Exponent of local ring extensions of Galois rings and digraphs of the kth power mapping

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4 July 2016

Overview

Digraph of the kth power mapping

Let R be a finite commutative ring with identity $1 \neq 0$. For an integer $k \geq 2$, the **kth power mapping digraph over R**, denoted by $G^{(k)}(R)$, is the digraph whose vertex set is R and there is a directed edge from a to b if and only if $a^k = b$.

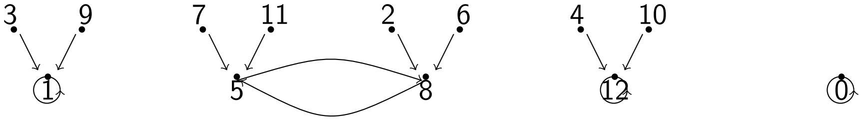
We consider two disjoint subdigraphs:

$G_1^{(k)}(R)$ induced on the set of vertices in the unit group $R^\times$ and

$G_2^{(k)}(R)$ induced on the remaining vertices which are not invertible.

Example

The digraph $G^{(3)}(\mathbb{Z}_{13})$.



Timeline

Křížek and Somer studied the digraphs $G^{(2)}(\mathbb{Z}_n)$ and $G^{(k)}(\mathbb{Z}_n)$.

2004

Křížek M., Somer L.: On a connection of number theory with graph theory, *Czechoslovak Math. J.* **54** (2004), 465–485.

2009

Křížek M., Somer L.: On symmetric digraphs of the congruences $x^k \equiv y \pmod{n}$, *Discrete Math.* **309** (2009), 1999–2009.

Křížek and Somer's tool

The Carmichael λ-function which is defined by a modification of the Euler's φ-function as follows:

1. $\lambda(1) = 1 = \varphi(1)$, $\lambda(2) = 1 = \varphi(2)$, $\lambda(4) = 2 = \varphi(4)$.
2. $\lambda(2^k) = 2^{k-2} = \frac{1}{2}\varphi(2^k)$, for $k \geq 3$.
3. $\lambda(p^k) = (p-1)p^{k-1} = \varphi(p^k)$, for any odd prime p and $k \geq 1$.
4. $\lambda(p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}) = \mathrm{lcm}(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots, \lambda(p_r^{k_r}))$, where $p_1, p_2, \dots, p_r$ are distinct primes and $k_i \geq 1$ for $i \in \{1, \dots, r\}$.

Timeline

Meemark and Wiroonsri worked on the digraphs $G^{(2)}(\mathbb{F}_{p^n}[x]/(f(x)))$ and $G^{(k)}(\mathbb{F}_{p^n}[x]/(f(x)))$ where $f(x)$ is a monic polynomial of degree ≥ 1 in $\mathbb{F}_{p^n}[x]$.

2010

Meemark Y., Wiroonsri N.: The quadratic digraph on polynomial rings over finite fields, *Finite Fields Appl.* **16** (2010), 334–346.

2012

Meemark Y., Wiroonsri N.: The digraphs of the kth power mapping of the quotient ring of polynomial ring over finite fields, *Finite Fields Appl.* **18** (2012), 179–191.

Timeline

Meemark found that we can replace Carmichael λ-function with the "exponent" of the unit group of $\mathbb{F}_{p^n}[x]/(f(x))$.

2011

Meemark Y., Maingam N.: The digraphs of the square mapping on quotient rings over the Gaussian integers, *Int. J. Number Theory*. **7** (2011), 835–852.

2012

Su H.D., Tang G.H., Wei Y.J.: The square mapping graphs of finite commutative rings, *Algeb. Collo.* **19** (3) (2012), 569–580.

Timeline

2014

Nan J.H., Tang G.H., Wei Y.J.: The iteration digraphs of group rings over finite fields, *Algebra and Its Appl.* **5** (2014), 1–19.

2015

Tang G.H., Wei Y.J.: The iteration digraphs of finite commutative rings, *Turk. J. Math.* **39** (2015), 872–883.

2015

Deng G., Somer L.: On the symmetric digraphs from the kth power mapping on a finite commutative ring, *Discrete Math., Algorithms and Appl.*, Vol.7, No.1 (2015), 1–15.

Exponent of a finite group

Let G be a finite group. The **exponent of** G, denoted by $\exp G$, is the least positive integer n such that $g^n = e$ for all $g \in G$.

1. $\exp G$ divides $|G|$
2. $\exp G = \mathrm{lcm}\{o(a) : a \in G\}$
3. If $G = G_1 \times G_2$, then $\exp G = \mathrm{lcm}(\exp G_1, \exp G_2)$.

E.g., $\exp \mathbb{Z}_n = n$ and $\exp S_4 = 12$.

Exponent of a finite ring

For a finite ring R with identity, we write $R^\times$ for the group of units of R. The **exponent of** R, denoted by $\lambda(R)$, is defined to be the exponent of the group of units of R. That is, $\lambda(R) = \exp(R^\times)$.

1. We can easily determine the exponent of R if the structure of the group of units is known, such as when R is the ring of integers modulo m, finite fields, Galois rings, and finite chain rings.
2. The exponent of the ring of integers modulo m is the "Carmichael λ-function".

Local rings whose unit group structure is known

A **local ring** R is a commutative ring with unique maximal ideal M. We call the field $k = R/M$ the **residue field**. E.g., every field is a local ring and $\mathbb{Z}_{p^n}$ is a local ring.

- Finite fields: $\mathbb{F}_q$
- $\mathbb{Z}_{p^s}$ where p is a prime and $s \in \mathbb{N}$
- Galois rings: $\mathbb{Z}_{p^n}[t]/(g(t))$ where $g(t)$ is irreducible in $\mathbb{Z}_p[t]$
- Finite chain rings: finite commutatitive local rings with unique *principal* maximal ideal (Hou X.D., Leung K.H., Ma S.L.: On the group of units of finite commutative chain rings, *Finite Fields Appl.* **9** (2003), 20–38.)

Galois rings

Let n, d be positive integers and p a prime.

1. We know that there exists a monic polynomial $g(t)$ in $\mathbb{Z}_{p^n}[t]$ of degree d such that the reduction $\overline{g}(t)$ in $\mathbb{Z}_p[t]$ is irreducible.
2. Consider the ring extension $\mathbb{Z}_{p^n}[t]/(g(t))$ of $\mathbb{Z}_{p^n}$. It is called a **Galois extension** of $\mathbb{Z}_{p^n}$.
3. Up to isomorphism the Galois extension with parameters n, d and p is unique. Hence, we may denote $\mathbb{Z}_{p^n}[t]/(g(t))$ by $GR(p^n, d)$, and call it the **Galois ring**.
4. Observe that $GR(p^n, 1) = \mathbb{Z}_{p^n}$ and $GR(p, d) = \mathbb{F}_{p^d}$.

Galois rings

Theorem

1. *$GR(p^n, d)$ is a finite local ring of characteristic p^n and of order p^{nd} with maximal ideal $M = p(GR(p^n, d))$, which is principal, and residue field $R/M \cong \mathbb{F}_{p^d}$.*
2. *The unit group $GR(p^n, d)^\times \cong H \times \mathbb{F}_{p^d}^\times$, where H is a group of order $p^{(n-1)d}$ such that:*
   a. *If (p is odd) or ($p = 2$ and $n \leq 2$), then H is a direct product of d cyclic groups each of order p^{n-1}, and so the exponent of $GR(p^n, d)$ in this case is $p^{n-1}(p^d - 1)$.*
   b. *If $p = 2$ and $n \geq 3$, then H is a direct product of a cyclic group of order 2, a cyclic group of order 2^{n-2} and $d - 1$ cyclic groups each of order 2^{n-1}, and so the exponent of $GR(2^n, d)$ in this case is $2^{n-1}(2^d - 1)$ for $d \geq 2$ and 2^{n-2} for $d = 1$, respectively.*

Local extensions

An extension ring S of a local ring R is called a **local extension** if S is a local ring. Hence, the Galois ring $GR(p^n, d)$ is a local extension of $\mathbb{Z}_{p^n}$. The next result is well known.

Theorem

Let R be a finite local ring, and $f(x)$ be a monic irreducible polynomial in $R[x]$. Then $R[x]/(f(x)^a)$ is a finite local ring for any $a \in \mathbb{N}$.

Main results

Consider a local extension of the Galois ring $GR(p^n, d)$ of the form

$$R = GR(p^n, d)[x]/(f(x)^a),$$

where $a \geq 1$ and $f(x)$ is a monic polynomial in $GR(p^n, d)[x]$ of degree r such that the reduction $\overline{f}(x)$ in $\mathbb{F}_{p^d}[x]$ is irreducible.

R is a local ring of characteristic p^n with maximal ideal

$$\begin{aligned} M &= (p, f(x))/(f(x)^a) \\ &= \{h(x) + f(x)l(x) + (f(x)^a) : h(x) \in pGR(p^n, d)[x], \\ &\qquad l(x) \in GR(p^n, d)[x], \deg h < r, \deg l < r(a-1)\}. \end{aligned}$$

We shall proceed to compute the "exponent of R" without completely determination of its unit group structure

$a = 1$

When $a = 1$, it turns out that R is still a Galois ring as a result of the next theorem.

Theorem

Let $f(x) \in GR(p^n, d)[x]$ be a monic polynomial of degree r such that the reduction $\overline{f}(x)$ in $\mathbb{F}_{p^d}[x]$ is irreducible. Then the ring $GR(p^n, d)[x]/(f(x))$ is isomorphic to a Galois ring $GR(p^n, dr)$.

Hence, $R = GR(p^n, d)[x]/(f(x)) \cong GR(p^n, dr)$ and the exponent of R is presented in previous theorem.

$a \geq 2$

Deng and Somer (2015) considered the exponent of the ring $\mathbb{F}_{p^n}[x]/(f(x)^a)$, where $a \geq 2$ and $f(x)$ is an irreducible polynomial in $\mathbb{F}_{p^n}[x]$ of degree r in the following theorem.

Theorem

Let $f(x)$ be an irreducible polynomial in $\mathbb{F}_{p^n}[x]$ of degree r and $a \geq 2$. Then

$$\lambda(\mathbb{F}_{p^n}[x]/(f(x)^a)) = p^s(p^{nr} - 1),$$

where $p^{s-1} < a \leq p^s$ for some $s \in \mathbb{N} \cup \{0\}$.

1. Since R is a local ring with maximal ideal M, we have $R^\times \cong (1+M) \times \mathbb{F}_{p^{dr}}^\times$ and $\mathbb{F}_{p^{dr}}^\times$ is cyclic of order $p^{dr}-1$, so it suffices to determine the exponent of the p-group $1+M$.

2. Following Deng and Somer, let s be the positive integer such that $p^{s-1} < a \leq p^s$. We shall show that every element in $1+M$ is of order not exceeding p^{s+n-1} and the order of $1+f(x)+(f(x)^a)$ is p^{s+n-1}, so the exponent of the group $1+M$ is p^{s+n-1}.

3. However, our computation is more complicated because the characteristic of the ring R is p^n and the binomial coefficients do not disappear easily like in the extension of fields case where it is of characteristic p.

For $m \in \mathbb{N}$, we write $e_p(m)$ for **the maximum power of p in m**, that is, $p^{e_p(m)} \mid m$ but $p^{e_p(m)+1} \nmid m$.

The proof is started by deriving some facts on the maximum power of p is binomial coefficients using de Polignac formula.

Theorem (de Polignac formula)

Let $m \in \mathbb{N}$ and p be a prime. Then

$$e_p(m!) = \sum_{i=1}^{\infty} [\frac{m}{p^i}].$$

We divide the computation into four lemmas as follows.

Lemma 1

Lemma

$e_p(\binom{p^n}{l_1}) = e_p(\binom{p^n}{l_2})$, *where* $1 \leq l_1, l_2 \leq p-1$ *and* $n \in \mathbb{N}$.
Moreover, $e_p(\binom{p^n}{l_1}) = n$.

Lemma 2

Lemma

Let $a \geq 2$, *and* $s, n \in \mathbb{N}$, *where* $p^{s-1} < a \leq p^s$. *For,* $0 \leq i \leq s-2$, $1 \leq k \leq (p-1)p^{s-2-i} - 1$. *Then:*

1. $e_p(\binom{p^{s+n-1}}{p^{s-1-i}}) \geq n$.
2. $e_p(\binom{p^{s+n-1}}{p^{s-1-i}+l_1}) = e_p(\binom{p^{s+n-1}}{p^{s-1-i}+l_2})$, *where* $1 \leq l_1, l_2 \leq p-1$. *Moreover,* $e_p(\binom{p^{s+n-1}}{p^{s-1-i}+l_1}) \geq n$.
3. $e_p(\binom{p^{s+n-1}}{p^{s-1-i}+kp}) \geq n$.
4. $e_p(\binom{p^{s+n-1}}{p^{s-1-i}+kp+l_1}) = e_p(\binom{p^{s+n-1}}{p^{s-1-i}+kp+l_2})$, *where* $1 \leq l_1, l_2 \leq p-1$. *Moreover,* $e_p(\binom{p^{s+n-1}}{p^{s-1-i}+kp+l_1}) \geq n$.

Lemmas 3–4

Lemma

Let $a \geq 2$, and $s, n \in \mathbb{N}$, where $p^{s-1} < a \leq p^s$. Let $f(x)$ be a monic polynomial in $GR(p^n, d)[x]$ such that the reduction $\overline{f}(x)$ in $\mathbb{F}_{p^d}[x]$ is irreducible. Then:

1. $e_p(\binom{p^{s+n-1-t}}{p^{s-1}}) = n - t$ *for all* $t \in \mathbb{N}$.
2. $(1 + f(x) + (f(x)^a))^{p^{s+n-1-t}} \neq 1 + (f(x)^a)$ *for all* $t \in \mathbb{N}$.

Lemma

$e_p(m!) < \frac{m}{p-1}$ *for all* $m \in \mathbb{N}$.

Now, we are ready to compute the exponent of $GR(p^n, d)[x]/(f(x)^a)$, when $a \geq 2$.

Theorem

Let $f(x) \in GR(p^n, d)[x]$ be a monic polynomial of degree r such that the reduction $\overline{f}(x)$ in $\mathbb{F}_{p^d}[x]$ is irreducible, and $a \geq 2$. If s is the positive integer such that $p^{s-1} < a \leq p^s$, then

$$\lambda(GR(p^n, d)[x]/(f(x)^a)) = p^{s+n-1}(p^{dr} - 1).$$

Proof

Let $h(x) \in pGR(p^n, d)[x]$, and $l(x) \in GR(p^n, d)[x]$, where $\deg h < r$, and $\deg l < r(a-1)$. Then

$$\begin{aligned}(1+h+fl+(f^a))^{p^{s+n-1}} &= (1+fl)^{p^{s+n-1}} + \binom{p^{s+n-1}}{1}(1+fl)^{p^{s+n-1}-1}h \\ &+ \cdots + \binom{p^{s+n-1}}{p^{s+n-1}-1}(1+fl)h^{p^{s+n-1}-1} \\ &+ h^{p^{s+n-1}} + (f^a).\end{aligned}$$

Proof

Since $h(x) \in pGR(p^n, d)[x]$, Lemma 4 forces that

$$\binom{p^{s+n-1}}{1} h = \cdots = \binom{p^{s+n-1}}{p^{s+n-1}-1} h^{p^{s+n-1}-1} = h^{p^{s+n-1}} = 0.$$

Thus,

$$\begin{aligned}(1 + h + fl + (f^a))^{p^{s+n-1}} &= (1 + fl)^{p^{s+n-1}} + (f^a) \\ &= 1 + \binom{p^{s+n-1}}{1} fl + \cdots + \binom{p^{s+n-1}}{p^{s-1}} (fl)^{p^{s-1}} \\ &\quad + \cdots + \binom{p^{s+n-1}}{a-1} (fl)^{a-1} + (f^a).\end{aligned}$$

Lemmas 1 and 2 imply that $p^n \mid \binom{p^{s+n-1}}{i}$ for all i.

Hence, $(1 + h + fl + (f^a))^{p^{s+n-1}} = 1 + (f^a)$.

Proof

Thus, Lemma 3 implies that p^{s+n-1} is the order of $1+f+(f^a) \in 1+M$, so $\exp(1+M) = p^{s+n-1}$.

Therefore,

$$\begin{aligned} \lambda(GR(p^n,d)[x]/(f(x)^a)) &= \mathrm{lcm}(\exp(1+M), \exp \mathbb{F}_{p^{dr}}^{\times}) \\ &= p^{s+n-1}(p^{dr}-1) \end{aligned}$$

as desired. □

Meemark Y., Tocharoenirattisai I.: Exponent of local ring extension of Galois rings and digraphs of the kth power mapping, *Turk. J. Math.*, to appear.

The End

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