Divisor-class groups of monadic submonoids of $\text{Int}(\mathbb{R})$

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Let $R$ be an integral domain with quotient field $K$ and $X$ an indeterminate over $K$. Set $R^\bullet = R \setminus \{0\}$, called the monoid of nonzero elements of $R$. 

$\text{Int}(R) = \{f \in K[X] \mid f(x) \in R \text{ for all } x \in R\}$ is called the ring of integer-valued polynomials over $R$. 


Let $R$ be an integral domain with quotient field $K$ and $X$ an indeterminate over $K$. Set $R^\bullet = R \setminus \{0\}$, called the monoid of nonzero elements of $R$.

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The arithmetic of Krull monoids is well understood. It can be described in terms of the divisor-class group of the Krull monoid. Is it possible to study the arithmetic of $\text{Int}(R)$ for “interesting domains” $R$ by using the theory of Krull monoids?
**Theorem (Cahen, Gabelli, Houston, 2000)**

Let $R$ be an integral domain and $X$ an indeterminate over $R$. Then $\text{Int}(R)$ is a Krull domain if and only if $R$ is a Krull domain and $\text{Int}(R) = R[X]$. 
Theorem (Cahen, Gabelli, Houston, 2000)

Let $R$ be an integral domain and $X$ an indeterminate over $R$. Then $\text{Int}(R)$ is a Krull domain if and only if $R$ is a Krull domain and $\text{Int}(R) = R[X]$.

What about finding suitable submonoids of $\text{Int}(R)^\bullet$ that are Krull monoids?
In this talk a monoid is always a commutative (multiplicative) cancellative semigroup with identity.
Definition

Let $H$ be a monoid, $K$ a quotient group of $H$ and $X \subseteq K$.

1. Set $X^{-1} = \{ z \in K \mid zX \subseteq H \}$ and $X_v = (X^{-1})^{-1}$.

2. Set $\mathcal{F}_v(H) = \{ X \subseteq K \mid X_v = X \text{ and } xX \subseteq H \text{ for some } x \in H \}$, called the set of fractional $v$-ideals of $H$.

3. Set $\mathcal{F}_v(H)^\times = \{ X \in \mathcal{F}_v(H) \mid (XX^{-1})_v = H \}$, called the set of $v$-invertible fractional $v$-ideals. It forms a group under $v$-multiplication.
Definition

Let $H$ be a monoid, $K$ a quotient group of $H$ and $X \subseteq K$.

1. Set $X^{-1} = \{z \in K \mid zX \subseteq H\}$ and $X_v = (X^{-1})^{-1}$.
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3. Set $\mathcal{F}_v(H)\times = \{X \in \mathcal{F}_v(H) \mid (XX^{-1})_v = H\}$, called the set of $v$-invertible fractional $v$-ideals. It forms a group under $v$-multiplication.
4. Set $\mathcal{H}(H) = \{xH \mid x \in K\}$, called the set of fractional principal ideals of $H$. It is a subgroup of $\mathcal{F}_v(H)\times$ under $v$-multiplication.
5. Set $\mathcal{C}_v(H) = \mathcal{F}_v(H)\times / \mathcal{H}(H)$, called the divisor-class group of $H$. 

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Divisor-class groups of monadic submonoids of $\text{Int}(\mathbb{R})$
Definition

Let $H$ be a monoid and $X \subseteq H$.

1. Set $\mathcal{A}(H) = \{ x \in H \setminus H^\times \mid \text{for all } u, v \in H \text{ with } x = uv \text{ it follows that } u \in H^\times \text{ or } v \in H^\times \}$, called the set of atoms of $H$.

2. $X$ is called a (prime) $s$-ideal of $H$ if $XH = X$ (and $xy \in X$ implies that $x \in X$ or $y \in X$ for all $x, y \in H$).
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<td><strong>3.</strong> Set $\mathcal{I}_v(H) = {I \in \mathcal{F}_v(H) \mid I \subseteq H}$, called the set of $v$-ideals of $H$.</td>
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<td><strong>4.</strong> Let $\mathcal{X}(H)$ <strong>be the set of minimal non-empty prime $s$-ideals of</strong> $H$, <strong>called the set of height-one prime ideals of</strong> $H$.</td>
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Saturated submonoids

Definition

Let $H$ be a monoid and $T \subseteq H$ a submonoid. We say $T \subseteq H$ is saturated if the following equivalent conditions are satisfied:

a. For all $x, y \in T$ with $x \mid_H y$ it follows that $x \mid_T y$.

b. For every $I \in \mathcal{I}_v(T)$ we have $I_{v_H} \cap T = I$.

c. For every $x \in T$, $xH \cap T = xT$. 

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Divisor-class groups of monadic submonoids of $\text{Int}(\mathbb{R})$
**Definition**

Let $H$ be a monoid and $T \subseteq H$ a submonoid.

1. We say $T \subseteq H$ is divisor-closed if for all $x, y \in H$ such that $xy \in T$ it follows that $x, y \in T$.

2. We say $T \subseteq H$ is monadic $H$ if $T = \{g \in H \mid g \mid_{H} x^{k} \text{ for some } k \in \mathbb{N}\}$ for some $x \in H$.

3. For $x \in H$ set $[x]_{H} = \{g \in H \mid g \mid_{H} x^{k} \text{ for some } k \in \mathbb{N}\}$, called the monadic submonoid of $H$ generated by $x$. 

The divisor-closed submonoids of an integral domain are precisely the proper saturated multiplicatively closed subsets in the terminology of Kaplansky’s book. Every monadic submonoid is divisor-closed and every divisor-closed submonoid is saturated.
Divisor-closed and monadic submonoids

Definition

Let $H$ be a monoid and $T \subseteq H$ a submonoid.

1. We say $T \subseteq H$ is divisor-closed if for all $x, y \in H$ such that $xy \in T$ it follows that $x, y \in T$.

2. We say $T \subseteq H$ is monadic if $T = \{g \in H \mid g \mid_{H} x^{k} \text{ for some } k \in \mathbb{N}\}$ for some $x \in H$.

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The divisor-closed submonoids of an integral domain are precisely the proper saturated multiplicatively closed subsets in the terminology of Kaplansky’s book. Every monadic submonoid is divisor-closed and every divisor-closed submonoid is saturated.
Let $R$ be an integral domain and $X$ an indeterminate over $R$.

- $R^\bullet \subseteq R[X]^\bullet$ and $R^\bullet \subseteq \text{Int}(R)^\bullet$ are divisor-closed submonoids.
- $R^\bullet \subseteq R[X]^\bullet$ is monadic if and only if $R^\bullet \subseteq \text{Int}(R)^\bullet$ is monadic if and only if $R$ is a $G$-domain (i.e., the intersection of all nonzero prime ideals of $R$ is nonzero).
- $R^\bullet \subseteq R[X]^\bullet$ is saturated, but not divisor-closed.
Remark

Let $H$ be a monoid.

1. $H$ is called a Krull monoid if every non-empty (fractional) $v$-ideal of $H$ is $v$-invertible and $H$ satisfies the ACC on $v$-ideals. If $H$ is a Krull monoid, then every $v$-ideal of $H$ is a finite $v$-product of height-one prime ideals of $H$.
Some remarks

Remark

Let $H$ be a monoid.

1. $H$ is called a Krull monoid if every non-empty (fractional) $\nu$-ideal of $H$ is $\nu$-invertible and $H$ satisfies the ACC on $\nu$-ideals. If $H$ is a Krull monoid, then every $\nu$-ideal of $H$ is a finite $\nu$-product of height-one prime ideals of $H$.

2. $H$ is called a quasi finitely generated monoid if $H$ satisfies the ACC on $s$-ideals (equivalently, $H$ is atomic and $\{uH \mid u \in \mathcal{A}(H)\}$ is finite). If $H$ is a quasi finitely generated Krull monoid, then $\mathcal{X}(H)$ is finite and $\mathcal{C}_\nu(H)$ is a finitely generated abelian group.
Remark

Let $H$ be a monoid.

1. $H$ is called a Krull monoid if every non-empty (fractional) $v$-ideal of $H$ is $v$-invertible and $H$ satisfies the ACC on $v$-ideals. If $H$ is a Krull monoid, then every $v$-ideal of $H$ is a finite $v$-product of height-one prime ideals of $H$.

2. $H$ is called a quasi finitely generated monoid if $H$ satisfies the ACC on $s$-ideals (equivalently, $H$ is atomic and $\{uH \mid u \in A(H)\}$ is finite). If $H$ is a quasi finitely generated Krull monoid, then $X(H)$ is finite and $C_v(H)$ is a finitely generated abelian group.

3. If $x \in H$ is such that $[x]$ is a Krull monoid, then $[x]$ is a quasi finitely generated monoid.
Some remarks

Remark

Let $R$ be an integral domain.

1. $R^\bullet$ is quasi finitely generated if and only if $R$ is a Cohen-Kaplansky domain.

2. $R^\bullet$ is a quasi finitely generated Krull monoid if and only if $R$ is a semilocal PID. The divisor-class group of $R^\bullet$ is trivial in this case.
When monadic submonoids of $\text{Int}(R)$ are Krull

**Theorem (R, 2014)**

Let $R$ be a factorial domain. Then every monadic submonoid of $\text{Int}(R)^\bullet$ is a Krull monoid.
When monadic submonoids of $\text{Int}(R)$ are Krull

**Theorem (R, 2014)**

Let $R$ be a factorial domain. Then every monadic submonoid of $\text{Int}(R)^\cdot$ is a Krull monoid.

**Theorem (Frisch)**

Let $R$ be a Krull domain. Then every monadic submonoid of $\text{Int}(R)^\cdot$ is a Krull monoid.
When monadic submonoids of \( \text{Int}(R) \) are Krull

**Theorem (R, 2014)**

Let \( R \) be a factorial domain. Then every monadic submonoid of \( \text{Int}(R)^\bullet \) is a Krull monoid.

**Theorem (Frisch)**

Let \( R \) be a Krull domain. Then every monadic submonoid of \( \text{Int}(R)^\bullet \) is a Krull monoid.

Goal: Describe the structure of the divisor-class groups of monadic submonoids \( \llbracket f \rrbracket_{\text{Int}(R)^\bullet} \) of \( \text{Int}(R)^\bullet \) if \( R \) is factorial. This can be done by studying atoms and height-one prime ideals and the relations between them.
Definition

Let $R$ be a factorial domain and $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$. Let $d : \text{Int}(R)^* \to R^*$ be defined by
\[
d(g) = \prod_{p \in \mathcal{P}} p^{\min\{v_p(g(x)) | x \in R\}}
\]
for all $g \in \text{Int}(R)^*$. 

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Definition
Let $R$ be a factorial domain and $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$. Let $d : \text{Int}(R)^\bullet \to R^\bullet$ be defined by $d(g) = \prod_{p \in \mathcal{P}} p^{\min\{v_p(g(x)) \mid x \in R\}}$ for all $g \in \text{Int}(R)^\bullet$.

Lemma
Let $R$ be a factorial domain, $X$ an indeterminate over $R$ and $f \in R[X]^\bullet$. Then there are some $a \in R^\bullet$, $n \in \mathbb{N}$ and $f$ an $n$-tuple of pairwise non-associated non-constant atoms of $R[X]$ such that $[f] = [a \prod_{i=1}^{n} f_i]$.
Proposition

Let $R$ be a factorial domain, $a \in R^\bullet$, $n \in \mathbb{N}$ and $f$ an $n$-tuple of pairwise non-associated non-constant atoms of $R[X]$. Set $f = a \prod_{i=1}^{n} f_i$. Then there is some finite $E \subseteq \mathbb{N}_0^n \setminus \{0\}$ such that

$$\{ u[f] \mid u \in \mathcal{A}(f) \} = \{ u[f] \mid u \in \mathcal{A}(R), u R d(f) \} \cup$$

$$\{ \frac{\prod_{i=1}^{n} f_i^{y_i}}{d(\prod_{i=1}^{n} f_i^{y_i})} [f] \mid y \in E \}.$$
Proposition

Let $R$ be a factorial domain, $a \in R^\bullet$, $n \in \mathbb{N}$ and $f$ an $n$-tuple of pairwise non-associated non-constant atoms of $R[X]$. Set $f = a \prod_{i=1}^{n} f_i$. Then there is some finite $E \subseteq \mathbb{N}_0^n \setminus \{0\}$ such that

$$\{u[f] \mid u \in A([f])\} = \{u[f] \mid u \in A(R), u \mid_R d(f)\} \cup \left\{ \frac{\prod_{i=1}^{n} f_i^{y_i}}{d(\prod_{i=1}^{n} f_i^{y_i})} [f] \mid y \in E \right\}.$$

Suppose that $f \in \text{Int}(R)^\bullet$. Then $f = \frac{g}{c}$ for some $g \in R[X]^\bullet$ and $c \in R^\bullet$ and $[f]$ is a monadic submonoid of $[g]$. 
Proposition

Let $R$ be a factorial domain, $a \in R^\bullet$, $n \in \mathbb{N}$ and $f$ an $n$-tuple of pairwise non-associated non-constant atoms of $R[X]$. Set $f = a \prod_{i=1}^{n} f_i$. Then there is some finite $E \subseteq \mathbb{N}^n_0 \setminus \{0\}$ such that

$$\{ u[f] \mid u \in \mathcal{A}(\lbrack f \rbrack) \} = \{ u[f] \mid u \in \mathcal{A}(R), u \mid_{R} d(f) \} \cup \{ \frac{\prod_{i=1}^{n} f_{i}^{y_{i}}}{d(\prod_{i=1}^{n} f_{i}^{y_{i}})} \lbrack f \rbrack \mid y \in E \}.$$ 

Suppose that $f \in \text{Int}(R)^\bullet$. Then $f = \frac{g}{c}$ for some $g \in R[X]^\bullet$ and $c \in R^\bullet$ and $\lbrack f \rbrack$ is a monadic submonoid of $\lbrack g \rbrack$.

Lemma

Let $H$ be a monoid and $T \subseteq H$ a divisor-closed submonoid. Then $\mathcal{A}(T) = \mathcal{A}(H) \cap T$. 
A simple observation

Lemma

Let $R$ be a factorial domain and $f \in \text{Int}(R)^\bullet$

1. If $u \in \mathcal{A}([f]) \cap R$, then $u[f]$ is a radical ideal of $[f]$.
2. If $P \in \mathcal{X}([f])$ and $u, w \in P \cap \mathcal{A}(R)$, then $u \simeq_{[f]} w$. 
Proposition

Let $R$ be a factorial domain, $K$ a field of quotients of $R$, $X$ an indeterminate over $K$ and $f \in R[X]^\cdot$. Then

$$\{ P \in \mathcal{X}([f]) \mid P \cap R = \emptyset \} = \{ gK[X] \cap [f] \mid g \in [f] \cap A(K[X]) \}.$$
Description of height-one prime ideals I

Proposition

Let $R$ be a factorial domain, $K$ a field of quotients of $R$, $X$ an indeterminate over $K$ and $f \in R[X]^\cdot$. Then
$$\{P \in \mathcal{X}(\lbrack f \rbrack) \mid P \cap R = \emptyset\} = \{gK[X] \cap \lbrack f \rbrack \mid g \in \lbrack f \rbrack \cap \mathcal{A}(K[X])\}.$$

Definition

Let $R$ be a factorial domain, $f \in \text{Int}(R)^\cdot$, $a \in \lbrack f \rbrack \cap R$ and $A \subseteq B \subseteq \lbrack f \rbrack$. We say that $A$ is $a$-compatible if there is some $w \in \lbrack f \rbrack$ such that $a \mid_R \frac{d(uw)}{d(w)}$ for all $u \in A$. Moreover, $A$ is called maximal $a$-compatible in $B$ if $A$ is maximal (with respect to inclusion) among the $a$-compatible subsets of $B$. 
Description of height-one prime ideals II

**Proposition**

Let $R$ be a factorial domain, $f \in \text{Int}(R)^\bullet$, $S$ a system of representatives of $\mathcal{A}([f]) \setminus R$, $p \in \mathcal{A}([f]) \cap R$ and $A \subseteq [f]$. Then
\[
\{ P \in \mathcal{X}([f]) \mid p \in P \} = \{(Q \cup \{p\})[f] \mid Q \subseteq S, Q \text{ is maximal } p\text{-compatible in } S \}.
\]
Proposition

Let \( R \) be a factorial domain, \( f \in \text{Int}(R)\setminus 1 \), \( S \) a system of representatives of \( A([f]) \setminus R \), \( p \in A([f]) \cap R \) and \( A \subseteq [f] \). Then
\[
\{ P \in \mathcal{X}(f) \mid p \in P \} = \{ (Q \cup \{ p \})[f] \mid Q \subseteq S, Q \text{ is maximal } p\text{-compatible in } S \}.
\]

Lemma

Let \( H \) be a Krull monoid and \( T \subseteq H \) a saturated submonoid. Then \( T \) is a Krull monoid and for every \( P \in \mathcal{X}(T) \) there is some \( Q \in \mathcal{X}(H) \) such that \( P = Q \cap T \). In particular, \( \mathcal{X}(T) \) is the set of minimal elements of \( \{ Q \cap T \mid Q \in \mathcal{X}(H) \} \setminus \{ \emptyset \} \).
Connection between atoms and height-one prime ideals I

**Definition**

Let $R$ be a factorial domain, $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$ and $f \in \text{Int}(R)^{\bullet}$. Let $e_f : [f] \to R^{\bullet}$ be defined by

$$e_f(g) = \prod_{p \in \mathcal{P}} p^{\max\{v_p\left(\frac{d(gh)}{d(h)}\right) | h \in [f]\}}$$

for all $g \in [f]$. 


Connection between atoms and height-one prime ideals

Definition

Let $R$ be a factorial domain, $\mathcal{P}$ a system of representatives of $A(R)$ and $f \in \text{Int}(R)^\bullet$. Let $e_f : [f] \to R^\bullet$ be defined by

$$e_f(g) = \prod_{p \in \mathcal{P}} p^{\max \{ v_p\left(\frac{d(gh)}{d(h)}\right) | h \in [f] \}}$$

for all $g \in [f]$.

Proposition

Let $R$ be a factorial domain, $K$ a field of quotients of $R$, $X$ an indeterminate over $K$ and $f \in R[X]^\bullet$. For $g \in [f]$ set $P_g = gK[X] \cap [f]$.

1. If $g \in [f] \cap A(K[X])$, then $v_{P_g}(x[f]) = v_g(x)$ for all $x \in [f]$.
2. If $Q \in \mathcal{X}([f])$ and $q \in Q \cap A(R)$, then $v_Q(x[f]) \leq v_q(e_f(x))$ for all $x \in [f]$.
Proposition

Let $H$ be a Krull monoid, $T \subseteq H$ a saturated submonoid, $I \in \mathcal{I}_v(T) \setminus \{\emptyset\}$ and $P \in \mathcal{X}(T)$. Then

$$v_P(I) = \max\left\{\left\lceil \frac{v_Q(I_{vH})}{v_Q(P_{vH})} \right\rceil \mid Q \in \mathcal{X}(H), Q \cap T = P\right\}.$$
Quand les idéaux premiers de hauteur un sont principaux 1

**Proposition**

Let \( R \) be a factorial domain and \( f \in \text{Int}(R)^\bullet \). The following conditions are equivalent:

1. Every \( P \in \mathcal{X}([f]) \) such that \( P \cap R \neq \emptyset \) is principal.
2. For every \( P \in \mathcal{X}([f]) \) such that \( P \cap R \neq \emptyset \) there is some \( n \in \mathbb{N} \) such that \((P^n)_v\) is principal.
3. Every \( u \in \mathcal{A}([f]) \cap R \) is a prime element of \([f]\).
4. \( d(gh) = d(g)d(h) \) for all \( g, h \in [f] \).

If \( C_v([f]) \) is finite, then these conditions are satisfied.
Proposition

Let $R$ be a factorial domain, $K$ a field of quotients of $R$, $X$ an indeterminate over $K$, $f \in R[X]^\bullet$ and $g \in \mathbb{[f]}$. The following conditions are equivalent:

1. $gK[X] \cap \mathbb{[f]}$ is a principal ideal of $\mathbb{[f]}$.
2. $gK[X] \cap \mathbb{[f]} = \frac{g}{d(g)} \mathbb{[f]}$.
3. $d(gh) = d(g)d(h)$ for all $h \in \mathbb{[f]}$. 
Theorem

Let $R$ be a factorial domain, $K$ a field of quotients of $R$, $X$ an indeterminate over $K$ and $f \in R[X]^\bullet$. Set
\[
  r = |\{P \in \mathcal{X}([f]) \mid P \cap R \neq \emptyset\}| - |\{u[f] \mid u \in \mathcal{A}([f]) \cap R\}|.
\]
Then $C_v([f]) \cong \mathbb{Z}^r$ and the following conditions are equivalent:

1. $[f]$ is factorial.
2. $C_v([f])$ is finite.
3. $d(gh) = d(g)d(h)$ for all $g, h \in [f]$. 
A realization theorem

Definition
Let $G$ be an abelian group and $G_0 \subseteq G$. Then $D(G_0) = \sup\{|A| \mid A \text{ is a nonempty minimal zero-sum sequence in } G_0\}$ is called the Davenport constant of $G_0$. 

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A realization theorem

Definition
Let $G$ be an abelian group and $G_0 \subseteq G$. Then $D(G_0) = \sup \{ |A| \mid A \text{ is a nonempty minimal zero-sum sequence in } G_0 \}$ is called the Davenport constant of $G_0$.

Theorem
Let $R$ be a factorial domain, $X$ an indeterminate over $R$, $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$, $n \in \mathbb{N}$, $a \in R^n$ and $p \in \mathcal{P}^n$ such that for all $i \in [1, n]$, $p_1 \mid_R a_i - 1$, $a_i + p_k R \in (R/p_k R)\times$ for all $k \in [2, n]$ and if $i > 1$, then $n = |\{p_j + p_1 R \mid j \in [1, n]\}| = |R/p_1 R| < |R/p_i R|$. Set $H = \prod_{i=1}^n (a_i X - p_i)$. Then $C_v(H) \cong \mathbb{Z}^{n-1}$, $\{[P] \mid P \in \mathcal{X}(H)\} = \{[P^{-1}] \mid P \in \mathcal{X}(H)\}$ and $D(\{[P] \mid P \in \mathcal{X}(H)\}) \geq n$. 
Theorem

Let $R$ be a factorial domain, $X$ an indeterminate over $R$, $a \in R$ and $f, g \in R[X]^\bullet$ such that $\text{GCD}_{R[X]}(f, g) = R[X]^\times$, $\text{GCD}_R(f(a), g(a)) = R^\times$ and $d(rs) = d(r)d(s)$ for all $r \in \llbracket f \rrbracket$ and $s \in \llbracket g \rrbracket$. Then $C_v(\llbracket fg \rrbracket) \cong C_v(\llbracket f \rrbracket) \times C_v(\llbracket g \rrbracket)$.
Products of divisor-class groups

Theorem

Let $R$ be a factorial domain, $X$ an indeterminate over $R$, $a \in R$ and $f, g \in R[X]^{\bullet}$ such that $\text{GCD}_{R[X]}(f, g) = R[X]^\times$, $\text{GCD}_{R}(f(a), g(a)) = R^\times$ and $d(rs) = d(r)d(s)$ for all $r \in \llbracket f \rrbracket$ and $s \in \llbracket g \rrbracket$. Then $C_v(\llbracket fg \rrbracket) \cong C_v(\llbracket f \rrbracket) \times C_v(\llbracket g \rrbracket)$.

Theorem

Let $R$ be a factorial domain, $X$ an indeterminate over $R$ and $f, g \in R[X]^{\bullet}$ such that $\text{GCD}_{R[X]}(f, g) = R[X]^\times$, $\text{GCD}_{R}(f(x), g(x)) = R^\times$ for all but finitely many $x \in R$ and $d(rs) = d(r)d(s)$ for all $r \in \llbracket f \rrbracket$ and $s \in \llbracket g \rrbracket$. If $x \in \llbracket f \rrbracket$ and $y \in \llbracket g \rrbracket$, then $C_v(\llbracket xy \rrbracket) \cong C_v(\llbracket x \rrbracket) \times C_v(\llbracket y \rrbracket)$. 
When is $d(rs) = d(r)d(s)$?

Remark

Let $R$ be a factorial domain, $X$ an indeterminate over $R$, $a \in R$ and $f, g \in R[X]^{\bullet}$ such that $\text{GCD}_R(f(a), g(a)) = R^\times$ and for all $p \in A(R)$ and $h \in A(R[X])$ with $(p |_R f(a)$ and $h |_{R[X]} g)$ or $(p |_R g(a)$ and $h |_{R[X]} f)$ it follows that $p |_{R[X]} h - h(a)$. Then $d(rs) = d(r)d(s)$ for all $r \in \llbracket f \rrbracket$ and $s \in \llbracket g \rrbracket$. 
Example

Let $X$ be an indeterminate over $\mathbb{Q}$. Set

$g = (95095X + 2)(95095X + 3)$ and

$h = (6X + 5)(6X + 7)(6X + 11)(6X + 13)(6X + 19)$. Then

$C_v([gh]) \cong C_v([g]) \times C_v([h]) \cong \mathbb{Z} \times \mathbb{Z}^4 \cong \mathbb{Z}^5$. 
Example

Let $X$ be an indeterminate over $\mathbb{Q}$. Set

$g = (95095X + 2)(95095X + 3)$ and

$h = (6X + 5)(6X + 7)(6X + 11)(6X + 13)(6X + 19)$. Then

$C_v([gh]) \cong C_v([g]) \times C_v([h]) \cong \mathbb{Z} \times \mathbb{Z}^4 \cong \mathbb{Z}^5$.

Example

Let $X$ be an indeterminate over $\mathbb{Q}$. Set $a = 5713492603$, $f = (aX + 1)(aX + 2)(aX + 3)$, $g = \frac{f(aX+2)}{12}$ and

$h = (6Xf + 7)(6Xf + 13)(6Xf + 19)(6Xf + 31)(6Xf + 37)(6Xf + 43)(6Xf + 67)$. Then

$C_v([gh]) \cong C_v([g]) \times C_v([h]) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^6$. 
Motivational aspect

Theorem (Frisch, 2013)

Every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of $\text{Int}(\mathbb{Z})$. 

Theorem (Kainrath, 1999)

Let $H$ be a Krull monoid with infinite divisor-class group $G$ such that every class of $G$ contains a height-one prime ideal. Then every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of $H$.

Is the former theorem a consequence of the latter theorem?
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Elasticity and tame degree

Definition

Let \( H \) be an atomic monoid.

1. \( \rho(H) = \sup\{ \frac{\sup L(a)}{\min L(a)} \mid a \in H \setminus H^\times \} \) is called the elasticity of \( H \).

2. \( t(H) = \sup\{ \inf\{ N \in \mathbb{N}_0 \mid \text{for all } z \in Z(a) \text{ there is some } z' \in Z(a) \text{ with } uH^\times \mid z' \text{ and } d(z, z') \leq N \} \mid a \in H, u \in \mathcal{A}(H), u \mid a \} \) is called the tame degree of \( H \).
Proposition (Geroldinger, Halter-Koch, 2006)

Let $H$ be a Krull monoid such that
$$\{[P] \mid P \in \mathcal{X}(H)\} = \{[P^{-1}] \mid P \in \mathcal{X}(H)\}$$
and
$$D = D(\{[P] \mid P \in \mathcal{X}(H)\}) \geq 2.$$ 
Then $\rho(H) \geq \frac{D}{2}$ and $t(H) \geq D$. 
A corollary

Proposition (Geroldinger, Halter-Koch, 2006)
Let $H$ be a Krull monoid such that
\[
\{[P] \mid P \in \mathcal{X}(H)\} = \{[P^{-1}] \mid P \in \mathcal{X}(H)\}
\]
and
\[
D = D(\{[P] \mid P \in \mathcal{X}(H)\}) \geq 2.
\]
Then $\rho(H) \geq \frac{D}{2}$ and $t(H) \geq D$.

Corollary
Let $R$ be a factorial domain, $X$ an indeterminate over $R$ and $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$. Let $(\mathcal{P}_i)_{i \in \mathbb{N}}$ be a sequence of finite subsets of $\mathcal{P}$ such that for every $i \in \mathbb{N}$ there is some $p \in \mathcal{P}_i$ for which $i < |\mathcal{P}_i| = |\{r + pR \mid r \in \mathcal{P}_i\}| = |R/pR| < |R/qR| < \infty$ for all $q \in \mathcal{P}_i \setminus \{p\}$. Then $\rho(\text{Int}(R)) = t(\text{Int}(R)) = \infty$. 
Remark

**Theorem (Cahen, Chabert, 1995)**

Let $R$ be a Krull domain and $I$ a pseudo-principal ideal of $R$ (i.e., some power of $I$ is contained in a proper principal ideal of $R$) such that $R/I$ is finite. Then $\rho(\text{Int}(R)) = \infty$. 
**Theorem (Cahen, Chabert, 1995)**

Let $R$ be a Krull domain and $I$ a pseudo-principal ideal of $R$ (i.e., some power of $I$ is contained in a proper principal ideal of $R$) such that $R/I$ is finite. Then $\rho(\text{Int}(R)) = \infty$.

**Corollary (Cahen, Chabert, 1995)**

Let $R$ be a PID with at least one finite residue field. Then $\rho(\text{Int}(R)) = \infty$.
Remark

Let $H$ be a monoid such that every monadic submonoid of $H$ is a Krull monoid (i.e., $H$ is a monadically Krull monoid). Then $H$ is a completely integrally closed FF-monoid. If $T \subseteq H$ is a saturated submonoid, then $T$ is monadically Krull.
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In particular, if $R$ is an integral domain such that $\text{Int}(R) \cdot$ is monadically Krull, then $R \cdot$ is monadically Krull. It is an open problem whether the converse is true.
Let \( R \) be a factorial domain, \( f \in \text{Int}(R)^\bullet \) and \( g \in \llbracket f \rrbracket \). Is there some group epimorphism \( \phi : C_V(\llbracket f \rrbracket) \to C_V(\llbracket g \rrbracket) \)?
Let $R$ be a factorial domain, $f \in \text{Int}(R)^\bullet$ and $g \in \llbracket f \rrbracket$. Is there some group epimorphism $\phi : C_v(\llbracket f \rrbracket) \rightarrow C_v(\llbracket g \rrbracket)$?

Let $G$ be a finitely generated abelian group. Is there some $f \in \text{Int}(\mathbb{Z})^\bullet$ such that $C_v(\llbracket f \rrbracket) \cong G$?