Star reductions of ideals and Prüfer v-multiplication domains

Evan Houston
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Definitions. Let R be a domain and I a nonzero ideal of R.

- (Northcott/Rees, 1954) An ideal $J \subseteq I$ is said to be a *reduction* of I if $JI^n = I^{n+1}$ for some positive integer n.
- (Northcott/Rees) I is *basic* if it has no proper reductions.
- R has the *basic ideal property* if each ideal of R is basic.

It is difficult for a domain R to be basic:

- For $a, b \in R$, we have $(a^2, b^2)(a, b) = (a, b)^3$.
- Hence $(a^2, b^2)(a, b)^2 = ((a, b)^2)^2$,
- so (a^2, b^2) is a reduction of $(a, b)^2$.

Theorem. But in a Prüfer domain, we do have $(a^2, b^2) = (a, b)^2$ for all a, b.

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2. $I \subseteq I^*$ and $I \subseteq J \Rightarrow I^* \subseteq J^*$ for all I, J.
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Lemma (Hays). Let R have the finite $\star$-basic ideal property. Then R is integrally closed.

Proof. Suppose $a, b, \{r_i\} \in R$ with
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- $I^{-1} = (R : I) = \{x \in K \mid xI \subseteq R\}$.
- $I_v = (I^{-1})^{-1}$.
- $I_t = \bigcup\{J_v \mid J \subseteq I,\ J \text{ finitely generated }\}$.
-
$$\begin{aligned} I_w &= \bigcup\{(I : J) \mid J \text{ finitely generated }, J_v = R\} \\ &= \{x \in K \mid xJ \subseteq I \text{ for some finitely generated ideal } J \text{ with } J_v = R\} \\ &= \bigcap\{IR_M \mid M \in \mathrm{Max}_t(R)\} \end{aligned}$$
.
- A domain R is a Prüfer v-multiplication domain (PvMD) if it satisfies any of the following equivalent conditions:
  - Each nonzero finitely generated ideal of R is t-invertible.
  - R_M is a valuation domain for each maximal t-ideal M of R.
  - (Kang) R is integrally closed and $t = w$.

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Recall that a domain R is *completely integrally closed* if every nonzero ideal of R is v-invertible.

Proposition. Let $\star$ be a star operation on an integral domain R.

1. If R has the $\star$-basic ideal property, then R is completely integrally closed.
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So here's the correct result:

Theorem. Let R be a domain.

1. R is a Prüfer v-multiplication domain $\Leftrightarrow$ R has the finite w-basic ideal property.
2. R is a Prüfer domain v-multiplication domain of t-dimension 1 $\Leftrightarrow$ R has the (full) w-basic ideal property.

Sketch of proof:

1. ($\Rightarrow$) It is easy to show that $\star$-invertible ideals are $\star$-basic. Since finitely generated ideals are t-invertible, they are $t = w$-basic. ($\Leftarrow$) R has the finite w-basic ideal property implies that R is integrally closed. Let $M \in \mathrm{Max}_t(R)$, and let $a, b \in M$. Since (a^2, b^2) is a reduction of $(a, b)^2$, we have $(a^2, b^2)_w = ((a, b)^2)_w$ and hence $(a^2, b^2)R_M = (a, b)^2 R_M$. Therefore, R is a PvMD.
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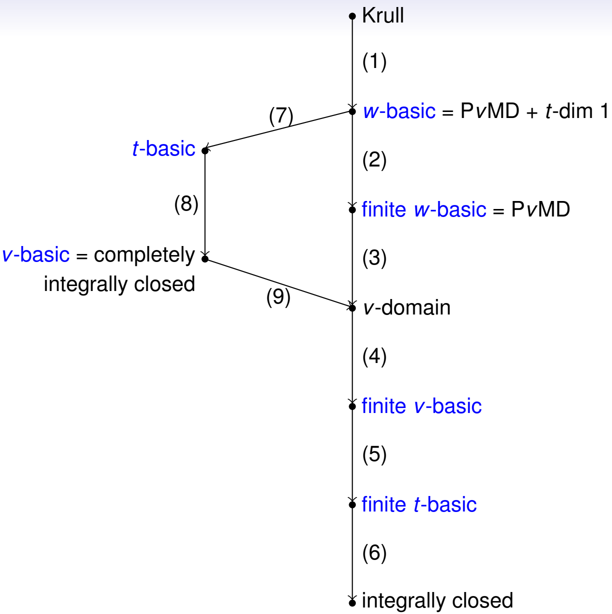
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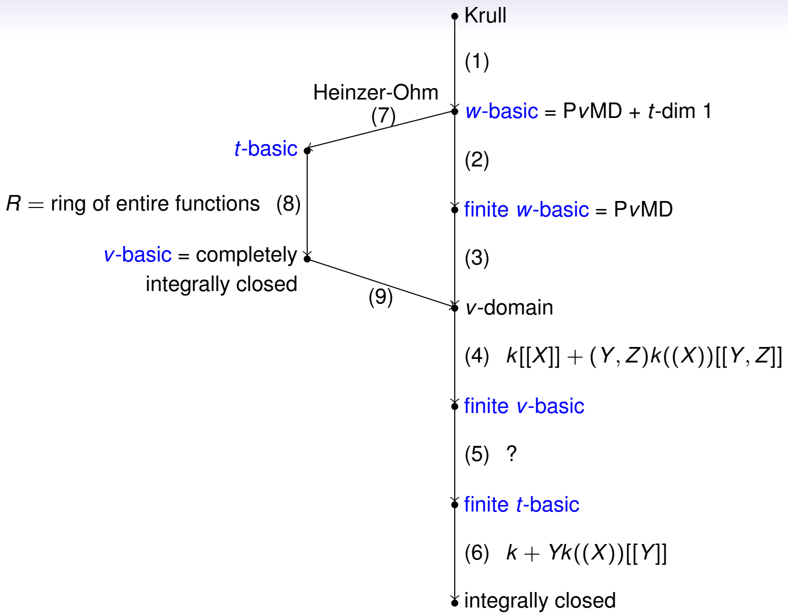
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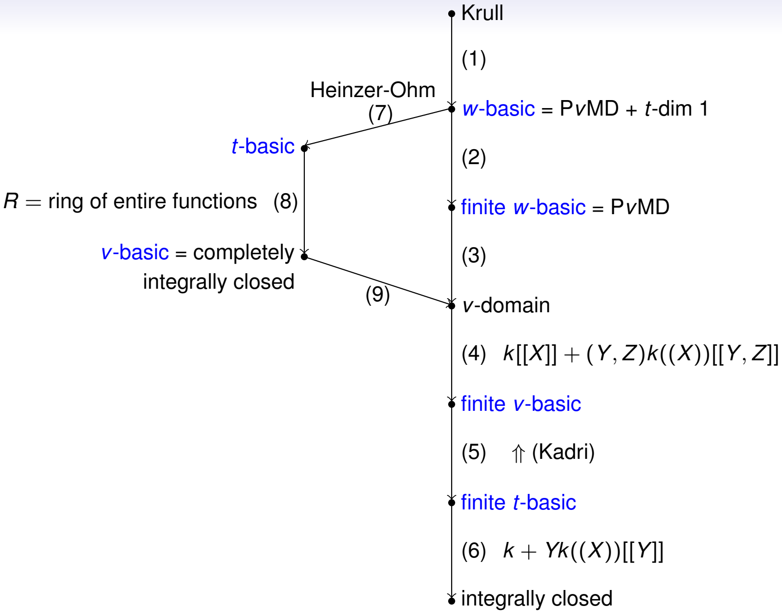
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