrm{Spec}(R)$ for some ring R, we write for short $\mathcal{X}(R)$ instead of $\mathcal{X}(\mathrm{Spec}(R))$.

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• We define *a Zariski topology on the space $\boldsymbol{\mathcal{X}}(X)$* by taking, as subbasis of open sets, the sets of the form

$$\boldsymbol{\mathcal{U}}(\Omega) := \{Y \in \boldsymbol{\mathcal{X}}(X) \mid Y \subseteq \Omega\},$$

where Ω varies among the quasi-compact open subspaces of X.

• Note that *the previous subbasis is in fact a basis*, since

$$\boldsymbol{\mathcal{U}}(\Omega) \cap \boldsymbol{\mathcal{U}}(\Omega') = \boldsymbol{\mathcal{U}}(\Omega \cap \Omega')$$

and $\Omega \cap \Omega'$ is a quasi-compact open subspace of X, for any pair Ω, Ω' of quasi-compact open subspaces of X.

• Moreover, $\emptyset \neq \boldsymbol{\mathcal{U}}(\Omega)$ because $\Omega \in \boldsymbol{\mathcal{U}}(\Omega)$, since a quasi-compact open subset Ω of X is a closed in the inverse topology of X.

• Note also that, when $X = \mathrm{Spec}(R)$, for some ring R, a generic basic open set of the Zariski topology on $\boldsymbol{\mathcal{X}}(R)$ is of the form

$$\boldsymbol{\mathcal{U}}(J) := \boldsymbol{\mathcal{U}}(\mathtt{D}(J)) = \{Y \in \boldsymbol{\mathcal{X}}(R) \mid Y \subseteq \mathtt{D}(J)\},$$

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Theorem 1

Let X be a spectral space.

(1) *The space $\boldsymbol{\mathcal{X}}(X)^{\mathrm{zar}}$, i.e. the set of all nonempty inverse-closed subspaces of X, $\boldsymbol{\mathcal{X}}(X)$, endowed with the Zariski topology, is a spectral space.*

(2) *The canonical map $\varphi : X \to \boldsymbol{\mathcal{X}}(X)^{\mathrm{zar}}$, defined by $\varphi(x) := \{x\}^{\mathrm{gen}}$, for each $x \in X$, is a spectral embedding (and, in particular, an order-preserving embedding between ordered sets, with the ordering induced by the Zariski topologies).*

Note that, for each quasi-compact open subset Ω of X, it is easy to see that

$$\varphi^{-1}(\boldsymbol{\mathcal{U}}(\Omega)) = \Omega\,.$$

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In order to prove that a nonempty topological space $\mathcal{X}$ is a spectral space, we use the characterization given by C. A. Finocchiaro in 2014.

- We recall that if $\mathcal{B}$ is a nonempty family of subsets of $\mathcal{X}$, for a given subset $\mathcal{Y}$ of $\mathcal{X}$ and an ultrafilter $\mathcal{U}$ on $\mathcal{Y}$, we set

$$\mathcal{Y}_{\mathcal{B}}(\mathcal{U}) := \{x \in \mathcal{X} \mid \text{ for each } B \in \mathcal{B}, \text{ it happens that } x \in B \Leftrightarrow B \cap \mathcal{Y} \in \mathcal{U}\}$$

- The subset $\mathcal{Y}$ of $\mathcal{X}$ is called *$\mathcal{B}$-ultrafilter closed* if $\mathcal{Y}_{\mathcal{B}}(\mathcal{U}) \subseteq \mathcal{Y}$, for each ultrafilter $\mathcal{U}$ on $\mathcal{Y}$.
- The $\mathcal{B}$-ultrafilter closed subsets of $\mathcal{X}$ are the closed subspaces of a topology on $\mathcal{X}$ called the *$\mathcal{B}$-ultrafilter topology on $\mathcal{X}$*.

Finocchiaro's Lemma

For a topological space $\mathcal{X}$ being a spectral space is equivalent to $\mathcal{X}$ being a T_0-space having a subbasis for the open sets $\mathcal{S}$ such that $\mathcal{X}_{\mathcal{S}}(\mathcal{U}) \neq \emptyset$, for each ultrafilter $\mathcal{U}$ on $\mathcal{X}$.

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• *The space of closed subsets of a spectral space*

Let X be a spectral space and let $\mathtt{Cl}(Y)$ denote the closure of a subspace Y in the given (spectral) topology of X.
Let $\mathcal{X}'(X)$ be the space of nonempty closed sets of X (in the given spectral topology of X), i.e.,

$$\mathcal{X}'(X) := \{Y \subseteq X \mid Y \neq \emptyset,\ Y = \mathtt{Cl}(Y)\}.$$

• Endow it with a topology, called the *Zariski topology on the space* $\mathcal{X}'(X)$ whose subbasic open sets are the family of sets

$$\mathcal{U}'(\Omega) := \{Y \in \mathcal{X}'(X) \mid Y \cap \Omega = \emptyset\},$$

as Ω ranges among the quasi-compact open subspaces of X.
• Note that *the family of sets of the type* $\mathcal{U}'(\Omega)$ *forms a basis, since* $\mathcal{U}'(\Omega_1) \cap \mathcal{U}'(\Omega_2) = \mathcal{U}'(\Omega_1 \cup \Omega_2)$.

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• *The space of closed subsets of a spectral space*

Let X be a spectral space and let $\mathtt{Cl}(Y)$ denote the closure of a subspace Y in the given (spectral) topology of X.

Let $\boldsymbol{\mathcal{X}}'(X)$ be the space of nonempty closed sets of X (in the given spectral topology of X), i.e.,

$$\boldsymbol{\mathcal{X}}'(X) := \{Y \subseteq X \mid Y \neq \emptyset,\ Y = \mathtt{Cl}(Y)\}.$$

• Endow it with a topology, called the *Zariski topology on the space* $\boldsymbol{\mathcal{X}}'(X)$ whose subbasic open sets are the family of sets

$$\boldsymbol{\mathcal{U}}'(\Omega) := \{Y \in \boldsymbol{\mathcal{X}}'(X) \mid Y \cap \Omega = \emptyset\},$$

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The notation used above for the space $\mathcal{X}'(X)$ is chosen in analogy and for coherence with the construction of the space $\mathcal{X}(X)$, which is sketched in Section 1. More precisely,

Theorem 2

Let X be a spectral space.

(1) *The space $\mathcal{X}'(X)^{\rm zar}$ is a spectral space.*

(2) *The canonical injective map $\varphi' : X^{\rm inv} \to \mathcal{X}'(X)^{\rm zar}$, defined by $\varphi'(x) := \{x\}^{\rm sp}$, for each $x \in X$, is a spectral embedding (and, in particular, an order-preserving embedding between ordered sets, with the ordering induced by the Zariski topologies).*

(3) *$\mathcal{X}'(X)$ coincides as a set with $\mathcal{X}(X^{\rm inv})$, thus the spectral embedding φ' coincides with $\varphi^{\rm inv} : X^{\rm inv} \to \mathcal{X}(X^{\rm inv})^{\rm zar}$.*

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Remarks and sketch of the proof of the previous theorem

- Keeping in mind the Hochster's duality (i.e., sketchy, $(X^{\mathtt{inv}})^{\mathtt{inv}} = X$), the set $\boldsymbol{\mathcal{X}}(X^{\mathtt{inv}})$ consists of all the nonempty closed sets of X, with respect to the given spectral topology, i.e., $\boldsymbol{\mathcal{X}}(X^{\mathtt{inv}}) = \boldsymbol{\mathcal{X}}'(X)$.

- Keeping in mind that the quasi-compact open subspaces $\mho$ of $X^{\mathtt{inv}}$ are precisely the complements of the quasi-compact open subspaces of X, i.e., $\mho = X \setminus \Omega$ for some quasi-compact open subspace Ω of X, it follows immediately, by definition, that the Zariski topology of $\boldsymbol{\mathcal{X}}'(X)$ has as a basis of open sets the collection of the sets of the type:

$$\begin{aligned}\boldsymbol{\mathcal{U}}'(\Omega) = &\ \{C \in \boldsymbol{\mathcal{X}}'(X) \mid C \cap \Omega = \emptyset\} = \{C \in \boldsymbol{\mathcal{X}}'(X) \mid C \cap (X \setminus \mho) = \emptyset\} \\ = &\ \{C \in \boldsymbol{\mathcal{X}}'(X) \mid C \subseteq \mho\} = \{C \in \boldsymbol{\mathcal{X}}(X^{\mathtt{inv}}) \mid C \subseteq \mho\} \\ = &\ \boldsymbol{\mathcal{U}}(\mho),\end{aligned}$$

for Ω varying among the quasi-compact open subspaces of X.

- Note that for the canonical injective map $\varphi' : X^{\mathtt{inv}} \to \boldsymbol{\mathcal{X}}'(X)^{\mathtt{zar}}$, we have

$$\varphi'^{-1}(\boldsymbol{\mathcal{U}}'(\Omega)) = \{x \in X^{\mathtt{sp}} \mid \{x\}^{\mathtt{sp}} \cap \Omega = \emptyset\} = X \setminus \Omega = \mho.$$

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§2. Spectral spaces of ideals and modules

Next goal is to introduce a natural well behaved topology on the set of all ideals of a ring R, which induces the Zariski topology on the subset of prime ideals, and makes it a spectral space.

We follow a general approach in terms of R-modules.

• Let R be a ring and M be an R-module. On the set $\mathtt{SMod}(M|R)$ of R-submodules of M we can define an *hull-kernel topology* (or, *Jacobson topology* (*)) having, as a subbasis for the closed sets, the subsets of the form

$$\boldsymbol{V}(x_1, x_2, \ldots, x_m) := \{N \in \mathtt{SMod}(M|R) \mid x_1, x_2, \ldots, x_m \in N\},$$

where $x_1, x_2, \ldots, x_m$ varies among all finite subsets of M.

Note that *the hull-kernel topology is clearly* T_0 and, by definition, *the order induced by this topology on* $\mathtt{SMod}(M|R)$ *coincides with the order provided by the set-theoretic inclusion* $\subseteq$.

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Let R be a ring, M an R-module, and $x_1, x_2, \dots, x_m \in M$.
Set

$$\boldsymbol{D}(x_1, x_2, \dots, x_m) := \texttt{SMod}(M|R) \setminus \boldsymbol{V}(x_1, x_2, \dots, x_m).$$

Proposition 3

• *For any ring R and for any R-module M, $\texttt{SMod}(M|R)$ endowed with the hull-kernel topology is a spectral space.*

Moreover,

• *the collection of sets $\mathcal{S} := \{\boldsymbol{D}(x_1, x_2, \dots, x_m) \mid x_1, x_2, \dots, x_m \in M\}$ is a subbasis of quasi-compact open subspaces of $\texttt{SMod}(M|R)$.*

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• Given a ring R and a R-module M, *a closure operation* on $\texttt{SMod}(M|R)$ is a map $(-)^c : \texttt{SMod}(M|R) \to \texttt{SMod}(M|R)$ that is
– *extensive* (i.e., $N \subseteq N^c$),
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• We also say that a *closure operation c is of finite type* if, for any $N \in \texttt{SMod}(M|R)$,
$N^c = \bigcup\{L^c \mid L \subseteq N, L \in \texttt{SMod}(M|R), L \text{ is finitely generated}\}$.

For a deeper insight on this topic see, for example, N. Epstein (2012, 2015), J. Elliott (2010), and J. Vassilev (2009).

Proposition 4

Let M be an R-module and c a closure operation of finite type on $\texttt{SMod}(M|R)$.

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For a deeper insight on this topic see, for example, N. Epstein (2012, 2015), J. Elliott (2010), and J. Vassilev (2009).

Proposition 4

Let M be an R-module and c a closure operation of finite type on $\mathtt{SMod}(M|R)$.

• *The set $\mathtt{SMod}^c(M|R) := \{N \in \mathtt{SMod}(M|R) \mid N = N^c\}$ is a spectral space. Moreover,*

• *$\mathtt{SMod}^c(M|R)$ is closed in $\mathtt{SMod}(M|R)$, endowed with the constructible topology.*

• Given a ring R and a R-module M, *a closure operation* on $\mathtt{SMod}(M|R)$ is a map $(-)^c : \mathtt{SMod}(M|R) \to \mathtt{SMod}(M|R)$ that is
– *extensive* (i.e., $N \subseteq N^c$),
– *order-preserving* (i.e., $N_1 \subseteq N_2$ implies $N_1^c \subseteq N_2^c$) and
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As particular cases of the spectral space of the submodules of a given module, we can consider the following distinguished cases.

- Given any ring R, let

$$\mathtt{Id}(R) := \mathtt{SMod}(R|R),$$
$$\mathtt{Id}_{\bullet}(R) := \mathtt{Id}(R) \setminus \{R\},$$

where $\mathtt{Id}(R)$ (respectively, $\mathtt{Id}_{\bullet}(R)$) is the set of all ideals (respectively, the set of all proper ideals) of R.

- As usual, let $\mathtt{rad}(I)$ denote the radical of an ideal I of R.

- Note that $\mathtt{rad}$ is a closure operation on $\mathtt{Id}(R)$ (and on $\mathtt{Id}_{\bullet}(R)$) and, in fact, $\mathtt{rad}$ *is a closure operation of finite type* since, for $x \in \mathtt{rad}(I)$, $x \in \mathtt{rad}(x^n)$ for some $x^n \in I$, with $n \geq 1$.

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• Set $\mathtt{Rd}(R) := \mathtt{Id}^c_\bullet(R) = \{I \in \mathtt{Id}_\bullet(R) \mid I = \mathtt{rad}(I)\}$, where $c = \mathtt{rad}$ is a closure operation of finite type.

Corollary 5

Let R be a ring.

• *The set $\mathtt{Id}(R)$ (respectively, $\mathtt{Id}_\bullet(R)$), endowed with the hull-kernel topology, is a spectral space, inducing on the subset $\mathtt{Spec}(R)$ the Zariski topology.*

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$$\mathtt{Spec}(R)^{\mathrm{zar}} \subseteq \mathtt{Rd}(R)^{\mathrm{hk}} \subseteq \mathtt{Id}_\bullet(R)^{\mathrm{hk}} \subseteq \mathtt{Id}(R)^{\mathrm{hk}}.$$

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§3. A topological version of Hilbert Nullstellensatz

Given a ring R, we have shown that the set of radical ideals of R endowed with the hull-kernel topology, $\mathtt{Rd}(R)^{\mathrm{hk}}$, is a spectral space, so we can introduce on this space the inverse topology that give rise on the same underlying set to another spectral space.

We simply set

$$\mathtt{Rd}(R)^{\mathrm{inv}} := (\mathtt{Rd}(R)^{\mathrm{hk}})^{\mathrm{inv}},$$

and, inside $\mathcal{X}'(R)$, we denote by $\mathcal{X}'_{\mathrm{irr}}(R)$ the subset consisting of all nonempty irreducible closed subspaces of $\mathrm{Spec}(R)^{\mathrm{zar}}$.

Now, we are in condition to state a "topological version" of the Hilbert Nullstellensatz.

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Theorem 7

Let R be a ring and let $\boldsymbol{\mathcal{X}}'(R) := \boldsymbol{\mathcal{X}}'(\mathtt{Spec}(R))$ be the topological space of the non-empty Zariski closed subspaces of $\mathtt{Spec}(R)$, endowed with the Zariski topology. Let $\mathtt{Rd}(R)$ be the spectral space of all proper radical ideals of R with the inverse topology. Then, the canonical map

$$\begin{aligned} \mathcal{J} \colon \boldsymbol{\mathcal{X}}'(R)^{\mathrm{zar}} &\to \mathtt{Rd}(R)^{\mathrm{inv}} \\ C &\mapsto \bigcap \{P \in \mathtt{Spec}(R) \mid P \in C\} \end{aligned}$$

is a homeomorphism.
Moreover, changing the topologies, the same map on the same underlying sets $\mathcal{J}$ defines a homeomorphism

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Thanks for your attention!