Module analogues of coincidence of nilpotent elements of a ring and its prime radical

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Abstract

It is a well-known fact that in a commutative ring, the set of all nilpotent elements coincides with the intersection of all prime ideals. McCasland and Moore generalised this notion to modules by defining modules that satisfy the radical formula. A not necessarily commutative ring is 2-primal if its nilpotent elements coincide with the intersection of all its prime ideals. Groenewald and Ssevviiri generalised 2-primal rings to 2-primal modules. In this talk, I compare modules that satisfy the radical formula and 2-primal modules.

About rings and modules in this talk

- All rings are unital.
- All modules are left unital modules defined over rings.

Definitions and Notation

Definition

P is a prime ideal of a ring R if for all ideals A, B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Definition

P is a completely prime ideal of a ring R if for all elements $a, b \in R$, $ab \in P$ implies $a \in P$ or $b \in P$.

- Any completely prime ideal is prime.
- We denote the prime radical (resp. completely prime radical) of R by $\beta(R)$ (resp. $\beta_{co}(R)$).

Ring theoretic notions to generalize to modules

- **Commutative ring case**:
  - $\mathcal{N}(R) = \beta(R)$.
- **Not necessarily commutative ring case:**
  - In general, $\mathcal{N}(R) \neq \beta(R)$.
  - If $\mathcal{N}(R) = \beta(R)$, we say R is *2-primal* - Birkernmeier.
  - R is 2-primal iff every minimal prime ideal of R is completely prime - Shin
  - R is 2-primal iff $\beta(R) = \beta_{co}(R)$ - Birkernmeier.
  - Commutative rings, reduced rings are 2-primal.

Definitions in the module setting

Definition

A proper submodule P of an R-module M for which $RM \not\subseteq P$ is

1. *completely prime* if $am \in P$ implies $m \in P$ or $aM \subseteq P$, for all $a \in R$ and $m \in M$;
2. *prime* if for all ideals $\mathcal{A}$ of R and submodules N of M, $\mathcal{A}N \subseteq P$ implies $N \subseteq P$ or $\mathcal{A}M \subseteq P$.

- R is a prime (resp. completely prime) ring iff ${}_RR$ is a prime (resp. completely prime) module.
- Any completely prime submodule is prime.

Definitions contnd

Definition

A proper submodule P of an R-module M for which $RM \not\subseteq P$ is *completely semiprime* (resp. *semiprime*) if $a^2m \in P$ (resp. $aRam \subseteq P$) implies $am \in P$, for all $a \in R$ and $m \in M$.

Defns contnd and some Notation defined

Definition

The envelope of a submodule N of an R-module M is the set

$$E_M(N) := \{rm \ : \ r \in R, m \in M \text{ and } r^k m \in N \text{ for some } \ k \in \mathbb{N}\}.$$

- $E_M(0)$ is the module analogue of $\mathcal{N}(R)$.
- For a commutative ring R, $E_R(0) = \mathcal{N}(R)$.
- $E_M(N)$ is in general not a submodule of M.
- $\langle E_M(N) \rangle$ denotes a submodule of M generated by $E_M(N)$.
- $\beta(M)$ - prime radical of M
- $\beta_{co}(M)$ - completely prime radical of M

When does $E_M(N)$ become a submodule of M?

- When $M = R$, $N = 0$ and R is commutative.
- When N is a completely semiprime submodule of an R-module M we get $E_M(N) = N$.
- If M is a 2-primal module, then $E_M(\beta(M)) = \beta(M)$. In particular, $E_M(\beta(M))$ is a submodule of M.

Modules that satisfy the radical formula (s.t.r.f)

Definition

A submodule N of an R-module M s.t.r.f if $\langle E_M(N)\rangle = \beta(N)$.

- A module M s.t.r.f if every submodule of M satisfies the radical formula.
- A ring R s.t.r.f if every R-module s.t.r.f.
- Many authors studied modules that s.t.r.f, see [1, 2, 5, 7, 8, 10, 11] among others
- Unlike commutative rings for which $\sqrt{I} = \beta(I)$ for any ideal I, not all modules over commutative rings s.t.r.f.

2-primal modules

Definition

A submodule N of an R-module M is 2-primal if $\beta_{co}(M/N) = \beta(M/N)$.

Definition

An R-module M is 2-primal if $\beta_{co}(M) = \beta(M)$.

- Any module over a commutative ring is 2-primal.
- A projective module over a 2-primal ring is 2-primal.

(Sub)modules that s.t.r.f Vs 2-primal (sub)modules

Proposition

Any 2-primal submodule N of an R-module M for which $\beta(N) = N$ s.t.r.f.

Proposition

If M is a 2-primal R-module such that $\beta(M) = \beta(R)M$ or $\beta_{co}(M) = \beta_{co}(R)M$, then the zero submodule of M s.t.r.f.

Some Definitions

Definition

An R-module M is

1. reduced (Lee and Zhou in [6]), if for all $a \in R$ and every $m \in M$, $am = 0$ implies $Rm \cap aM = 0$.
2. symmetric if $abm = 0$ implies $bam = 0$ for $a, b \in R$ and $m \in M$.
3. IFP (i.e., it has the insertion-of-factor-property) if whenever $am = 0$ for $a \in R$ and $m \in M$, we have $aRm = 0$.
4. semi-symmetric if for all $a \in R$ and every $m \in M$, $a^2m = 0$ implies $(a)^2m = 0$ where (a) is the ideal of R generated by $a \in R$.

Implications

$$\begin{array}{ccccccc} & & R \text{ commutative} & & & & \text{2-primal} \\ & & \Downarrow & & & & \Uparrow \\ \text{Lee-Zhou} & \Rightarrow & \text{symmetric} & \Rightarrow & \text{IFP} & \Rightarrow & \text{semi-symmet} \\ \text{completely semiprime} & & & & & & \end{array}$$

Lemma

Any one of the following statements implies that the zero submodule of M s.t.r.f:

1. M *is 2-primal and free,*
2. M *is semi-symmetric and free,*
3. M *is semi-symmetric and projective,*
4. M *is IFP and projective,*
5. M *is IFP and free,*
6. M *is symmetric and projective,*
7. M *is symmetric and free,*
8. M *is reduced and projective,*
9. M *is reduced and free,*
10. R *is commutative and* M *is projective,*
11. R *is commutative and* M *is free.*

s.t.r.f Vs 2-primal Contnd

Theorem

If the R-module M is any one of the modules given in Lemma 1 or it is 2-primal and projective, then M s.t.r.f.

Corollary

If R is a semisimple ring such that the R-module M is 2-primal, then M s.t.r.f.

Corollary

If R is a semisimple and commutative ring, then the R-module M s.t.r.f.

s.t.r.f Vs 2-primal Contnd

Theorem

The necessary and sufficient condition for the zero submodule of an R-module M to s.t.r.f if and only if M is 2-primal is $\beta_{co}(M) \subseteq \langle E_M(0) \rangle$.

- The following modules satisfy the above condition:
  - the regular module ${}_RR$ when R is commutative,
  - a projective module M over a 2-primal ring R.

The first case is easy to see. For the second case, let $m \in \beta_{co}(M) = \beta_{co}(R)M = \beta(R)M$. Then, $m = \sum_{i=1}^{n} a_i m_i$ with $a_i^{k_i} m_i = 0$ for some positive integer k_i since each $a_i \in \beta(R)$ and $\beta(R)$ is nil. It follows that $a_i m_i \in E_M(0)$ for each i and hence $m \in \langle E_M(0) \rangle$.

Some question

- We know that all modules over commutative rings are 2-primal but not all s.t.r.f.
- Is there an example of a module that s.t.r.f but not 2-primal?

Conclusion

- "2-primal modules" is a better generalisation than " modules that s.t.r.f". This is because all modules over commutative rings are 2-primal just like all commutative rings are 2-primal. On the contrary, not all modules over commutative rings s.t.r.f.
- There was considerable research aimed at getting examples of modules that s.t.r.f, e.g., see [5, 7, 8, 9, 10, 11] among others. Now that there is a generalisation better than the notion of modules that s.t.r.f, i,e., that of 2-primal modules, it is hoped that there will be interest by different researchers to search for examples of 2-primal modules, in addition to those pointed out in [3].

Danke!

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