# On redundancy of ideal-projectivity for superfluous ideals in the finitely generated setting

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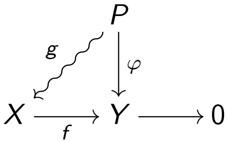
# Definitions

Let $R$ be a ring and $I$ its two-sided ideal contained in the Jacobson radical of $R$. A right $R$-module $P$ is called *projective* if for every epimorphism $f : X \to Y$ and for every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : M \to X$ such that $f \circ g = \varphi$, i.e. the following diagram commutes:




We say the factor-module $P/PI$ (resp. $P/P\mathcal{J}(R)$) is *the ideal factor* (resp. *the radical factor*) of $P$.

# Facts

## Properties of projectives

Let $R$ be a ring. Then:

(i) an $R$-module $P$ is projective iff every epimorphism with $P$ as a codomain splits iff $P$ is isomorphic to a direct summand of a free right module $F$

(ii) (Kaplansky, 1958) Every projective module is a direct sum of countably generated modules.

(iii) (Příhoda, 2007) Right projective $R$-modules $P$ and $Q$ are isomorphic if $P/P\mathcal{J}(R)$ and $Q/Q\mathcal{J}(R)$ are isomorphic.

# Definitions

- A submodule $N$ of $M$ *decomposes* $M$ (is *DM* in $M$) if there is a summand $S$ of $M$ such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule $X$ of $M$.
- A submodule $N$ of $M$ is called *SDM* in $M$ if there is a summand $S$ of $M$ such that $S \subseteq N$ and $M = S \oplus X$, whenever $N + X = M$ for a submodule $X$ of $M$.
- A submodule $N$ of $M$ is called *PDM* in $M$ if there is a projective summand $S$ of $M$ such that $S \subseteq N$ and $M = S + X$, whenever $N + X = M$ for a submodule $X$ of $M$.
- a submodule $N$ of $M$ is *superfluous* in $M$, denoted $N << M$ if $N + L \neq M$ for any proper submodule $L$ of $M$.

# Ideal-superfluity

**Observation**

Let $N$ be a submodule of $M$. Then $N << M$ if and only if $N \subseteq Rad(M)$ and $N$ is PDM in $M$.

It might make sense to define:
Let $N$ be a submodule of a module $M$. Then $N$ is $I$-superfluous in $M$ if $N \subseteq MI$ and $N$ is PDM in $M$, denoted $N <<_I M$.

**Redundancy of ideal-superfluity in the boundary case for projectives**

Let $M$ be a right $R$-module and $G$ be a submodule of $M$.

(i) $G <<_{\mathcal{J}(R)} M$ implies $G << M$.

(ii) if $M$ satisfies $Rad(M) = M\mathcal{J}(R)$, then also the reverse holds.

# Definitions

- A pair $(P, f)$ is called *a projective $I$-semicover* of $M$ if $P$ is projective and $f : P \to M$ is an epimorphism such that $\ker f \subseteq PI$.
- A pair $(P, f)$ is called a projective $I$-cover of $M$ if it is a projective $I$-semicover of $P$ and $\ker f$ decomposes $P$.

# Properties of projective ideal-covers

## Projective ideal-covers redundant in the boundary case

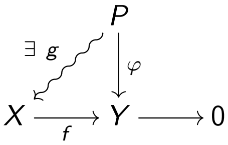
[Alkan, Nicholson, Özcan 2008] A right module $M$ has a projective $\mathcal{J}(R)$-cover if and only if $M$ has a projective cover.

## Relation to ideal-superfluity

Let $I \subseteq \mathcal{J}(R)$ and $M$ be a module. A projective module $P$ with a homomorphism $f : P \to M$ is a projective $I$-cover of $M$ if and only if $f$ in an epimorphism and $\ker(f)$ is $I$-superfluous in $P$.

# Ideal-semiprojectivity

Let $R$ be a ring and let $I$ be a two-sided ideal of $R$. Then an $R$-module $P$ is *$I$-semiprojective* if for every epimorphism $f : X \to Y$ such that $YI = 0$ and every morphism $\varphi : P \to Y$ there is a homomorphism $g : P \to X$ such that $\varphi = f \circ g$:




## Characterization lemma

Let $I$ be a two-sided ideal of $R$. Let $M$ be a module. Then $M$ is $I$-semiprojective if and only if for every epimorphism $f : X \to Y$ and every homomorphism $\varphi : M \to Y$ there exists a homomorphism $g : M \to X$ such that $(\varphi - f \circ g)(M) \subseteq YI$.

# Ideal-projectivity

- A right $R$-module $P$ is *$I$-projective* if for all right $R$-modules $X$ and $Y$, every $R$-epimorphism $f : X \to Y$ and every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : P \to X$ such that $(f \circ g - \varphi)(P) << _I Y$, i.e. the image of the triangle of the diagram is $I$-superfluous in $Y$.
- A right $R$-module $P$ is *radical-semiprojective* resp. *radical-projective* if for all right $R$-modules $X$ and $Y$, every $R$-epimorphism $f : X \to Y$ and every homomorphism $\varphi : P \to Y$ there exists a homomorphism $g : P \to X$ such that $(f \circ g - \varphi)(P) \subseteq Rad(Y)$ resp. $(f \circ g - \varphi)(P) << Rad(Y)$.

Note that projective modules are then just 0-projective modules.

# Ideal-projectivity for a finitely generated module

## Characterization for a finitely generated module $M$

Let $I \subseteq \mathcal{J}(R)$ and let $M$ be a finitely generated right $R$-module.

(i) [Izurdiaga 2004] if $M$ is $\mathcal{J}(R)$-semiprojective, then $M$ is radical-projective.

(ii) if $M$ is $I$-semiprojective, then $M$ is $I$-projective.

(iii) $M$ is $I$-(semi)projective if and only if for the canonical projection $\pi : M \to M/MI$ there exists a finitely generated module $F$ and a pair of homomorphisms $\alpha : P \to F$ and $\beta : F \to P$ such that $\pi = \pi \circ \beta \circ \alpha$

## Izurdiaga 2004, Example 3.11

There exist a (non-finitely generated) $\mathcal{J}(R)$-semiprojective module that is not radical-projective.

# Ideal-supplements

Let $R$ be a ring, $I$ be a two-sided ideal and $M$ be a right $R$-module.

- We say that submodule $K$ of $M$ is a *supplemented* submodule of $M$ if there is a submodule $G$ of $M$ such that $M = K + G$ and $G$ is minimal with this property. (i.e. $G$ is a *supplement* of $K$ if $K + G = M$ and $K \cap G$ superfluous in $G$.
- We say that a submodule $G$ of $M$ is an $I$-supplement if there is a submodule $K$ of $M$ such that $K + G = M$ and $K \cap G$ is $I$-superfluous in $G$. (note that 0-supplements are just direct summands)

**Ideal-supplements redundant in the boundary case for projectives**

Let $M$ be a module and $G$ be a submodule of $M$.

(i) if $G$ is a $\mathcal{J}(R)$-supplement then $G$ is a supplement.

(ii) if $G$ satisfies $Rad(G) = G\mathcal{J}(R)$, then also the reverse holds.

# Previous results

Mohammed-Sandomierski, J.Alg. 127, 206-217 (1989)

## Equivalence of redundancy in the boundary case (the f.g. setting)

Let $R$ be a ring, $\mathcal{J}(R)$ its Jacobson radical. Then FIE:

(1) every supplement submodule in a finitely generated left $R$-module is a direct summand

(2) if $M$ is a finitely generated left $R$-module such that the left $R/\mathcal{J}(R)$-module $M/\mathcal{J}(R)M$ is projective then $M$ is projective

(3) every finitely generated $\mathcal{J}(R)$-(semi)projective $R$-module is projective

## Redundancy for the prime radical

Let $R$ be a ring, let $I$ be a an ideal contained in the $\beta(R)$ of $R$. Every finitely generated $I$-(semi)projective $R$-module is projective.

# Previous results

Izurdiaga M.C.: *Supplement Submodules and Generalization of Projective Modules* J.Alg. **277** (2004)

**Equivalence of redundancy in the boundary case (the general setting)**

(1) every supplement submodule of a projective module is a direct summand

(2) for every set $\Gamma$, for every $\mathbb{A} \in RFM_\Gamma(R)$ with $\mathbb{A} - \mathbb{A}^2 \in \mathcal{J}(RFM_\Gamma(R))$ and such that there is $\mathbb{T} \in RFM_\Gamma(R)$ satisfying $\mathbb{T}\mathbb{A}^2 = \mathbb{A}$, $\mathbb{A}\mathbb{T}\mathbb{A} = \mathbb{A}$ holds.

(3) every radical-projective module is projective.

# Previous results

Facchini A., Herbera D., Shakhaev I.: "Flat modules and lifting of projective modules", Pac.J.Math 220/1, 49-67 (2005)

**Lifting of pure monos in the boundary case**

Let $Q$ and $Q'$ be projective right $R$-modules and let $\varphi : Q' \to Q$ be a homomorphism. If the mapping $\overline{\varphi} : Q'/Q'\mathcal{J}(R) \to Q/Q\mathcal{J}(R)$ induced by $\varphi$ is a pure monomorphism, then $\varphi$ is a pure monomorphism.

**Proposition 7.3**

Let $M$ be a finitely generated flat right module over a ring $R$ and let $P$ be a projective module. If $M/M\mathcal{J}(R) \simeq P/P\mathcal{J}(R)$, then $M \simeq P$.

# Lifting projectives modulo the radical factors

**FHS 2005 - Theorem 7.1**

Let $(R/\mathcal{J}(R))^n \simeq P \oplus Q$. Then FIE:

(i-ii) there exists a finitely generated (countably presented) flat $M_R$ such that the radical factor of M is isomorphic to $P$

(iii) there exists a projective $Q'_R$ such that the radical factor of $Q'$ is isomorphic to $Q$

(iv-v) there exists a finitely generated (countably presented) flat ${}_RN$ such that the radical factor of $N$ is isomorphic to $Hom_R(Q, R/\mathcal{J}(R))$

(vi) there exists a projective ${}_RP'$ such that the radical factor of $Q$ is isomorphic to $Hom_R(P, R/\mathcal{J}(R))$

# Lifting projectives modulo the superfluous ideal factors

Let $I$ be an ideal of $R$, $I \subseteq \mathcal{J}(R)$. If $(R/I)^n = P \oplus Q$, then FIE:

(L1-2) there exists a finitely generated (countably presented) flat $M_R$ such that the ideal factor of $M$ is isomorphic to $P$

(L3) there exists a projective $Q'_R$ such that the ideal factor of $Q'$ is isomorphic to $Q$

(L4-5) there exists a finitely generated (countably presented) flat ${}_RN$ such that the ideal factor of $N$ is isomorphic to $Hom_R(Q, R/I)$

(L6) there exists a projective ${}_RP'$ such that the ideal factor of $Q$ is isomorphic to $Hom_R(P, R/\mathcal{J}(R))$

## Corollary

Let $M$ be a f.g. flat right $R$-module and let $P$ be a projective right $R$-module. If $\gamma : P \to M$ is a projective $I$-cover, then $P \simeq M$.

# Main result

**Theorem - the f.g. setting**

Let $R$ be a ring and $I \subseteq \mathcal{J}(R)$ be a two-sided ideal of $R$. Then the following is equivalent:

(1) for every finitely generated projective right $R$-module $P$, every $I$-supplement is a direct summand.

(2) every finitely generated $I$-projective right $R$-module is projective

(3) every finitely generated flat right $R$-module $M$ with the right $R/I$-module $M/MI$ projective is itself projective

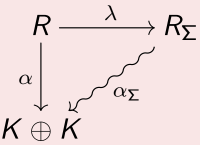
(4) for every projective right $R$-module $Q$, if the factor-module $Q/QI$ is finitely generated then $Q$ is finitely generated

The condition (4) for the boundary case is s.c. *Lazard's Conjecture*.

# Counterexample to Lazard's Conjecture

## Gerasimov-Sakhaev, 1984

Let $K$ be a field and let $R$ be a factor of the $K$-algebra $K\langle x, y\rangle$ by the ideal $I = \langle yx\rangle$. Consider the universal localization $R_\Sigma$ of $R$ with respect to a set $\Sigma \subseteq M(R) = \bigcup_{n=1}^{\infty} M_n(R)$ of all matrices with $\widetilde{\alpha}$-image invertible, where $\alpha(x) \mapsto (1, 0)$ and $\alpha(y) \mapsto (0, 1)$ and $\widetilde{\alpha} : M(R) \to M(K \oplus K)$ induced by $\alpha$. Then there is a commutative diagram in the category of rings:




where $\lambda$ is an embedding, so $\lambda(y)\lambda(x) = 0_{R_\Sigma}$

# Counterexample to Lazard's Conjecture

$$\begin{pmatrix} \mathbf{p} & r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix} \times \begin{pmatrix} & r' \\ & \end{pmatrix} = \begin{pmatrix} \mathbf{p} & rr' \\ \mathbb{A} & \mathbf{q}^T r' \end{pmatrix},$$

$$\begin{pmatrix} & r'' \\ & \end{pmatrix} \times \begin{pmatrix} \mathbf{p} & r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix} = \begin{pmatrix} r''\mathbf{p} & r''r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{p} & r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix} + \begin{pmatrix} \mathbf{p} & r' \\ \mathbb{A} & \mathbf{q}'^T \end{pmatrix} = \begin{pmatrix} \mathbf{p} & r + r' \\ \mathbb{A} & \mathbf{q}^T + \mathbf{q}'^T \end{pmatrix},$$

$$\begin{pmatrix} \mathbf{p} & r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix} + \begin{pmatrix} & s \\ & \end{pmatrix} = \begin{pmatrix} \mathbf{p} & r + s \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix}$$

## Criterion for being zero

An element $t$ equals zero in $R_\Sigma$ if and only if there are $\mathbf{b} \in R^n, \mathbf{c} \in R^k, \mathbb{B}, \mathbb{C} \in M(R)$ such that:

$$t = \begin{pmatrix} \mathbf{p} & r \\ \mathbb{A} & \mathbf{q}^T \end{pmatrix} = \begin{pmatrix} \mathbf{b} & \\ \mathbb{B} & \end{pmatrix} \times \begin{pmatrix} & \\ \mathbb{C} & \mathbf{c}^T \end{pmatrix}$$

# Counterexample to Lazard's Conjecture

## Dubrovin, Příhoda, Puninski, 2008, Corollary 2.6

Let $x, y \in R$ with $yx = 0$, $1 - x - y \in \mathcal{J}(R)$, denote $u := x + y$, $s := u - y$ (so $yx = 0$ implies $s = u^{-1}s^2$), let $r_i := u^{-i-1}xu^i$, $i \geq 0$ and $P = \bigcup_{i=0}^{\infty} r_i R$. Then $P$ is a projective module, $P/P\mathcal{J}(R)$ is 1-generated by $\bar{x}$ and $P$ is finitely generated if and only if $xu^{-1}y = 0$ if and only if $su^{-1}s = s$.
(Note that $s - s^2 = (1 - u)s \in \mathcal{J}(R)$)

## Properties of $R_\Sigma$

The ring homomorphism $\widetilde{\alpha}$ induces an isomorphism $R_\Sigma/\mathcal{J}(R_\Sigma) \simeq K \oplus K$. In particular, $R_\Sigma$ is semilocal, $\alpha(1 - x - y) = 0$ implies $1 - x - y \in \mathcal{J}(R_\Sigma)$ and $x(x + y)^{-1}y \neq 0$.

# Lifting idempotents

## Lifting idempotents in matrix rings

Let $R$ be a ring and let $I \subseteq \mathcal{J}(R)$ be an ideal of $R$. Then FIE:

(L1') for every $n \in \mathbb{N}$ every direct summand of a right R-module $R^{(n)}/I^{(n)}$ has a projective $I$-cover

(L2') for every $n \in \mathbb{N}$, if $P$ is a direct summand of $(R/I)^{(n)}$, then there is a direct summand $P'$ of $R^{(n)}$ such that $P = P' + I^{(n)}/I^{(n)}(\simeq P'/P'I)$

(L3') $M_n(I)$ is (strongly) lifting in $M_n(R)$ for every $n \in \mathbb{N}$

(L4') every direct summand of a finitely generated right $R$-module with a projective $I$-cover has a projective $I$-cover.

# Lifting idempotents

## Koethe's Conjecture equivalents, [Krempa, 1972]

(K1) the Koethe's radical contains every one-sided nil ideal

(K2) the sum of two one-sided nil ideals is a nil ideal

(K3) $\mathcal{N}(M_n(R)) = M_n(\mathcal{N}(R))$ for every ring $R$ and every $n \in \mathbb{N}$

(K4) $\mathcal{J}(R[x]) = \mathcal{N}(R)[x]$ for every ring $R$

## Lifting idempotents modulo nil ideals

Let $R$ be a ring and $N$ a nil ideal. If $\overline{f}$ is an idempotent element in $R/N$, then there is an idempotent element $e$ in $R$ such that $\pi(e) = \overline{f}$, where $\pi : R \to R/N$ is the canonical projection.

# Application

## Cases when the main result affirmative

If $I$ is an ideal of $R$ is contained in a radical $\gamma \subseteq \mathcal{N}$ that is matrix-extensible, then the condition (4) holds for $I$. In particular, for $\beta(R), \mathcal{L}(R)$.

## Proof.

Let $P$ be projective with the finitely generated ideal factor. By the assumption $M_n(I) \subseteq M_n(\gamma(R)) = \gamma(M_n(R)) \subseteq \mathcal{N}(M_n(R))$. By (K3) and idempotent lifting modulo nil ideals we infer that (L3') is true. Then (L2') is true and it gives us a finitely generated summand $Q$ with the ideal factor isomorphic to the ideal factor of $P$. The fact that projectives are determined by the superfluous ideal factors would lead to the conclusion that $P \simeq Q$. ∎

## Interests

- Are the conditions satisfied for the Koethe's radical? I.e., are the conditions an approximation of a positive solution to the Koethe's problem?
- Is it possible to state the main result analogically in the general case, i.e. also in the non-finitely generated setting?

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