Precovers and preenvelopes by phantom and Ext-phantom morphisms

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Introduction

Covers and envelopes

According to Enochs, 1981, let $\mathcal{A}$ be any category and $\mathcal{B}$ a class of objects in $\mathcal{A}$. A morphism $\phi : B \to A$ in $\mathcal{A}$ is called a $\mathcal{B}$-precover of A if $B \in \mathcal{B}$ and, for any morphism $f : B' \to A$ with $B' \in \mathcal{B}$, there is a morphism $g : B' \to B$ such that $\phi g = f$.

A $\mathcal{B}$-precover $\phi : B \to A$ is said to be a $\mathcal{B}$-cover of A if every endomorphism $g : B \to B$ such that $\phi g = \phi$ is an isomorphism.

Dually we have the definitions of a $\mathcal{B}$-preenvelope and a $\mathcal{B}$-envelope.

Ideal approximation theory has been recently introduced and developed by Fu, Guil Asensio, Herzog and Torrecillas (Ideal approximation theory, Adv. Math. 244 (2013), 750-790).

An additive subbifunctor of the bifunctor $\mathrm{Hom}_R(-,-) : R\text{-Mod}^{op} \times R\text{-Mod} \to \mathrm{Ab}$ is called an ideal $\mathcal{I}$ of R-Mod.

Let $\mathcal{I}$ be an ideal of R-Mod. Recall that a morphism $\phi : M \to N$ in $\mathcal{I}$ is an $\mathcal{I}$-precover of N if for any morphism $\psi : C \to N$ in $\mathcal{I}$, there is a morphism $\theta : C \to M$ such that $\phi\theta = \psi$.
An $\mathcal{I}$-precover $\phi : M \to N$ is called an $\mathcal{I}$-cover if every endomorphism h of M such that $\phi h = \phi$ is an isomorphism.
An $\mathcal{I}$-preenvelope and an $\mathcal{I}$-envelope are defined dually.

In ideal approximation theory, it is crucial to study the existence of (pre)covers and (pre)envelopes with respect to some special ideals.
An important instance of ideals in R-Mod is the ideal of phantom morphisms.

Herzog called a morphism $f : M \to N$ in R-Mod a phantom morphism if the induced morphism $\mathrm{Tor}_1^R(A, f) : \mathrm{Tor}_1^R(A, M) \to \mathrm{Tor}_1^R(A, N)$ is 0 for every (finitely presented) right R-module A.
Similarly, a morphism $g : M \to N$ in R-Mod is said to be an Ext-phantom morphism if the induced morphism $\mathrm{Ext}_R^1(B, g) : \mathrm{Ext}_R^1(B, M) \to \mathrm{Ext}_R^1(B, N)$ is 0 for every finitely presented left R-module B.

One can easily observe that the class of phantom (Ext-phantom) morphisms in R-Mod forms an ideal.
It is known that every module has a phantom cover in R-Mod (2007, Herzog) and has an Ext-phantom preenvelope (2013, Mao).

In the talk, we prove that every left R-module has a phantom preenvelope if R is a right coherent ring and every left R-module has an Ext-phantom cover if R is a left coherent ring. In addition, we investigate the properties of precovers and preenvelopes by phantom and Ext-phantom morphisms under change of rings.

Section 1.
When does every module have a phantom preenvelope or an Ext-phantom precover?

Lemma 1. Let R be a ring. Then

1. R is a right coherent ring if and only if the class of phantom morphisms in R-Mod is closed under direct products.
2. R is a left coherent ring if and only if the class of Ext-phantom morphisms in R-Mod is closed under direct limits.

Lemma 2. Consider the following commutative diagram with pure exact rows in R-Mod:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & M_1 & \xrightarrow{\beta_1} & L_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & K_2 & \xrightarrow{\alpha_2} & M_2 & \xrightarrow{\beta_2} & L_2 & \longrightarrow & 0. \end{array}$$

1. If φ is a phantom morphism in R-Mod, then both ψ and γ are phantom morphisms in R-Mod.
2. If R is a left coherent ring and φ is an Ext-phantom morphism in R-Mod, then both ψ and γ are Ext-phantom morphisms in R-Mod.

Let R-Mor denote the category whose objects are left R-module morphisms and the morphism from a left R-module morphism $M_1 \xrightarrow{f} M_2$ to a left R-module morphism $N_1 \xrightarrow{g} N_2$ is a pair of left R-module morphisms $(M_1 \xrightarrow{d} N_1, M_2 \xrightarrow{s} N_2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} M_1 & \xrightarrow{d} & N_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M_2 & \xrightarrow{s} & N_2. \end{array}$$

If $\mathcal{C}$ is a locally finitely presented additive category with products, then a full subcategory $\mathcal{D}$ which is closed under products, direct limits and pure subobjects is said to be a definable subcategory.
The following result describes when the full subcategory of (Ext-)phantom morphisms in R-Mor is definable.

Proposition 3. Let R be a ring. Then

1. R is a right coherent ring if and only if the full subcategory of phantom morphisms in R-Mor is definable.
2. R is a left coherent ring if and only if the full subcategory of Ext-phantom morphisms in R-Mor is definable.

Theorem 4. Let R be a ring.

1. Every left R-module morphism has a phantom cover in R-Mor.
2. R is a right coherent ring if and only if every left R-module morphism has a phantom preenvelope in R-Mor.
3. If R is a left coherent ring, then every left R-module morphism has an Ext-phantom cover and an Ext-phantom preenvelope in R-Mor.

Next we will investigate the existence of phantom preenvelopes and Ext-phantom precovers in R-Mod. The key technique is establishing the bijective correspondence between an $\mathcal{I}$-preenvelope (resp. precover) of a left R-module and the usual $\mathcal{I}$-preenvelope (resp. precover) of a left R-module morphism in R-Mor.

Lemma 5. Let $\mathcal{I}$ be an ideal of R-Mod and $\varphi : M \to N$ be a left R-module morphism. The following conditions are equivalent:

1. $\varphi : M \to N$ is an $\mathcal{I}$-prenvelope of M in R-Mod.
2. $(1_M, \varphi) : 1_M \to \varphi$ is an $\mathcal{I}$-preenvelope of 1_M in R-Mor.
3. 1_M has an $\mathcal{I}$-preenvelope $(f, \varphi) : 1_M \to \psi$ in R-Mor.

Lemma 6. Let $\mathcal{I}$ be an ideal of R-Mod and $\varphi : M \to N$ be a left R-module morphism. The following conditions are equivalent:

1. $\varphi : M \to N$ is an $\mathcal{I}$-precover of N in R-Mod.
2. $(\varphi, 1_N) : \varphi \to 1_N$ is an $\mathcal{I}$-precover of 1_N in R-Mor.
3. 1_N has an $\mathcal{I}$-precover $(\varphi, f) : \psi \to 1_N$ in R-Mor.

Theorem 7. Let R be a ring.

1. Every left R-module has a phantom cover in R-Mod.
2. If R is a right coherent ring, then every left R-module has a phantom preenvelope in R-Mod.
3. If R is a left coherent ring, then every left R-module has an Ext-phantom cover in R-Mod.

Section 2.

Precovers and preenvelopes by phantom and Ext-phantom morphisms under change of rings

Let $R \longrightarrow S$ be a ring homomorphism. Then S is an R-R-bimodule in a canonical way. Moreover any left (resp. right) S-module can be regarded as a left (resp. right) R-module and any left (resp. right) S-module morphism can be regarded as a left (resp. right) R-module morphism.

We first consider the phantom precovers and preenvelopes under change of rings.

Lemma 8. Let $R \to S$ be a ring homomorphism.

1. If ${}_RS$ is flat and $\varphi : {}_SM \to {}_SN$ is a phantom morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a phantom morphism in R-Mod.
2. If S_R is flat and $\psi : {}_RA \to {}_RB$ is a phantom morphism in R-Mod, then $1 \otimes_R \psi : S \otimes_R A \to S \otimes_R B$ is a phantom morphism in S-Mod.

Corollary 9. Let S be a multiplicative subset of a commutative ring R.

1. If $\varphi : {}_{S^{-1}R}M \to {}_{S^{-1}R}N$ is a phantom morphism in $S^{-1}R$-Mod, then $\varphi : {}_RM \to {}_RN$ is a phantom morphism in R-Mod.
2. If $\psi : {}_RA \to {}_RB$ is a phantom morphism in R-Mod, then $S^{-1}\psi : S^{-1}A \to S^{-1}B$ is a phantom morphism in $S^{-1}R$-Mod.

Recall that a phantom precover $\varphi : M \to N$ in R-Mod is special if $\ker(\varphi)$ is a pure-injective left R-module.

Theorem 10. Let $R \to S$ be a ring homomorphism with ${}_R S$ and S_R flat.

1. If a left S-module morphism $\varphi : {}_S M \to {}_S N$ is a (resp. special) phantom precover in S-Mod, then $\varphi : {}_R M \to {}_R N$ is a (resp. special) phantom precover in R-Mod.
2. If a left R-module morphism $\psi : {}_R A \to {}_R B$ is a phantom preenvelope in R-Mod, then $1 \otimes_R \psi : S \otimes_R A \to S \otimes_R B$ is a phantom preenvelope in S-Mod.

Corollary 11. Let S be a multiplicative subset of a commutative ring R.

1. If $\varphi : {}_{S^{-1}R}M \longrightarrow {}_{S^{-1}R}N$ is a (resp. special) phantom precover in $S^{-1}R$-Mod, then $\varphi : {}_RM \longrightarrow {}_RN$ is a (resp. special) phantom precover in R-Mod.
2. If $\psi : {}_RA \longrightarrow {}_RB$ is a phantom preenvelope in R-Mod, then $S^{-1}\psi : S^{-1}A \longrightarrow S^{-1}B$ is a phantom preenvelope in $S^{-1}R$-Mod.

Corollary 12. Let $R \to S$ be a surjective ring homomorphism with S_R and ${}_RS$ flat.

1. A left S-module morphism $\varphi : {}_SM \to {}_SN$ is a (resp. special) phantom precover in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is a (resp. special) phantom precover in R-Mod.
2. A left S-module morphism $\varphi : {}_SM \to {}_SN$ is a phantom cover in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is a phantom cover in R-Mod.

Next we study the Ext-phantom precovers and preenvelopes under change of rings.

Lemma 13. Let $R \to S$ be a ring homomorphism.

1. If S_R is flat and $\varphi : {}_SM \to {}_SN$ is an Ext-phantom morphism in S-Mod, then $\varphi : {}_RM \to {}_RN$ is an Ext-phantom morphism in R-Mod.
2. If ${}_RS$ is finitely generated projective and $\psi : {}_RU \to {}_RV$ is an Ext-phantom morphism in R-Mod, then $\psi_* : \mathrm{Hom}_R(S, U) \to \mathrm{Hom}_R(S, V)$ is an Ext-phantom morphism in S-Mod.

Theorem 14. Let $R \to S$ be a ring homomorphism with ${}_RS$ finitely generated projective and S_R flat.

1. If a left S-module morphism $\varphi : {}_SM \to {}_SN$ is a (resp. special) Ext-phantom preenvelope in S-Mod, then $\varphi : {}_RM \to {}_RN$ is a (resp. special) Ext-phantom preenvelope in R-Mod.
2. If a left R-module morphism $\psi : {}_RU \to {}_RV$ is an Ext-phantom precover in R-Mod, then $\psi_* : \mathrm{Hom}_R(S, U) \to \mathrm{Hom}_R(S, V)$ is an Ext-phantom precover in S-Mod.

Proposition 15. Let $R \to S$ be a surjective ring homomorphism with $_RS$ projective and S_R flat.

1. A left S-module morphism $\varphi : {}_SM \to {}_SN$ is a (resp. special) Ext-phantom preenvelope in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is a (resp. special) Ext-phantom preenvelope in R-Mod.
2. A left S-module morphism $\varphi : {}_SM \to {}_SN$ is an Ext-phantom envelope in S-Mod if and only if $\varphi : {}_RM \to {}_RN$ is an Ext-phantom envelope in R-Mod.

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Thank You !