Divisor-class groups of monadic submonoids of $\mathrm{Int}(R)$

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Krull monoids and rings of integer-valued polynomials

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Krull monoids and rings of integer-valued polynomials

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The arithmetic of Krull monoids is well understood. It can be described in terms of the divisor-class group of the Krull monoid.

Is it possible to study the arithmetic of $\mathrm{Int}(R)$ for "interesting domains" R by using the theory of Krull monoids?

Theorem (Cahen, Gabelli, Houston, 2000)

Let R be an integral domain and X an indeterminate over R. Then $\mathrm{Int}(R)$ is a Krull domain if and only if R is a Krull domain and $\mathrm{Int}(R) = R[X]$.

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What about finding suitable submonoids of $\mathrm{Int}(R)^\bullet$ that are Krull monoids?

In this talk a monoid is always a commutative (multiplicative) cancellative semigroup with identity.

Definition

Let H be a monoid, K a quotient group of H and $X \subseteq K$.

1. Set $X^{-1} = \{z \in K \mid zX \subseteq H\}$ and $X_v = (X^{-1})^{-1}$.
2. Set $\mathcal{F}_v(H) = \{X \subseteq K \mid X_v = X$ and $xX \subseteq H$ for some $x \in H\}$, called the set of fractional v-ideals of H.
3. Set $\mathcal{F}_v(H)^\times = \{X \in \mathcal{F}_v(H) \mid (XX^{-1})_v = H\}$, called the set of v-invertible fractional v-ideals. It forms a group under v-multiplication.

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4. Set $\mathcal{H}(H) = \{xH \mid x \in K\}$, called the set of fractional principal ideals of H. It is a subgroup of $\mathcal{F}_v(H)^\times$ under v-multiplication.
5. Set $\mathcal{C}_v(H) = \mathcal{F}_v(H)^\times / \mathcal{H}(H)$, called the divisor-class group of H.

Definition

Let H be a monoid and $X \subseteq H$.

1. Set $\mathcal{A}(H) = \{x \in H \setminus H^\times \mid$ for all $u, v \in H$ with $x = uv$ it follows that $u \in H^\times$ or $v \in H^\times\}$, called the set of atoms of H.
2. X is called a (prime) s-ideal of H if $XH = X$ (and $xy \in X$ implies that $x \in X$ or $y \in X$ for all $x, y \in H$).

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3. Set $\mathcal{I}_v(H) = \{I \in \mathcal{F}_v(H) \mid I \subseteq H\}$, called the set of v-ideals of H.
4. Let $\mathfrak{X}(H)$ be the set of minimal non-empty prime s-ideals of H, called the set of height-one prime ideals of H.

Saturated submonoids

Definition

Let H be a monoid and $T \subseteq H$ a submonoid. We say $T \subseteq H$ is saturated if the following equivalent conditions are satisfied:

a. For all $x, y \in T$ with $x \mid_H y$ it follows that $x \mid_T y$.

b. For every $I \in \mathcal{I}_v(T)$ we have $I_{v_H} \cap T = I$.

c. For every $x \in T$, $xH \cap T = xT$.

Divisor-closed and monadic submonoids

Definition

Let H be a monoid and $T \subseteq H$ a submonoid.

1. We say $T \subseteq H$ is divisor-closed if for all $x, y \in H$ such that $xy \in T$ it follows that $x, y \in T$.
2. We say $T \subseteq H$ is monadic H if $T = \{g \in H \mid g \mid_H x^k \text{ for some } k \in \mathbb{N}\}$ for some $x \in H$.
3. For $x \in H$ set $[\![x]\!]_H = \{g \in H \mid g \mid_H x^k \text{ for some } k \in \mathbb{N}\}$, called the monadic submonoid of H generated by x.

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The divisor-closed submonoids of an integral domain are precisely the proper saturated multiplicatively closed subsets in the terminology of Kaplansky's book. Every monadic submonoid is divisor-closed and every divisor-closed submonoid is saturated.

Familiar situations

Lemma

Let R be an integral domain and X an indeterminate over R.

- $R^\bullet \subseteq R[X]^\bullet$ and $R^\bullet \subseteq \mathrm{Int}(R)^\bullet$ are divisor-closed submonoids.
- $R^\bullet \subseteq R[X]^\bullet$ is monadic if and only if $R^\bullet \subseteq \mathrm{Int}(R)^\bullet$ is monadic if and only if R is a G-domain (i.e., the intersection of all nonzero prime ideals of R is nonzero).
- $R^\bullet \subseteq R[\![X]\!]^\bullet$ is saturated, but not divisor-closed.

Some remarks

Remark

Let H be a monoid.

1. H is called a Krull monoid if every non-empty (fractional) v-ideal of H is v-invertible and H satisfies the ACC on v-ideals. If H is a Krull monoid, then every v-ideal of H is a finite v-product of height-one prime ideals of H.

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2. H is called a quasi finitely generated monoid if H satisfies the ACC on s-ideals (equivalently, H is atomic and $\{uH \mid u \in \mathcal{A}(H)\}$ is finite). If H is a quasi finitely generated Krull monoid, then $\mathfrak{X}(H)$ is finite and $\mathcal{C}_v(H)$ is a finitely generated abelian group.

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3. If $x \in H$ is such that $[\![x]\!]$ is a Krull monoid, then $[\![x]\!]$ is a quasi finitely generated monoid.

Some remarks

Remark

Let R be an integral domain.

1. $R^\bullet$ is quasi finitely generated if and only if R is a Cohen-Kaplansky domain.
2. $R^\bullet$ is a quasi finitely generated Krull monoid if and only if R is a semilocal PID. The divisor-class group of $R^\bullet$ is trivial in this case.

When monadic submonoids of $\mathrm{Int}(R)$ are Krull

Theorem (R, 2014)

Let R be a factorial domain. Then every monadic submonoid of $\mathrm{Int}(R)^\bullet$ is a Krull monoid.

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Goal: Describe the structure of the divisor-class groups of monadic submonoids $[\![f]\!]_{\mathrm{Int}(R)^\bullet}$ of $\mathrm{Int}(R)^\bullet$ if R is factorial. This can be done by studying atoms and height-one prime ideals and the relations between them.

Description of atoms I

Definition

Let R be a factorial domain and $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$. Let $d : \mathrm{Int}(R)^\bullet \to R^\bullet$ be defined by $d(g) = \prod_{p \in \mathcal{P}} p^{\min\{\mathrm{v}_p(g(x)) | x \in R\}}$ for all $g \in \mathrm{Int}(R)^\bullet$.

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Lemma

Let R be a factorial domain, X an indeterminate over R and $f \in R[X]^\bullet$. Then there are some $a \in R^\bullet$, $n \in \mathbb{N}$ and $\underline{f}$ an n-tuple of pairwise non-associated non-constant atoms of $R[X]$ such that $[\![f]\!] = [\![a \prod_{i=1}^{n} f_i]\!]$.

Description of atoms II

Proposition

Let R be a factorial domain, $a \in R^\bullet$, $n \in \mathbb{N}$ and $\underline{f}$ an n-tuple of pairwise non-associated non-constant atoms of $R[X]$. Set $f = a\prod_{i=1}^n f_i$. Then there is some finite $E \subseteq \mathbb{N}_0^n \setminus \{\underline{0}\}$ such that $\{u[\![f]\!] \mid u \in \mathcal{A}([\![f]\!])\} = \{u[\![f]\!] \mid u \in \mathcal{A}(R), u \mid_R d(f)\} \cup \{\frac{\prod_{i=1}^n f_i^{y_i}}{d(\prod_{i=1}^n f_i^{y_i})}[\![f]\!] \mid \underline{y} \in E\}$.

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Suppose that $f \in \mathrm{Int}(R)^\bullet$. Then $f = \frac{g}{c}$ for some $g \in R[X]^\bullet$ and $c \in R^\bullet$ and $[\![f]\!]$ is a monadic submonoid of $[\![g]\!]$.

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Lemma

Let H be a monoid and $T \subseteq H$ a divisor-closed submonoid. Then $\mathcal{A}(T) = \mathcal{A}(H) \cap T$.

A simple observation

Lemma

Let R be a factorial domain and $f \in \mathrm{Int}(R)^\bullet$.

1. If $u \in \mathcal{A}(\llbracket f \rrbracket) \cap R$, then $u\llbracket f \rrbracket$ is a radical ideal of $\llbracket f \rrbracket$.
2. If $P \in \mathfrak{X}(\llbracket f \rrbracket)$ and $u, w \in P \cap \mathcal{A}(R)$, then $u \simeq_{\llbracket f \rrbracket} w$.

Description of height-one prime ideals I

Proposition

Let R be a factorial domain, K a field of quotients of R, X an indeterminate over K and $f \in R[X]^\bullet$. Then $\{P \in \mathfrak{X}([\![f]\!]) \mid P \cap R = \emptyset\} = \{gK[X] \cap [\![f]\!] \mid g \in [\![f]\!] \cap \mathcal{A}(K[X])\}$.

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Definition

Let R be a factorial domain, $f \in \mathrm{Int}(R)^\bullet$, $a \in [\![f]\!] \cap R$ and $A \subseteq B \subseteq [\![f]\!]$. We say that A is a-compatible if there is some $w \in [\![f]\!]$ such that $a \mid_R \frac{d(uw)}{d(w)}$ for all $u \in A$. Moreover, A is called maximal a-compatible in B if A is maximal (with respect to inclusion) among the a-compatible subsets of B.

Description of height-one prime ideals II

Proposition

Let R be a factorial domain, $f \in \mathrm{Int}(R)^{\bullet}$, $\mathcal{S}$ a system of representatives of $\mathcal{A}(\llbracket f \rrbracket) \setminus R$, $p \in \mathcal{A}(\llbracket f \rrbracket) \cap R$ and $A \subseteq \llbracket f \rrbracket$. Then $\{P \in \mathfrak{X}(\llbracket f \rrbracket) \mid p \in P\} = \{(Q \cup \{p\})\llbracket f \rrbracket \mid Q \subseteq \mathcal{S}, Q$ is maximal p-compatible in $\mathcal{S}\}$.

Description of height-one prime ideals II

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Let R be a factorial domain, $f \in \mathrm{Int}(R)^\bullet$, $\mathcal{S}$ a system of representatives of $\mathcal{A}([\![f]\!]) \setminus R$, $p \in \mathcal{A}([\![f]\!]) \cap R$ and $A \subseteq [\![f]\!]$. Then $\{P \in \mathfrak{X}([\![f]\!]) \mid p \in P\} = \{(Q \cup \{p\})[\![f]\!] \mid Q \subseteq \mathcal{S}, Q$ is maximal p-compatible in $\mathcal{S}\}$.

Lemma

Let H be a Krull monoid and $T \subseteq H$ a saturated submonoid. Then T is a Krull monoid and for every $P \in \mathfrak{X}(T)$ there is some $Q \in \mathfrak{X}(H)$ such that $P = Q \cap T$. In particular, $\mathfrak{X}(T)$ is the set of minimal elements of $\{Q \cap T \mid Q \in \mathfrak{X}(H)\} \setminus \{\emptyset\}$.

Connection between atoms and height-one prime ideals I

Definition

Let R be a factorial domain, $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$ and $f \in \mathrm{Int}(R)^\bullet$. Let $e_f : [\![f]\!] \to R^\bullet$ be defined by $e_f(g) = \prod_{p \in \mathcal{P}} p^{\max\{\mathrm{v}_p(\frac{d(gh)}{d(h)}) \mid h \in [\![f]\!]\}}$ for all $g \in [\![f]\!]$.

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Proposition

Let R be a factorial domain, K a field of quotients of R, X an indeterminate over K and $f \in R[X]^\bullet$. For $g \in [\![f]\!]$ set $P_g = gK[X] \cap [\![f]\!]$.

1. If $g \in [\![f]\!] \cap \mathcal{A}(K[X])$, then $\mathrm{v}_{P_g}(x[\![f]\!]) = \mathrm{v}_g(x)$ for all $x \in [\![f]\!]$.
2. If $Q \in \mathfrak{X}([\![f]\!])$ and $q \in Q \cap \mathcal{A}(R)$, then $\mathrm{v}_Q(x[\![f]\!]) \le \mathrm{v}_q(e_f(x))$ for all $x \in [\![f]\!]$.

Connection between atoms and height-one prime ideals II

Proposition

Let H be a Krull monoid, $T \subseteq H$ a saturated submonoid, $I \in \mathcal{I}_v(T) \setminus \{\emptyset\}$ and $P \in \mathfrak{X}(T)$. Then $\mathrm{v}_P(I) = \max\{\lceil \frac{\mathrm{v}_Q(I_{v_H})}{\mathrm{v}_Q(P_{v_H})} \rceil \mid Q \in \mathfrak{X}(H), Q \cap T = P\}$.

When height-one prime ideals are principal I

Proposition

Let R be a factorial domain and $f \in \mathrm{Int}(R)^\bullet$. The following conditions are equivalent:

1. Every $P \in \mathfrak{X}([\![f]\!])$ such that $P \cap R \neq \emptyset$ is principal.
2. For every $P \in \mathfrak{X}([\![f]\!])$ such that $P \cap R \neq \emptyset$ there is some $n \in \mathbb{N}$ such that $(P^n)_v$ is principal.
3. Every $u \in \mathcal{A}([\![f]\!]) \cap R$ is a prime element of $[\![f]\!]$.
4. $d(gh) = d(g)d(h)$ for all $g, h \in [\![f]\!]$.

If $\mathcal{C}_v([\![f]\!])$ is finite, then these conditions are satisfied.

When height-one prime ideals are principal II

Proposition

Let R be a factorial domain, K a field of quotients of R, X an indeterminate over K, $f \in R[X]^\bullet$ and $g \in [\![f]\!]$. The following conditions are equivalent:

1. $gK[X] \cap [\![f]\!]$ is a principal ideal of $[\![f]\!]$.
2. $gK[X] \cap [\![f]\!] = \frac{g}{d(g)}[\![f]\!]$.
3. $d(gh) = d(g)d(h)$ for all $h \in [\![f]\!]$.

Main result

Theorem

Let R be a factorial domain, K a field of quotients of R, X an indeterminate over K and $f \in R[X]^\bullet$. Set $r = |\{P \in \mathfrak{X}([\![f]\!]) \mid P \cap R \neq \emptyset\}| - |\{u[\![f]\!] \mid u \in \mathcal{A}([\![f]\!]) \cap R\}|$. Then $\mathcal{C}_v([\![f]\!]) \cong \mathbb{Z}^r$ and the following conditions are equivalent:

1. $[\![f]\!]$ is factorial.
2. $\mathcal{C}_v([\![f]\!])$ is finite.
3. $d(gh) = d(g)d(h)$ for all $g, h \in [\![f]\!]$.

A realization theorem

Definition

Let G be an abelian group and $G_0 \subseteq G$. Then $D(G_0) = \sup\{|A| \mid A \text{ is a nonempty minimal zero-sum sequence in } G_0\}$ is called the Davenport constant of G_0.

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Theorem

Let R be a factorial domain, X an indeterminate over R, $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$, $n \in \mathbb{N}$, $\underline{a} \in R^n$ and $\underline{p} \in \mathcal{P}^n$ such that for all $i \in [1, n]$, $p_1 \mid_R a_i - 1$, $a_i + p_k R \in (R/p_k R)^\times$ for all $k \in [2, n]$ and if $i > 1$, then $n = |\{p_j + p_1 R \mid j \in [1, n]\}| = |R/p_1 R| < |R/p_i R|$. Set $H = [\![\prod_{i=1}^{n}(a_i X - p_i)]\!]$. Then $\mathcal{C}_v(H) \cong \mathbb{Z}^{n-1}$, $\{[P] \mid P \in \mathfrak{X}(H)\} = \{[P^{-1}] \mid P \in \mathfrak{X}(H)\}$ and $D(\{[P] \mid P \in \mathfrak{X}(H)\}) \geq n$.

Products of divisor-class groups

Theorem

Let R be a factorial domain, X an indeterminate over R, $a \in R$ and $f, g \in R[X]^\bullet$ such that $\mathrm{GCD}_{R[X]}(f, g) = R[X]^\times$, $\mathrm{GCD}_R(f(a), g(a)) = R^\times$ and $d(rs) = d(r)d(s)$ for all $r \in [\![f]\!]$ and $s \in [\![g]\!]$. Then $\mathcal{C}_v([\![fg]\!]) \cong \mathcal{C}_v([\![f]\!]) \times \mathcal{C}_v([\![g]\!])$.

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Theorem

Let R be a factorial domain, X an indeterminate over R and $f, g \in R[X]^\bullet$ such that $\mathrm{GCD}_{R[X]}(f,g) = R[X]^\times$, $\mathrm{GCD}_R(f(x), g(x)) = R^\times$ for all but finitely many $x \in R$ and $d(rs) = d(r)d(s)$ for all $r \in [\![f]\!]$ and $s \in [\![g]\!]$. If $x \in [\![f]\!]$ and $y \in [\![g]\!]$, then $\mathcal{C}_v([\![xy]\!]) \cong \mathcal{C}_v([\![x]\!]) \times \mathcal{C}_v([\![y]\!])$.

When is $d(rs) = d(r)d(s)$?

Remark

Let R be a factorial domain, X an indeterminate over R, $a \in R$ and $f, g \in R[X]^\bullet$ such that $\mathrm{GCD}_R(f(a), g(a)) = R^\times$ and for all $p \in \mathcal{A}(R)$ and $h \in \mathcal{A}(R[X])$ with $(p \mid_R f(a)$ and $h \mid_{R[X]} g)$ or $(p \mid_R g(a)$ and $h \mid_{R[X]} f)$ it follows that $p \mid_{R[X]} h - h(a)$. Then $d(rs) = d(r)d(s)$ for all $r \in [\![f]\!]$ and $s \in [\![g]\!]$.

Examples

Example

Let X be an indeterminate over $\mathbb{Q}$. Set
$g = (95095X + 2)(95095X + 3)$ and
$h = (6X + 5)(6X + 7)(6X + 11)(6X + 13)(6X + 19)$. Then
$\mathcal{C}_v([\![gh]\!]) \cong \mathcal{C}_v([\![g]\!]) \times \mathcal{C}_v([\![h]\!]) \cong \mathbb{Z} \times \mathbb{Z}^4 \cong \mathbb{Z}^5$.

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Example

Let X be an indeterminate over $\mathbb{Q}$. Set $a = 5713492603$, $f = (aX + 1)(aX + 2)(aX + 3)$, $g = \frac{f(aX+2)}{12}$ and $h = (6Xf + 7)(6Xf + 13)(6Xf + 19)(6Xf + 31)(6Xf + 37)(6Xf + 43)(6Xf + 67)$. Then $\mathcal{C}_v([\![gh]\!]) \cong \mathcal{C}_v([\![g]\!]) \times \mathcal{C}_v([\![h]\!]) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^6$.

Motivational aspect

Theorem (Frisch, 2013)

Every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of $\mathrm{Int}(\mathbb{Z})$.

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Every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of $\mathrm{Int}(\mathbb{Z})$.

Theorem (Kainrath, 1999)

Let H be a Krull monoid with infinite divisor-class group G such that every class of G contains a height-one prime ideal. Then every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of H.

Motivational aspect

Theorem (Frisch, 2013)

Every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of $\mathrm{Int}(\mathbb{Z})$.

Theorem (Kainrath, 1999)

Let H be a Krull monoid with infinite divisor-class group G such that every class of G contains a height-one prime ideal. Then every finite non-empty subset of $\mathbb{N}_{\geq 2}$ is the set of lengths of some element of H.

Is the former theorem a consequence of the latter theorem?

Elasticity and tame degree

Definition

Let H be an atomic monoid.

1. $\rho(H) = \sup\{\frac{\sup L(a)}{\min L(a)} \mid a \in H \setminus H^\times\}$ is called the elasticity of H.
2. $t(H) = \sup\{\inf\{N \in \mathbb{N}_0 \mid \text{for all } z \in Z(a) \text{ there is some } z' \in Z(a) \text{ with } uH^\times \mid z' \text{ and } d(z, z') \leq N\} \mid a \in H, u \in \mathcal{A}(H), u \mid a\}$ is called the tame degree of H.

A corollary

Proposition (Geroldinger, Halter-Koch, 2006)

Let H be a Krull monoid such that $\{[P] \mid P \in \mathfrak{X}(H)\} = \{[P^{-1}] \mid P \in \mathfrak{X}(H)\}$ and $D = D(\{[P] \mid P \in \mathfrak{X}(H)\}) \geq 2$. Then $\rho(H) \geq \frac{D}{2}$ and $t(H) \geq D$.

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Corollary

Let R be a factorial domain, X an indeterminate over R and $\mathcal{P}$ a system of representatives of $\mathcal{A}(R)$. Let $(\mathcal{P}_i)_{i \in \mathbb{N}}$ be a sequence of finite subsets of $\mathcal{P}$ such that for every $i \in \mathbb{N}$ there is some $p \in \mathcal{P}_i$ for which $i < |\mathcal{P}_i| = |\{r + pR \mid r \in \mathcal{P}_i\}| = |R/pR| < |R/qR| < \infty$ for all $q \in \mathcal{P}_i \setminus \{p\}$. Then $\rho(\mathrm{Int}(R)) = t(\mathrm{Int}(R)) = \infty$.

Remark

Theorem (Cahen, Chabert, 1995)

Let R be a Krull domain and I a pseudo-principal ideal of R (i.e., some power of I is contained in a proper principal ideal of R) such that R/I is finite. Then $\rho(\mathrm{Int}(R)) = \infty$.

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Corollary (Cahen, Chabert, 1995)

Let R be a PID with at least one finite residue field. Then $\rho(\mathrm{Int}(R)) = \infty$.

Open problems I

Remark

Let H be a monoid such that every monadic submonoid of H is a Krull monoid (i.e., H is a monadically Krull monoid). Then H is a completely integrally closed FF-monoid. If $T \subseteq H$ is a saturated submonoid, then T is monadically Krull.

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In particular, if R is an integral domain such that $\mathrm{Int}(R)^\bullet$ is monadically Krull, then $R^\bullet$ is monadically Krull. It is an open problem whether the converse is true.

Open problems II

- Let R be a factorial domain, $f \in \mathrm{Int}(R)^\bullet$ and $g \in [\![f]\!]$. Is there some group epimorphism $\phi : \mathcal{C}_v([\![f]\!]) \to \mathcal{C}_v([\![g]\!])$?

Open problems II

- Let R be a factorial domain, $f \in \mathrm{Int}(R)^\bullet$ and $g \in [\![f]\!]$. Is there some group epimorphism $\phi : \mathcal{C}_v([\![f]\!]) \to \mathcal{C}_v([\![g]\!])$?
- Let G be a finitely generated abelian group. Is there some $f \in \mathrm{Int}(\mathbb{Z})^\bullet$ such that $\mathcal{C}_v([\![f]\!]) \cong G$?