Star reductions of ideals and Prüfer $\nu$-multiplication domains

Evan Houston
Salah Kabbaj
Abdeslam Mimouni
Definitions. Let $R$ be a domain and $I$ a nonzero ideal of $R$.

- (Northcott/Rees, 1954) An ideal $J \subseteq I$ is said to be a reduction of $I$ if $JI^n = I^{n+1}$ for some positive integer $n$.
- (Northcott/Rees) $I$ is basic if it has no proper reductions.
- $R$ has the basic ideal property if each ideal of $R$ is basic.

It is difficult for a domain $R$ to be basic:

- For $a, b \in R$, we have $(a^2, b^2)(a, b) = (a, b)^3$.
- Hence $(a^2, b^2)(a, b)^2 = ((a, b)^2)^2$,
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- An ideal $J \subseteq I$ is a $\star$-reduction of $I$ if $(JI^n)^\star = (I^{n+1})^\star$ for some $n$.
- $I$ is $\star$-basic if it has no proper $\star$-reductions, i.e., if an equation above implies $J^\star = I^\star$.
- $R$ has the $\star$-basic ideal property if each nonzero ideal of $R$ is $\star$-basic.
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What we thought we could prove:

(Very Wrong) “Theorem”. Let $R$ be a domain.

- $R$ is a Prüfer $v$-multiplication domain $\iff R$ has the finite $t$-basic ideal property.
- $R$ is a Prüfer $v$-multiplication domain of $t$-dimension 1 $\iff R$ has the (full) $t$-basic ideal property.
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2. $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$ for all $I, J$.
3. $I^{**} = I^*$ for all $I$.

**Lemma (Hays).** Let $R$ have the finite $*$-basic ideal property. Then $R$ is integrally closed.

**Proof.** Suppose $a, b, \{r_i\} \in R$ with

$$(a/b)^n + r_{n-1}(a/b)^{n-1} + \cdots + r_0 = 0.$$ 

Then $a^n = -(r_{n-1}a^{n-1}b + \cdots + r_0b^n) \in (a^{n-1}, \ldots, b^{n-1})(b)$

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**Proof.** Suppose $a, b, \{r_i\} \in R$ with $(a/b)^n + r_{n-1}(a/b)^{n-1} + \cdots + r_0 = 0$. Then $a^n = -(r_{n-1}a^{n-1}b + \cdots + r_0b^n) \in (a^{n-1}, \ldots, b^{n-1})(b)
\Rightarrow (a/b)^n = (a/b)^{n-1}(b)
\Rightarrow (a/b) = (b)
\Rightarrow a/b \in R.$
More definitions. Let $I$ be a fractional ideal of $F$.

- $I^{-1} = (R : I) = \{x \in K \mid xI \subseteq R\}$.
- $I_v = (I^{-1})^{-1}$.
- $I_t = \bigcup \{J_v \mid J \subseteq I, J \text{ finitely generated}\}$.
- $I_w = \bigcup \{(I : J) \mid J \text{ finitely generated}, J_v = R\}$
  
  $= \{x \in K \mid xJ \subseteq I \text{ for some finitely generated ideal } J \text{ with } J_v = R\}$
  
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A domain $R$ is a Prüfer $\nu$-multiplication domain (Prüfer $\nu$-MD) if it satisfies any of the following equivalent conditions:

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Recall that a domain $R$ is completely integrally closed if every nonzero ideal of $R$ is $v$-invertible.

**Proposition.** Let $\star$ be a star operation on an integral domain $R$.

1. If $R$ has the $\star$-basic ideal property, then $R$ is completely integrally closed.

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**Proposition.** A ν-domain (nonzero finitely generated ideals are ν-invertible) has the finite ν-basic ideal property.

**Proposition.** If a domain \( R \) has the finite \( \star \)-basic ideal property, then \( R \) also has the finite \( \star_f \)-basic ideal property. In particular, if \( R \) has the finite ν-basic ideal property, then \( R \) also has the finite \( t \)-basic ideal property.

**Corollary.** A PνMD has the finite \( t \)-basic ideal property.
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**Corollary.** A PνMD has the finite $t$-basic ideal property.
So here’s the correct result:

**Theorem.** Let $R$ be a domain.

1. $R$ is a Prüfer $\nu$-multiplication domain $\iff$ $R$ has the finite $w$-basic ideal property.

2. $R$ is a Prüfer domain $\nu$-multiplication domain of $t$-dimension 1 $\iff$ $R$ has the (full) $w$-basic ideal property.

**Sketch of proof:**

1. $(\Rightarrow)$ It is easy to show that $\star$-invertible ideals are $\star$-basic. Since finitely generated ideals are $t$-invertible, they are $t = w$-basic. $(\Leftarrow)$ $R$ has the finite $w$-basic ideal property implies that $R$ is integrally closed. Let $M \in \text{Max}_t(R)$, and let $a, b \in M$. Since $(a^2, b^2)$ is a reduction of $(a, b)^2$, we have $(a^2, b^2)_w = ((a, b)^2)_w$ and hence $(a^2, b^2)_wR_M = (a, b)^2R_M$. Therefore, $R$ is a PvMD.

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Preliminaries

Correct result

Examples.

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(1) \( w\)-basic = \( v\)\(\nu\)MD + \( t\)-dim 1

(2) finite \( w\)-basic = \( v\)\(\nu\)MD

(3) \( v\)-domain

(4) finite \( v\)-basic

(5) finite \( t\)-basic

(6) integrally closed

\( v\)-basic = completely integrally closed

\( t\)-basic

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\( w\)-basic = P \( v\)MD + \( t\)-dim 1

\( v\)-domain

\( v\)-basic = completely integrally closed

R = ring of entire functions

\( t\)-basic

\( k[[X]] + (Y, Z)k((X))[Y, Z] \)

finite \( v\)-basic

finite \( t\)-basic

\( k + Yk((X))[[Y]] \)

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